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Axial-vector anomaly in lattice gauge theory

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An exact derivation of the anomalous Ward-Takahashi identities on a finite lattice is given. It is shown in a general way that the contribution from the fermion-degeneracy regularization in the limit leads to the continuum form of the anomaly term. Thus the interconnection is established independently of perturbation theory.

I. INTRODUCTION AND SUMMARY

For the nonperturbative analysis of gauge fields, the introduction of a lattice[1] turned out to be most promising. A better understanding of the fermion-degeneracy problem,[2-5] however, is important for the inclusion of fermions in the numerical calculations[6] of fundamental features of quantum chromodynamics (QCD), and a prerequisite for the extension of lattice methods to electroweak theories. The latter is desirable because dynamical symmetry breaking,[7] as an alternative to the difficulties with elementary scalars,[8] basically requires nonperturbative methods. Recently Karsten and Smit[9,10] showed in weak-coupling perturbation theory at the one-loop level that Wilson's fermion-degeneracy regularization[2] gives rise to the triangle anomaly.[11] This suggests that one should investigate whether such an interconnection can be established independently of perturbation theory, which would be a step forward with respect to the nonperturbative analysis.

In the present paper I give an exact derivation of the anomalous Ward-Takahashi identities on a finite lattice. Then I show in a general way that the contribution from Wilson's degeneracy regularization[2] in the limit leads to the continuum form of the anomaly term. Further, I point out that the alternative regularization of Osterwalder and Seiler[5] gives the same limit. The interconnection of interest is thus established independently of perturbation theory.

The crucial property of degeneracy regularizations so far was to ensure the correct limit for fermion loops in perturbation theory. A nonperturbative criterion is now that they must give the anomaly term correctly.

The limit of the exact identity derived here can be considered as the proper definition of the corresponding path-integral relations in continuum theory. Then one has a well-defined formulation with a γ_5-invariant measure of the integrals, the anomaly arising from the degeneracy regularization. This appears to be more satisfactory than the alternative approach of Fujikawa[12] who starts from an ill-defined theory and then regulates the measure to get the desired result.

The anomaly term in the following emerges in a form which may be viewed as a generalization of the representation $\mu\,\mathrm{tr}(\gamma_5 G_c)$ of Schwinger[13] and of Brown, Carlitz, and Lee,[14] in which G_c is the continuum fermion propagator and μ its mass. The role of μ is in the present context taken by the degeneracy regularization.

In Sec. II, after defining the formulation, the anomalous Ward-Takahashi identities are derived. Section III is devoted to the investigation of the continuum limit.

II. ANOMALOUS WARD-TAKAHASHI IDENTITIES

The finite lattice to be used here has $\mathfrak{N}=16N_1N_2N_3N_4$ sites in four-dimensional Euclidean space. Periodicity for $n_\lambda \to n_\lambda + 2N_\lambda$ in the numbering of the variables is imposed as a "boundary condition." The action is

$$S=v\sum_{n',n}\bar\psi_{n'}(\not{D}-W+M)_{n'n}\psi_n+S_G\,, \qquad (2.1)$$

where S_G is the pure gauge field part, $v=a_1a_2a_3a_4$, and $M_{n'n}=m\delta_{n'n}$. $\not{D}=\sum_\lambda\gamma_\lambda D_\lambda$ and $W=\sum_\lambda W_\lambda$ are given by

$$D_{\lambda n',n}=(U^\dagger_{\lambda n'}\delta_{n'+\lambda,n}-U_{\lambda n}\delta_{n',n+\lambda})/(2a_\lambda)\,, \qquad (2.2)$$

$$W_{\lambda n',n}=(U^\dagger_{\lambda n'}\delta_{n'+\lambda,n}+U_{\lambda n}\delta_{n',n+\lambda}-2\delta_{n'n})/(2a_\lambda)\,, \qquad (2.3)$$

where $U_{\lambda n}=\exp(iga_\lambda A_{\lambda n})$, with $A_{\lambda n}=\sum_l T^l A^l_{\lambda n}$ in the non-Abelian case.

