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String tension in (2+1)-dimensional compact lattice QED: Weak- and strong-coupling results; a variational calculation

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The Hamiltonian and wave functions are written as a function of the gauge-invariant magnetic field only. A variational calculation with an ansatz, separable in plaquette space, leads to Mathieu's equation for the wave functions and a self-consistency constraint. This we solve analytically in both the strong- and weak-coupling limit, and we calculate the string tension. We find the electric flux to be confined to a tube of finite radius for all couplings.

I. INTRODUCTION

Quark confinement in the context of quantum chromodynamics (QCD) is still an unsolved problem. It is not a perturbative effect. So new, nonperturbative approaches had to be developed to deal with gauge theories. One was the formulation of the theory on a lattice by Wilson[1] and by Kogut and Susskind.[2] For large enough coupling it is easy to see that QED and QCD confine charges and quarks,[1-4] respectively. In four space-time dimensions QED has a nonconfining phase for low enough coupling,[4-6] a result that was hoped for since in the continuum limit the lattice coupling constant goes to zero and lattice QED, a U(1) gauge theory, should go into continuum QED, which does not confine charges. On the other hand, QCD, an SU(n) gauge theory, should stay in the confining phase all the way down to zero coupling. Quarks, after all, are supposed to be confined.

Balian, Drouffe, and Itzykson[7,8] found, from a strong-coupling expansion extrapolated to weak coupling by using Padé approximants, that in high enough space-time dimensions SU(n) gauge theories undergo a phase transition to a nonconfining phase for small couplings. This suggests that the Abelian and non-Abelian theories have very similar confining properties: they show confinement for all couplings in few space-time dimensions, $d<4$ for the U(1) theory, $d\leq 4$ in the non-Abelian case, but they have a phase transition to a nonconfining phase in higher dimensions.

Compact QED, which because of its Abelian character is easier to handle than non-Abelian theories, serves therefore as an interesting model for studying confinement. It also arises naturally as the U(1)-invariant subgroup of a spontaneously broken non-Abelian theory, such as the unified models of weak and electromagnetic interactions.

Polyakov[5] was the first to show that compact QED confines charges, in three space-time dimensions, for all couplings. He sketched it for a lattice theory and later showed it, working in continuous space-time, for the spontaneously broken Georgi-Glashow model, where the mass of the vector bosons $W^{\pm}$ serves essentially as a momentum cutoff in the photon sector. Banks and collaborators[9] obtained the same result for a Villain approximation to the Abelian theory on a lattice. Both sets of authors used a path-integral approach in Euclidean space (imaginary time) and traced the mechanism of confinement to contributions of pseudoparticle solutions, monopoles, to the Euclidean action. These monopoles have long-range Coulomb interactions, but screening occurs that makes the Green's function of the monopole gas short range. As a result the monopoles cannot screen out completely the magnetic field inside the dipole sheet created by the stationary electric current of the Wilson loop, thereby causing confinement.

Drell and co-workers[10] approached the problem of confinement in (2+1)-dimensional compact lattice QED at small coupling with a variational calculation in the Hamiltonian formulation. They make heavy use of the periodicity of the Hamiltonian as a function of the magnetic field. For the trial wave functions they use a separable ansatz in momentum space, consisting of a product of "periodic Gaussian" wave functions, and find confinement to be a consequence of the tunneling between neighboring wells.

In this paper we will follow a similar approach. However, we choose for our ansatz a wave function that is separable in plaquette space. The single-plaquette wave functions are determined variationally. This leads us to a differential equation of the Mathieu type for these single-plaquette wave functions, together with a Hartree-Fock-type self-consistency condition, which incorporates correlations among nearest-neighbor plaquettes. This real-space product ansatz has two main advantages:

(i) It allows us to solve the problem from strong coupling all the way down to weak coupling. For two opposite charges, separated by a distance L, we find an energy of the form

$$E = TL \,, \tag{1.1}$$

where T is the string tension. In the strong-coupling limit we are able to reproduce the usual strong-coupling expansion result[2]

$$T = \tfrac{1}{2} g^2 + O\left(\frac{1}{g^6}\right). \tag{1.2}$$

In the weak-coupling limit we find the exponentially small string tension[5,9,10]

$$T \sim e^{-c/g^2} \,, \tag{1.3}$$

where c is a constant. In our calculation, $c = 2$.