W in (2.1) is the degeneracy regularization term introduced by Wilson[2] to overcome the problems related to the doubling of the fermion spectrum on the lattice. In particular, it guarantees the correct continuum limit of perturbation theory. Classically W vanishes in the limit. Osterwalder and Seiler[5] use $R=-i\gamma_5 W$ instead of W, which enables them to construct a Hilbert space with positive metric.

A general correlation function has the form $\int e^{-S}Q/\int e^{-S}$, where $\int$ means $\int_U\int_\psi$, with $\int_\psi$ standing for the Grassmann-variable integrations $\prod_{n,\beta}\int d\psi_{n\beta}d\bar\psi_{n\beta}$ and $\int_U$ similarly for the invariant integrations over the gauge group. The appropriate gauge-fixing factors are to be included in Q to correspond to the usual continuum forms. In the following it suffices to consider $\int_\psi e^{-S}P$, which can be obviously supplemented to $\int e^{-S}Q/\int e^{-S}$ at any stage. A simple example for a choice of P is $\psi_{n'\beta'}\bar\psi_{n\beta}$.

After the transformation to variables $\psi'_n = \exp(i\alpha_n\gamma_5)\psi_n$ and $\bar\psi'_n = \bar\psi_n\exp(i\alpha_n\gamma_5)$ in $\int_\psi e^{-S}P$, it follows that

$$-\frac{\partial}{\partial\alpha_n}\int_{\psi'} e^{-S}P = \int_{\psi'} e^{-S}\left(\frac{\partial S}{\partial\alpha_n}P - \frac{\partial P}{\partial\alpha_n}\right) = 0\,. \tag{2.4}$$

Performing the derivatives, and then going back to the variables $\psi_n, \bar\psi_n$, (2.4) can readily be cast into the form

$$\int_\psi e^{-S}\Bigg[\left(\sum_\lambda (J^5_{\lambda n} - J^5_{\lambda, n-\lambda})/a_\lambda - 2m\bar\psi_n i\gamma_5\psi_n\right)P + \frac{1}{v}\sum_{n',\beta}\left(-\frac{\partial P}{\partial\psi_{n'\beta}}(\delta_{n'n}i\gamma_5\psi_n)_\beta + (\bar\psi_n i\gamma_5\delta_{nn'})_\beta\frac{\partial P}{\partial\bar\psi_{n'\beta}}\right)\Bigg] + X_n = 0\,, \tag{2.5}$$

where left derivatives with respect to the Grassmann variables are understood and the abbreviations

$$J^5_{\lambda n} = \tfrac{1}{2}(\bar\psi_n i\gamma_\lambda\gamma_5 U^\dagger_{\lambda n}\psi_{n+\lambda} + \bar\psi_{n+\lambda}i\gamma_\lambda\gamma_5 U_{\lambda n}\psi_n)\,, \tag{2.6}$$

$$X_n = \int_\psi e^{-S}\sum_{n',\lambda}(\bar\psi_{n'}W_{\lambda n'n}i\gamma_5\psi_n + \bar\psi_n i\gamma_5 W_{\lambda nn'}\psi_{n'})P \tag{2.7}$$

have been introduced. To proceed further, by exploiting the general property $\int_\psi(\partial/\partial\psi_{n\beta})Q = 0$ of $\int_\psi$ for the case $Q = e^{-S}\psi_{n'\beta'}P$, the relation

$$\sum_{n'',\beta''}\int_\psi e^{-S}\psi_{n'\beta'}\bar\psi_{n''\beta''}P(\not D - W + M)_{n''\beta''n\beta} = \frac{1}{v}\delta_{n'n}\delta_{\beta'\beta}\int_\psi e^{-S}P - \frac{1}{v}\int_\psi e^{-S}\psi_{n'\beta'}\frac{\partial P}{\partial\psi_{n\beta}} \tag{2.8}$$

is derived. It is to be noted that from P only even Grassmann elements contribute to the integrals in (2.8). With $G = (\not D - W + M)^{-1}$ one obtains from (2.8)

$$\int_\psi e^{-S}\bar\psi_{n\beta}\psi_{n'\beta'}P = -\frac{1}{v}G_{n'\beta'n\beta}\int_\psi e^{-S}P - \frac{1}{v}\sum_{n'',\beta''}\int_\psi e^{-S}\frac{\partial P}{\partial\psi_{n''\beta''}}G_{n''\beta''n\beta}\psi_{n'\beta'}\,. \tag{2.9}$$