(ii) It gives directly a nice picture of the electric flux configuration, leading to confinement. The flux is confined to a tube of finite radius r between the charges. For strong coupling this radius is only a few lattice spacings, whereas in the weak-coupling limit it grows exponentially like

$$r \sim e^{2/g^2} \,. \tag{1.4}$$

Outside this tube the electric field is screened, a consequence of the self-interactions of the photon which reflects itself in the compactness and hence periodicity of the Hamiltonian. Tunneling between neighboring potential wells makes the self-consistency condition of the form of a nonlinear Debye equation, preventing the electric flux from spreading out in a Coulomb-type fashion.

Originally we followed the approach of Drell *et al.*[10] for writing gauge-invariant states. They separated out the longitudinal, Coulomb part from the Hamiltonian and the states. Investigating the periodicity of the remaining Hamiltonian, they found a Bloch-type behavior with phases ϵ, which in turn depend on the classical Coulomb electric field. But at the end of our calculation we found that all dependence on the Coulomb terms had canceled. Adam Schwimmer then pointed out to us a simple way of writing gauge-invariant states and it is this approach which we will use in this paper.

The outline of the paper is as follows. In Sec. II we set up the problem and show how to write gauge-invariant states. In Sec. III we do the variational calculation that leads to Mathieu's equation for the single-plaquette wave functions and the consistency condition. The solution to this is shown in Sec. IV for strong coupling and in Sec. V for the weak-coupling limit. Section VI contains some concluding remarks.

II. GAUGE-INVARIANT STATES

We start with the usual Hamiltonian for compact QED on a lattice in the temporal gauge, $A_0 = 0$,

$$H = \frac{g^2}{2} \sum_{\vec{p},a} (E^a_{\vec{p}})^2 + \frac{1}{g^2} \sum_{\vec{p}} (1 - \cos B_{\vec{p}}) \,. \tag{2.1}$$

The Hamiltonian is written in terms of scaled, dimensionless variables, introduced by Drell *et al.*,[10] defined by

$$\begin{aligned} &e a^{1/2} \to g \,, \\ &e a A \to A \,, \\ &e a^2 B \to B \,, \\ &\frac{1}{e} a E \to E \,, \\ &a H \to H \,. \end{aligned} \tag{2.2}$$

The electric field $E^a_{\vec{p}}$ and the vector potential $A^a_{\vec{p}}$ are link variables. They are labeled by a site $\vec{p} = (p_1, p_2)$ and a direction $a = 1, 2$. They are located on the link between sites $\vec{p}$ and $\vec{p} + \hat{a}$, where $\hat{a}$ is the unit vector in the direction a. There are two ways of labeling a link, $(\vec{p}, a)$ or $(\vec{p} + \hat{a}, -a)$. The corresponding link variables are related by

$$A^{-a}_{\vec{p}+\hat{a}} = -A^a_{\vec{p}} \,, \quad E^{-a}_{\vec{p}+\hat{a}} = -E^a_{\vec{p}} \,. \tag{2.3}$$

$B_{\vec{p}}$ is the lattice magnetic field, a one-dimensional axial vector, normal to the plane of the lattice:

$$\begin{aligned} B_{\vec{p}} = (\vec{\nabla} \times \vec{A})_{\vec{p}} &= A^1_{\vec{p}} + A^2_{\vec{p}+\hat{1}} - A^1_{\vec{p}+\hat{2}} - A^2_{\vec{p}} \\ &= \sum_{a,b=1,2} \epsilon_{ab} (A^a_{\vec{p}} - A^a_{\vec{p}+\hat{b}}) \,. \end{aligned} \tag{2.4}$$

We will call such variables plaquette variables. The lattice curl and all subsequently introduced difference operators are defined such that their relationship to the differential operators in the continuum limit is given by

$$\vec{\nabla}_{\text{lattice}} \to a \vec{\nabla}_{\text{continuum}} \text{ for } a \to 0 \,, \tag{2.5}$$

where a is the lattice spacing. The electric field and the vector potential are canonically conjugate variables,

$$[A^a_{\vec{p}}, E^b_{\vec{p}'}] = -i \delta_{\vec{p},\vec{p}'} \delta^{a,b} \,. \tag{2.6}$$

The Hamiltonian is periodic in the vector potential with period 2π and it is assumed to operate on a Hilbert space of wave functions $\psi(\{A^a_{\vec{p}}\})$ with the same periodicity. This periodicity expresses the compactness of the theory: the trivial U(1) Lie algebra is realized on a circle, not a line. Confinement is intimately related to compactness since one can now see that the electric field is quantized. On our Hilbert space $E^a_{\vec{p}}$ is realized by

$$\hat{E}^a_{\vec{p}} = i \frac{\partial}{\partial A^a_{\vec{p}}} \,. \tag{2.7}$$

Eigenfunctions of the operator are $e^{-iE^a_{\vec{p}} A^a_{\vec{p}}}$, where $E^a_{\vec{p}}$ is the eigenvalue of the operator $\hat{E}^a_{\vec{p}}$. Periodicity implies

$$E^a_{\vec{p}} = n^a_{\vec{p}}\ , \quad \text{an integer}\ . \tag{2.8}$$

We can span our Hilbert space with a basis of eigenvectors of the electric field on each link, $|\{n^b_{\vec{q}}\}\rangle$,

$$\hat{E}^a_{\vec{p}}\,|\{n^b_{\vec{q}}\}\rangle = n^a_{\vec{p}}\,|\{n^b_{\vec{q}}\}\rangle\ . \tag{2.9}$$

$e^{\mp i A^a_{\vec{p}}}$ are raising and lowering operators:

$$e^{\mp i A^a_{\vec{p}}}\,|\{n^b_{\vec{q}}\}\rangle = |\{n^b_{\vec{q}} \pm \delta_{\vec{p},\vec{q}}\delta^{a,b}\}\rangle\ . \tag{2.10}$$

The canonical equations of motion give the compact versions of Ampére's and Faraday's laws, but Coulomb's law does not appear because it contains no time derivatives. On the other hand,

$$[H, (\vec{\nabla}\cdot\vec{E})_{\vec{p}}] = 0\ , \tag{2.11}$$

where the lattice divergence is

$$(\vec{\nabla}\cdot\vec{E})_{\vec{p}} = E^1_{\vec{p}} + E^2_{\vec{p}} - E^1_{\vec{p}-\hat{1}} - E^2_{\vec{p}-\hat{2}} = \sum_a (E^a_{\vec{p}} - E^a_{\vec{p}-\hat{a}})\ . \tag{2.12}$$

We can therefore seek simultaneous eigenfunctions of H and $(\vec{\nabla}\cdot\vec{E})_{\vec{p}}$. The eigenvalues $\rho_{\vec{p}}$ of $(\vec{\nabla}\cdot\vec{E})_{\vec{p}}$ are static external charges. Because of the quantization of the electric field $E^a_{\vec{p}}$ the external charges must be quantized as integers as well.

The statement $A_0 = 0$ is only an incomplete gauge fixing. The temporal-gauge Hamiltonian is still invariant under time-independent gauge transformations

$$A^a_{\vec{p}} \to A^a_{\vec{p}} - (\vec{\nabla}\Lambda)^a_{\vec{p}}\ , \tag{2.13}$$

where $\Lambda_{\vec{p}}$ is a time-independent scalar defined on the site $\vec{p}$. The lattice gradient is

$$(\vec{\nabla}\Lambda)^a_{\vec{p}} = \Lambda_{\vec{p}+\hat{a}} - \Lambda_{\vec{p}}\ . \tag{2.14}$$

It is defined such that it obeys the usual law for the continuum gradient: $\vec{\nabla}\times(\vec{\nabla}\Lambda)_{\vec{p}} = 0$. The gauge transformation can be written

$$U(\{\Lambda_{\vec{q}}\})A^a_{\vec{p}}U^\dagger(\{\Lambda_{\vec{q}}\}) = A^a_{\vec{p}} - (\vec{\nabla}\Lambda)^a_{\vec{p}}\ , \tag{2.15}$$

where

$$U = \exp\left[-i\sum_{\vec{q},b}(\vec{\nabla}\Lambda)^b_{\vec{q}}E^b_{\vec{q}}\right]. \tag{2.16}$$

We will work on a periodic lattice, so $A^a_{\vec{p}}$ is periodic and, to preserve this for the gauge-transformed potentials, $\Lambda_{\vec{p}}$ has to be periodic too. Then we can sum by parts:

$$U = \exp\left[i\sum_{\vec{q}}\Lambda_{\vec{q}}(\vec{\nabla}\cdot\vec{E})_{\vec{q}}\right]. \tag{2.17}$$