It is understood here that G^{-1} either has no zero modes or that they are appropriately handled to keep G well defined. Using (2.9) and an analogous relation involving $\partial/\partial\bar\psi_{n\beta}$, (2.7) can be written

$$X_n = -\frac{i}{v}\operatorname{tr}[\gamma_5(GW+WG)_{nn}]\int_\psi e^{-S}P + \int_\psi e^{-S}\frac{1}{v}\sum_{n',\beta}\left(-\frac{\partial P}{\partial\psi_{n'\beta}}[(GW)_{n'n}i\gamma_5\psi_n]_\beta + [\bar\psi_n i\gamma_5(WG)_{nn'}]_\beta\frac{\partial P}{\partial\bar\psi_{n'\beta}}\right), \tag{2.10}$$

where tr refers to γ matrices as well as to internal-symmetry indices, while Tr (to be used below) applies only to the latter.

Now (2.5) combined with (2.10) gives the exact anomalous Ward-Takahashi identities on the finite lattice. The current (2.6) is associated to the link from n to $n+\lambda$, thus having a structure reminiscent of the point-separation forms of continuum theory.[13,15]

III. CONTINUUM LIMIT

From (2.5) with (2.10) it becomes obvious that the usual continuum result is obtained if in the limit

$$\frac{1}{v}\operatorname{tr}[\gamma_5(GW+WG)_{nn}] \to \frac{g^2}{16\pi^2}\operatorname{Tr}\sum_{\mu\nu\lambda\rho}\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}(x)F_{\lambda\rho}(x)\,, \tag{3.1}$$

$$\frac{1}{v}[\delta_{n'n} + (GW)_{n'n}] \to \delta^4(x'-x)\,. \tag{3.2}$$

This will be shown in the following.

For the evaluation of the left-hand side of (3.1), the expansion

$$G = (\not D + W - M)(\mathcal{G} - \mathcal{G}V\mathcal{G} + \mathcal{G}V\mathcal{G}V\mathcal{G} \mp \cdots) \tag{3.3}$$

with $\mathcal{G} = [D^2 - (W-M)^2]^{-1}$ and $V = \Sigma + \Gamma$ is used, where $\Sigma = (i/2)\sum_{\lambda',\lambda}\sigma_{\lambda'\lambda}[D_{\lambda'}, D_\lambda]$ and $\Gamma = \sum_{\lambda',\lambda}\gamma_{\lambda'}[D_{\lambda'}, W_\lambda]$ with

$$[D_{\lambda'}, D_\lambda]_{n'n} = -ig(F^{\mathrm{I}}_{\lambda'\lambda,n'}\delta_{n'+\lambda'+\lambda,n} + F^{\mathrm{II}}_{\lambda'\lambda,n'-\lambda}\delta_{n'+\lambda'-\lambda,n} + F^{\mathrm{III}}_{\lambda'\lambda,n'-\lambda'}\delta_{n'-\lambda'+\lambda,n} + F^{\mathrm{IV}}_{\lambda'\lambda,n'-\lambda'-\lambda}\delta_{n'-\lambda'-\lambda,n})/4 \tag{3.4}$$

and $[D_{\lambda'}, W_\lambda]_{n'n}$ differing from (3.4) only by having minus signs in front of F^{II} and F^{IV}. In (3.4) one has