Thus $(\vec{\nabla}\cdot\vec{E})$ is the generator of these time-independent gauge transformations. We are interested in gauge-covariant states that are eigenfunctions of U, $\vec{\nabla}\cdot\vec{E}$, and H. They form the physical subspaces of our Hilbert space. Such states are linear combinations of basis states given in (2.9) which satisfy the restriction

$$(\vec{\nabla}\cdot\vec{n})_{\vec{p}} = \rho_{\vec{p}}\ . \tag{2.18}$$

Each given charge distribution $\{\rho_{\vec{p}}\}$ specifies a different physical subspace of the Hilbert space. In writing a variational wave function we already make progress if we can exclude unwanted contributions from the start. This amounts to the task of removing the remaining gauge dependence in the temporal-gauge formulation, so that both the wave function and the Hamiltonian are gauge covariant.

Drell and his collaborators have one approach. Ours is the following.

A. Uncharged sectors

For the vacuum and its excitations ($\rho_{\vec{p}} = 0$) the electric flux is conserved at each site. The electric field pattern is therefore an arbitrary superposition of elementary loops where one unit of flux runs around the links bounding a single plaquette. The electric flux of such an elementary loop can be written as a lattice curl,

$$n^a_{\vec{p}} = (\vec{\nabla}\times\phi)^a_{\vec{p}} \quad \text{with } \phi_{\vec{p}} = \begin{cases} 1, & \vec{p} = \vec{p}_0 \\ 0, & \text{otherwise}\ . \end{cases} \tag{2.19}$$

Here the curl of a plaquette variable must be distinguished from the curl of a link variable, (2.4). It is

$$(\vec{\nabla}\times\phi)^a_{\vec{p}} = \sum_b \epsilon_{ab}(\phi_{\vec{p}} - \phi_{\vec{p}-\hat{b}}) = \begin{cases} \phi_{\vec{p}} - \phi_{\vec{p}-\hat{2}}\ , & a = 1 \\ -\phi_{\vec{p}} + \phi_{\vec{p}-\hat{1}}\ , & a = 2\ . \end{cases} \tag{2.20}$$

The definition is arranged so that $\vec{\nabla}\cdot(\vec{\nabla}\times\phi) = 0$. Then

$$n^1_{\vec{p}_0} = -n^1_{\vec{p}_0+\hat{2}} = -n^2_{\vec{p}_0} = n^2_{\vec{p}_0+\hat{1}} = 1\ .$$

Hence any distribution of the integer-valued plaquette field $\{\phi_{\vec{p}}\}$ yields a gauge-invariant state $|\{n^a_{\vec{p}}\}\rangle$. We are thankful to Adam Schwimmer for pointing this out to us.

Not all $\{\phi_{\vec{p}}\}$ fields give different states, but the degeneracy is easily characterized. Two fields give the same electric field if $\vec{\nabla}\times(\phi - \phi')_{\vec{p}} = 0$, from which it follows that $\phi_{\vec{p}} - \phi'_{\vec{p}} = \text{constant}$. Hence if we require

$$-\frac{N}{2} < \sum_{\vec{p}}\phi_{\vec{p}} \le \frac{N}{2}\ , \quad N = \text{number of lattice sites} \tag{2.21}$$

the relation is one to one, and gauge-invariant states in the uncharged sector can be written as $|\{\phi_{\vec{p}}\}\rangle$. The restriction (2.21) allows us to define these states as orthonormalized:

$$\langle\{\phi'_{\vec{p}}\}|\{\phi_{\vec{p}}\}\rangle = \prod_{\vec{p}} \delta_{\phi'_{\vec{p}},\phi_{\vec{p}}} . \qquad (2.22)$$

The operators appearing in the Hamiltonian have the effects on these states, from the construction

$$E^a_{\vec{p}}|\{\phi_{\vec{q}}\}\rangle = (\vec{\nabla}\times\phi)^a_{\vec{p}}|\{\phi_{\vec{q}}\}\rangle \qquad (2.23)$$

and

$$e^{\mp iB_{\vec{p}}}|\{\phi_{\vec{q}}\}\rangle = |\{\phi_{\vec{q}} \pm \delta_{\vec{p},\vec{q}}\}\rangle . \qquad (2.24)$$

The second relation follows from the fact that $B_{\vec{p}}$ is an elementary loop of the vector potential around plaquette $\vec{p}$ and $e^{\mp iA^a_{\vec{p}}}$ raises (lowers) the eigenvalue of the electric field by one unit.

B. Charged sectors

Two opposite unit charges. Now we must have a string of unit flux from the positive to the negative charge, plus arbitrary closed loops of electric flux. Such a state can be characterized by a string of unit flux and an arbitrary $\{\phi_{\vec{p}}\}$ field that again satisfies the restriction (2.21). The path of the string is immaterial since an appropriate $\{\phi_{\vec{p}}\}$ field will reroute it in any allowed way. We will put the positive and negative charges on sites $\vec{p}_+ = (-M/2, 0)$ and $\vec{p}_- = (M/2, 0)$, respectively, and a string of unit strength $\vec{n}^0_{\vec{p}}$ running along the x axis between them.

In general. We characterize gauge-invariant states by the shortest strings $\vec{n}^0_{\vec{p}}$ between opposite charges, that satisfy $(\vec{\nabla}\cdot\vec{n}^0)_{\vec{p}} = \rho_{\vec{p}}$, plus an arbitrary $\phi_{\vec{p}}$ field, $|\{\vec{n}^0_{\vec{q}}\},\{\phi_{\vec{q}}\}\rangle$. As before we find the effects of the relevant operators:

$$E^a_{\vec{p}}|\{\vec{n}^0_{\vec{q}}\},\{\phi_{\vec{q}}\}\rangle = [n^{0a}_{\vec{p}} + (\vec{\nabla}\times\phi)^a_{\vec{p}}]|\{\vec{n}^0_{\vec{q}}\},\{\phi_{\vec{q}}\}\rangle , \qquad (2.25)$$

$$e^{\mp iB_{\vec{p}}}|\{\vec{n}^0_{\vec{q}}\},\{\phi_{\vec{q}}\}\rangle = |\{\vec{n}^0_{\vec{q}}\},\{\phi_{\vec{q}} \pm \delta_{\vec{p},\vec{q}}\}\rangle . \qquad (2.26)$$

A general gauge-invariant state can be written as a superposition of our gauge-invariant basis states:

$$|\psi\rangle = \sum_{\{\phi_{\vec{q}}\}}{}' \Psi_{\{\vec{n}^0_{\vec{q}}\}}(\{\phi_{\vec{q}}\})|\{\vec{n}^0_{\vec{q}}\},\{\phi_{\vec{q}}\}\rangle ; \qquad (2.27)$$

the prime indicates that the sum goes only over $\phi_{\vec{q}}$ fields satisfying the restriction (2.21). H can now be written as an operator on the wave functions,

$$H\Psi_{\{\vec{n}^0_{\vec{q}}\}}(\{\phi_{\vec{q}}\}) = \frac{g^2}{2}\sum_{\vec{p},a}[n^{0a}_{\vec{p}} + (\vec{\nabla}\times\phi)^a_{\vec{p}}]^2\Psi_{\{\vec{n}^0_{\vec{q}}\}}(\{\phi_{\vec{q}}\}) + \frac{1}{g^2}\sum_{\vec{p}}[\Psi_{\{\vec{n}^0_{\vec{q}}\}}(\{\phi_{\vec{q}}\}) - \tfrac{1}{2}\Psi_{\{\vec{n}^0_{\vec{q}}\}}(\{\phi_{\vec{q}} - \delta_{\vec{p},\vec{q}}\}) - \tfrac{1}{2}\Psi_{\{\vec{n}^0_{\vec{q}}\}}(\{\phi_{\vec{q}} + \delta_{\vec{p},\vec{q}}\})] . \qquad (2.28)$$

These wave functions which depend on integer arguments are not very physical. We introduce a continuous plaquette field $B_{\vec{p}}$, $-\pi \le B_{\vec{p}} \le \pi$, via the generating functional,

$$\tilde{\Psi}_{\{\vec{n}^0_{\vec{q}}\}}(\{B_{\vec{q}}\}) = \sum_{\{\phi_{\vec{q}}\}}{}' \tilde{\Psi}_{\{\vec{n}^0_{\vec{q}}\}}(\{\phi_{\vec{q}}\})\exp\left(-i\sum_{\vec{q}}\phi_{\vec{q}}B_{\vec{q}}\right) . \qquad (2.29)$$