$$\begin{aligned} F^{\mathrm{I}}_{\lambda'\lambda,n} &= (U^\dagger_{\lambda n}U^\dagger_{\lambda',n+\lambda} - U^\dagger_{\lambda' n}U^\dagger_{\lambda,n+\lambda'})/(iga_{\lambda'}a_\lambda)\,,\\ F^{\mathrm{II}}_{\lambda'\lambda,n} &= (U^\dagger_{\lambda',n+\lambda}U_{\lambda,n+\lambda'} - U_{\lambda n}U^\dagger_{\lambda' n})/(iga_{\lambda'}a_\lambda)\,,\\ F^{\mathrm{III}}_{\lambda'\lambda,n} &= (U_{\lambda' n}U^\dagger_{\lambda n} - U^\dagger_{\lambda,n+\lambda'}U_{\lambda',n+\lambda})/(iga_{\lambda'}a_\lambda)\,,\\ F^{\mathrm{IV}}_{\lambda'\lambda,n} &= (U_{\lambda,n+\lambda'}U_{\lambda' n} - U_{\lambda',n+\lambda}U_{\lambda n})/(iga_{\lambda'}a_\lambda)\,. \end{aligned} \tag{3.5}$$

The $F^{\mathrm{I}}, \ldots, F^{\mathrm{IV}}$ are related to the four plaquettes having the point n in common, and in the limit all give $F_{\lambda'\lambda}(x)$. In the strict mathematical sense it is to be assumed that either (3.3) converges or that the formal expansion already suffices for the present purpose. When inserting (3.3) into (3.1) only terms of third and higher order in $\mathcal{G}$ contribute. If the higher-order ones as well as those with Γ vanish in the limit, as will turn out later, the relevant contribution to $\mathrm{tr}[\gamma_5(GW)_{nn}]$ is

$$\mathrm{Tr}\sum_{\mu\nu\lambda\rho}\epsilon_{\mu\nu\lambda\rho}((W-M)\mathcal{G}[D_\mu,D_\nu]\mathcal{G}[D_\lambda,D_\rho]\mathcal{G}W)_{nn}\,. \tag{3.6}$$

Now, commuting $[D_{\lambda'}, D_\lambda]$ with $\mathcal{G}$ as well as replacing $\mathcal{G}$ by $\mathcal{G}_0$ without gauge fields amounts to omitting higher orders. Similarly, the commutation with W and the replacement of W by W_0 corresponds to the omission of terms, which will be seen to vanish in the limit. Thus

$$([D_\mu,D_\nu](W_0-M)\mathcal{G}_0{}^3W_0[D_\lambda,D_\rho])_{nn} \tag{3.7}$$

may be considered instead of the matrix element in (3.6).

Inserting (3.4) into (3.7) one gets 16 terms; for example,

$$-g^2F^{\mathrm{I}}_{\mu\nu,n}((W_0-M)\mathcal{G}_0{}^3W_0)_{n+\mu+\nu,\,n-\lambda+\rho}F^{\mathrm{II}}_{\lambda\rho,n-\rho}/16\,. \tag{3.8}$$

By using the transformation $\mathfrak{N}^{-1/2}\exp(-\pi i\sum_\lambda r_\lambda n_\lambda/N_\lambda)$, for the matrix element in (3.8) the representation

$$((W_0-M)\mathcal{G}_0{}^3W_0)_{n+\mu+\nu,\,n-\lambda+\rho} = \mathfrak{N}^{-1}\sum_r \mathscr{G}_r(0) \tag{3.9}$$

is obtained, where for later convenience the abbreviation

$$\mathscr{G}_r(\alpha) = -\frac{(w-\alpha-m)(w-\alpha)}{\left(\sum_\lambda s_\lambda{}^2+(w-\alpha-m)^2\right)^3} \times\exp\left[i\pi\left(\frac{r_\mu}{N_\mu}+\frac{r_\nu}{N_\nu}+\frac{r_\lambda}{N_\lambda}-\frac{r_\rho}{N_\rho}\right)\right], \tag{3.10}$$

with $s_\lambda = \sin(\pi r_\lambda/N_\lambda)/a_\lambda$ and $w=\sum_\lambda[\cos(\pi r_\lambda/N_\lambda)-1]/a_\lambda$, has been introduced. So far the summation over r_λ in (3.9) is from $-N_\lambda+1+\eta_\lambda$ to $N_\lambda+\eta_\lambda$, where η_λ is some integer. By an appropriate choice of η_λ and a shift of the summation indices in one half of the intervals by N_λ, $\sum_r$ can be replaced by $\sum_r'$ for which the summations are restricted to $-N_\lambda/2 < r_\lambda \leq N_\lambda/2$. This gives

$$\sum_r \mathscr{G}_r(0) = \sum_r{}'\Big(\mathscr{G}_r(0) - \sum_\lambda \mathscr{G}_r(m_\lambda) + \sum_{\lambda'>\lambda}\mathscr{G}_r(m_{\lambda'}+m_\lambda) - \sum_{\lambda''>\lambda'>\lambda}\mathscr{G}_r(m_{\lambda''}+m_{\lambda'}+m_\lambda) + \mathscr{G}_r(m_1+m_2+m_3+m_4)\Big), \tag{3.11}$$

where $m_\lambda = 2\cos(\pi r_\lambda/N_\lambda)/a_\lambda$. The crucial point is now that for $\sum_r'$, in which $k_\lambda = \pi r_\lambda/(a_\lambda N_\lambda)$ and $s_\lambda = \sin(k_\lambda a_\lambda)/a_\lambda$ are uniquely related, the continuum limit can safely be obtained. Thereby $(\mathfrak{N}v)^{-1}\sum_r'\mathscr{G}_r(0)$ vanishes while for the seven positive and eight negative contributions to (3.11) with $\mathscr{G}_r$ depending on the m_λ, for $N_\lambda\to\infty$ and a_λ small, $-(\mathfrak{N}v)^{-1}\sum_r'\mathscr{G}_r$ tends versus

$$(2\pi)^{-4}\int d^4k\,\alpha^2(k^2+\alpha^2)^{-3}\,, \tag{3.12}$$

where α is a constant of order $1/a_\lambda$. Since the integral (3.12) independently of α equals $(32\pi^2)^{-1}$, this gives the limit. A further consequence of the α independence of (3.12) is that a replacement of W by cW in (2.1), where c is a finite constant, does not change the result. Because with a denominator $(k^2+\alpha^2)^{3+\nu}$ the integral is proportional to $1/\alpha^{2\nu}$, the higher orders vanish indeed. The Γ terms involving $[D_{\lambda'}, W_\lambda]$, those with $[W_\sigma, [D_{\lambda'}, D_\lambda]]$ arising from the interchange, and the ones with $W-W_0$ from omitting gauge fields do not contribute because (due to forming commutators and a difference, respectively) they have one driving factor less in the numerator.

It can now be seen that the matrix elements of $(W_0-M)\mathcal{G}_0{}^3W_0$ of the 16 terms in (3.7) all become $(32\pi^2)^{-1}$, and thus with (3.6) one gets in fact (3.1). One has, however, to realize that $F_{\mu\nu}(x)$ and $F_{\lambda\rho}(x)$ arise as the limits of $(F^{\mathrm{I}}_{\mu\nu,n}+F^{\mathrm{II}}_{\mu\nu,n-\nu}+F^{\mathrm{III}}_{\mu\nu,n-\mu}+F^{\mathrm{IV}}_{\mu\nu,n-\mu-\nu})/4$ and $(F^{\mathrm{I}}_{\lambda\rho,n-\lambda-\rho}+F^{\mathrm{II}}_{\lambda\rho,n-\lambda}+F^{\mathrm{III}}_{\lambda\rho,n-\rho}+F^{\mathrm{IV}}_{\lambda\rho,n})/4$, respectively (with $x_\lambda = a_\lambda n_\lambda$). While for classical gauge fields the interpretation of these limits would be the naive one, in the full quantum case one has to note that the $F^{\mathrm{I}}, \ldots, F^{\mathrm{IV}}$ are functions of the gauge variables.