This corresponds to a transformation of basis, for each plaquette, $|\psi_{\vec{q}}\rangle \to |B_{\vec{q}}\rangle$ with

$$\langle B_{\vec{q}}|\phi_{\vec{q}}\rangle = e^{-iB_{\vec{q}}\phi_{\vec{q}}} ,$$
$$\langle B'_{\vec{q}}|B_{\vec{q}}\rangle = 2\pi\delta(B'_{\vec{q}} - B_{\vec{q}}) , \qquad (2.30)$$
$$1_{\vec{q}} = \int_{-\pi}^{\pi}\frac{dB_{\vec{q}}}{2\pi}|B_{\vec{q}}\rangle\langle B_{\vec{q}}| .$$

The effects of the operators on the wave functions $\tilde{\Psi}$ are now

$$E^a_{\vec{p}}\tilde{\Psi}_{\{\vec{n}^0_{\vec{q}}\}}(\{B_{\vec{q}}\}) = \left[n^{0a}_{\vec{p}} + \left(\vec{\nabla}\times\frac{i\partial}{\partial B_{\vec{p}}}\right)^a\right]\tilde{\Psi}_{\{\vec{n}^0_{\vec{q}}\}}(\{B_{\vec{q}}\}) , \qquad (2.31)$$

$$e^{\mp iB_{\vec{p}}}{}_{\text{operator}}\,\tilde{\Psi}_{\{\vec{n}^0_{\vec{q}}\}}(\{B_{\vec{q}}\}) = e^{\mp iB_{\vec{p}}}\tilde{\Psi}_{\{\vec{n}^0_{\vec{q}}\}}(\{B_{\vec{q}}\}) , \qquad (2.32)$$

which shows that the plaquette field $B_{\vec{p}}$ is just the magnetic field. The Hamiltonian operating on $\tilde{\Psi}$ becomes

$$H\Psi_{\{\vec{n}^0_{\vec{q}}\}}(\{B_{\vec{q}}\}) = \left\{\frac{g^2}{2}\sum_{\vec{p}}\left[\vec{n}^0_{\vec{p}} + \left(\vec{\nabla}\times\frac{i\partial}{\partial B_{\vec{p}}}\right)\right]^2 + \frac{1}{g^2}\sum_{\vec{p}}(1 - \cos B_{\vec{p}})\right\}\tilde{\Psi}_{\{\vec{n}^0_{\vec{q}}\}}(\{B_{\vec{q}}\}) . \qquad (2.33)$$

At this point we see that both the Hamiltonian and the wave functions are functions only of the gauge-invariant magnetic field. Note that from the definition $\tilde{\Psi}$ is periodic in $B_{\vec{p}}$ with period 2π:

$$\tilde{\Psi}_{\{\vec{n}^0_{\vec{q}}\}}(\{B_{\vec{q}} + 2\pi\delta_{\vec{p},\vec{q}}\}) = \tilde{\Psi}_{\{\vec{n}^0_{\vec{q}}\}}(\{B_{\vec{q}}\}) . \qquad (2.34)$$

III. THE VARIATIONAL WAVE EQUATIONS

We want to find the ground-state energy of the Hamiltonian (2.33), which has no additional gauge freedom, in a variational way according to the Rayleigh-Ritz procedure. We choose an ansatz wave function that is separable in plaquette space:

$$\tilde{\Psi}(\{B_{\vec{q}}\}) = \prod_{\vec{q}} \psi_{\vec{q}}(B_{\vec{q}}) . \tag{3.1}$$

According to (2.34) each plaquette wave function is periodic in its magnetic field.

We consider the variational energy

$$E = \frac{\langle \tilde{\psi} | H | \tilde{\psi} \rangle}{\langle \tilde{\psi} | \tilde{\psi} \rangle} = \frac{g^2}{2} \sum_{\vec{p}} (\vec{n}^0_{\vec{p}})^2 + \frac{1}{g^2} \sum_{\vec{p}} 1 - 2g^2 \sum_{\vec{p}} \frac{\langle \psi_{\vec{p}} |}{\langle \psi_{\vec{p}} | \psi_{\vec{p}} \rangle} \left\{ \frac{\partial^2}{\partial B_{\vec{p}}^2} + \frac{1}{2} \left[\sum_a \frac{\langle \psi_{\vec{p}-\hat{a}} | i\partial / \partial B_{\vec{p}-\hat{a}} | \psi_{\vec{p}-\hat{a}} \rangle}{\langle \psi_{\vec{p}-\hat{a}} | \psi_{\vec{p}-\hat{a}} \rangle} - (\vec{\nabla} \times \vec{n}^0_{\vec{p}}) \right] \frac{i\partial}{\partial B_{\vec{p}}} + \frac{1}{2g^4} \cos B_{\vec{p}} \right\} | \psi_{\vec{p}} \rangle . \tag{3.2}$$

We vary it with respect to $\langle \psi_{\vec{p}} |$ and find a single-plaquette wave equation

$$\left(\frac{\partial^2}{\partial B_{\vec{p}}^2} + 2 s_{\vec{p}} \frac{i\partial}{\partial B_{\vec{p}}} + \frac{1}{2g^4} \cos B_{\vec{p}} + \Lambda_{\vec{p}} \right) \psi_{\vec{p}}(B_{\vec{p}}) = 0 . \tag{3.3}$$

The eigenvalue $\Lambda_{\vec{p}}$ comes from the variation of the norm and the coefficient $s_{\vec{p}}$ is given by

$$2 s_{\vec{p}} = \frac{1}{2} \left[\sum_{i=\pm1,\pm2} \frac{\langle \psi_{\vec{p}-\hat{a}} | i\partial / \partial B_{\vec{p}-\hat{a}} | \psi_{\vec{p}-\hat{a}} \rangle}{\langle \psi_{\vec{p}-\hat{a}} | \psi_{\vec{p}-\hat{a}} \rangle} - (\vec{\nabla} \times \vec{n}^0)_{\vec{p}} \right] . \tag{3.4}$$

It depends on the solutions of the wave equation on neighboring plaquettes and therefore is a Hartree-Fock-type self-consistency constraint.

The wave equation can be converted into Mathieu's equation with the following definitions:

$$\psi_{\vec{p}}(B_{\vec{p}}) = e^{-i s_{\vec{p}} B_{\vec{p}}} \phi_{\vec{p}}(B_{\vec{p}}) , \qquad \alpha(s_{\vec{p}}) = s_{\vec{p}}^2 + \Lambda_{\vec{p}} , \quad \delta = \frac{1}{2g^4} . \tag{3.5}$$

Then $\phi_{\vec{p}}$ satisfies Mathieu's equation

$$\left(\frac{d^2}{dx^2} + \alpha(s) + \delta \cos x \right) \phi(x) = 0 . \tag{3.6}$$

The periodicity of ψ implies the boundary condition of ϕ:

$$\phi(x + 2\pi) = e^{2\pi i s} \phi(x) . \tag{3.7}$$

We can solve (3.6) and (3.7) and find the ground-state eigenvalue $\alpha(s)$ for phase s, as well as the matrix element

$$\kappa(s) = \frac{\langle \phi | i d/dx | \phi \rangle}{\langle \phi | \phi \rangle} . \tag{3.8}$$

Having done this, the self-consistency constraint (3.4) becomes the equation for $s_{\vec{p}}$,

$$\nabla^2 [s + \kappa(s)]_{\vec{p}} + 4\kappa(s_{\vec{p}}) = (\vec{\nabla} \times \vec{n}^0)_{\vec{p}} , \tag{3.9}$$

where the lattice Laplacian is

$$(\nabla^2 s)_{\vec{p}} = \vec{\nabla} \cdot (\vec{\nabla} s)_{\vec{p}} = \sum_a (s_{\vec{p}+\hat{a}} + s_{\vec{p}-\hat{a}} - 2 s_{\vec{p}}) . \tag{3.10}$$

We can look at the self-consistency condition in another way. The periodic wave functions ψ satisfy

$$-\left(\frac{d^2}{dx^2} + 2s \frac{id}{dx} - s^2 + \delta \cos x \right) \psi(x) = \alpha(s) \psi(x) , \qquad \frac{\langle \psi | i d/dx | \psi \rangle}{\langle \psi | \psi \rangle} = s + \kappa(s) . \tag{3.11}$$

First we note that, making a small change $s \to s + \Delta s$ in (3.11) and using first-order perturbation theory, we can find the relation

$$\frac{d}{ds} \alpha(s) = -2\kappa(s) . \tag{3.12}$$

We next assume the phases $s_{\vec{p}}$ are not determined by condition (3.9) but rather we treat them as additional variational parameters. Thus we rewrite the variational energy as a function of the $s_{\vec{p}}$'s, using the wave equation (3.11):