To show (3.2) one proceeds similarly as for (3.1). The left-hand side of (3.2), omitting gauge fields, can be represented as

$$(\mathfrak{N}v)^{-1}\sum_r \exp\left(\pi i \sum_\lambda r_\lambda \frac{n'_\lambda - n_\lambda}{N_\lambda}\right)[1+\mathcal{K}_r(1,0)]\,, \tag{3.13}$$

where

$$\mathcal{K}_r(h,\alpha) = -\frac{\left(i\sum_{\lambda'} h_{\lambda'}\gamma_{\lambda'}s_{\lambda'} + w - \alpha - m\right)(w-\alpha)}{\sum_\lambda s_\lambda{}^2 + (w-\alpha-m)^2}\,, \tag{3.14}$$

with h defined such that $h_\lambda = \pm 1$ in an appropriate way. When turning over to $\sum'_r$, in addition to $[1+\mathcal{K}_r(1,0)]$ terms of type $(-1)^{n'_\lambda - n_\lambda}\{1 + \mathcal{K}_r[h(\lambda), m_\lambda]\}$, $(-1)^{n'_\lambda - n_\lambda + n'_\sigma - n_\sigma}\{1+\mathcal{K}_r[h(\lambda,\sigma), m_\lambda + m_\sigma]\}$, etc., occur. In the limit the 1 of the first term gives $\delta^4(x'-x)$ in the Fourier representation, while $\mathcal{K}_r(1,0)$ vanishes. The other terms, because of $(-1)^{n'_\lambda - n_\lambda} = \exp[i(\pi/a_\lambda)(x'_\lambda - x_\lambda)]$ are oscillated away (also without the $\mathcal{K}_r$). In addition, the $\mathcal{K}_r$ depending on the m_λ tend to -1. The terms arising from the omission of gauge fields are seen to vanish *a fortiori*.

Finally the alternative regularization of Osterwalder and Seiler is considered, which is obtained by replacing W by $R = -i\gamma_5 W$. The only essential change in the presented derivation is that instead of (3.3) the expansion

$$G = (\not{D} - R - M)(\mathcal{G}_R - \mathcal{G}_R V_5 \mathcal{G}_R + \mathcal{G}_R V_5 \mathcal{G}_R V_5 \mathcal{G}_R \mp \cdots) \tag{3.15}$$

with $\mathcal{G}_r = (D^2 + R^2 - M^2)^{-1}$ and $V_5 = \Sigma - i\gamma_5\Gamma$ is to be used. From (3.15) one then obtains

$$\mathrm{tr}(\gamma_5 G R) = \mathrm{tr}[\gamma_5(\not{D} - R - M)(\mathcal{G}_R V_5 \mathcal{G}_R V_5 \mathcal{G}_R \mp \cdots)R]\,, \tag{3.16}$$

which shows that one gets the same limit as before.

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[1]K. G. Wilson, Phys. Rev. D 10, 2445 (1974).
[2]K. G. Wilson, in *New Phenomena in Subnuclear Physics*, Proceedings of the 14th course of the International School of Subnuclear Physics, Erice, 1975, edited by A. Zichichi (Plenum, New York, 1977), part A, p. 69.
[3]J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975).
[4]S. D. Drell, M. Weinstein, and S. Yankielowicz, Phys. Rev. D 14, 1627 (1976).
[5]K. Osterwalder and E. Seiler, Ann. Phys. (N. Y.) 110, 440 (1978).
[6]M. Creutz, Phys. Rev. Lett. 45, 313 (1980); Phys. Rev. D 21, 2308 (1980); and references cited therein.
[7]For a recent survey see H. Pagels, Phys. Lett. 87B, 222 (1979).
[8]L. Susskind, Phys. Rev. D 20, 2619 (1979).
[9]L. H. Karsten, Ph.D. thesis, Amsterdam, 1979 (unpublished).
[10]L. H. Karsten and J. Smit, Stanford University Report No. ITP-677, 1980 (unpublished).
[11]S. L. Adler, Phys. Rev. 177, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969).
[12]K. Fujikawa, Phys. Rev. Lett. 42, 1195 (1979); Phys. Rev. D 21, 2848 (1980).
[13]J. Schwinger, Phys. Rev. 82, 664 (1951).
[14]L. S. Brown, R. D. Carlitz, and C. Lee, Phys. Rev. D 16, 417 (1977).
[15]K. Johnson, Nucl. Phys. 25, 431 (1961).