$$E = 2g^2 \sum_{\vec{p}} \Big\{ \tfrac{1}{4} (\vec{n}^0_{\vec{p}})^2 + \frac{1}{2g^4} + \alpha(s_{\vec{p}}) + s_{\vec{p}}^2 - \tfrac{1}{2} \sum_a s_{\vec{p}} s_{\vec{p}-\hat{a}} + \tfrac{1}{2} s_{\vec{p}} (\vec{\nabla} \times \vec{n}^0)_{\vec{p}} + 2 s_{\vec{p}} \kappa(s_{\vec{p}}) - \tfrac{1}{2} \sum_a [s_{\vec{p}-\hat{a}} \kappa(s_{\vec{p}}) + s_{\vec{p}} \kappa(s_{\vec{p}-\hat{a}})] + \tfrac{1}{2} (\vec{\nabla} \times \vec{n}^0)_{\vec{p}} \kappa(s_{\vec{p}}) - \tfrac{1}{2} \sum_a \kappa(s_{\vec{p}}) \kappa(s_{\vec{p}-\hat{a}}) \Big\} . \tag{3.13}$$

Minimizing this with respect to $s_{\vec{p}}$, $\partial E/\partial s_{\vec{p}}=0$ gives the condition

$$0=\left[1+\frac{d\kappa(s_{\vec{p}})}{ds_{\vec{p}}}\right]\{-\tfrac{1}{2}(\nabla^2 s)_{\vec{p}}-\tfrac{1}{2}[\nabla^2\kappa(s)]_{\vec{p}}-2\kappa(s_{\vec{p}})+\tfrac{1}{2}(\vec{\nabla}\times\vec{n}^0)_{\vec{p}}\}. \quad (3.14)$$

Therefore we find that the consistency condition (3.9) minimizes the energy as a function of the phases $s_{\vec{p}}$.

Making use of the consistency condition we can simplify the energy. After some resummations we find

$$E=2g^2\sum_{\vec{p}}\left\{\tfrac{1}{4}(\vec{n}^0_{\vec{p}})^2+\frac{1}{2g^4}+\alpha(s_{\vec{p}})+s_{\vec{p}}\kappa(s_{\vec{p}})+\tfrac{1}{4}\vec{n}^0_{\vec{p}}\cdot\vec{\nabla}\times[s+\kappa(s)]_{\vec{p}}\right\}. \quad (3.15)$$

In Sec. II we said that the path of the string $\vec{n}_{\vec{p}}$ is immaterial. We will now show that the energy (3.15) really does not depend on this path. We remark that the $s_{\vec{p}}$'s are phases. Therefore, $\alpha(s_{\vec{p}})$ and $\kappa(s_{\vec{p}})$ must be periodic functions of $s_{\vec{p}}$, since changing $s_{\vec{p}}$ by an integer does not change the phase. However, a change by integers in the phases $s_{\vec{p}}$ must be accompanied by an appropriate change of the string in such a way that the consistency equation (3.9) is still satisfied. This is achieved by

$$s_{\vec{p}}\to s_{\vec{p}}+l_{\vec{p}}\,,\quad l_{\vec{p}}\ \text{integers},\qquad \vec{n}_{\vec{p}}\to\vec{n}_{\vec{p}}-(\vec{\nabla}\times l)_{\vec{p}} \quad (3.16)$$

because the curl of the plaquette variable (2.20) satisfies the relation

$$\vec{\nabla}\times(\vec{\nabla}\times l)_{\vec{p}}=-(\nabla^2 l)_{\vec{p}}\,. \quad (3.17)$$

Thus changing $s_{\vec{p}}\to s_{\vec{p}}-1$ gives rise to a closed loop of unit strength on the links counterclockwise around plaquette $\vec{p}$. The string can be deformed in any possible way compatible with $(\vec{\nabla}\cdot\vec{n})_{\vec{p}}=\rho_{\vec{p}}$. $[\vec{\nabla}\cdot(\vec{\nabla}\times l)_{\vec{p}}=0.]$ Under the change (3.16), the energy (3.15) is unchanged:

$$E\to 2g^2\sum_{\vec{p}}\left\{\tfrac{1}{4}[\vec{n}^0_{\vec{p}}-(\vec{\nabla}\times l)_{\vec{p}}]^2+\frac{1}{2g^4}+\alpha(s_{\vec{p}})+(s_{\vec{p}}+l_{\vec{p}})\kappa(s_{\vec{p}})+\tfrac{1}{4}[\vec{n}^0_{\vec{p}}-(\vec{\nabla}\times l)_{\vec{p}}]\cdot\vec{\nabla}\times[s+l+\kappa(s)]_{\vec{p}}\right\}.$$

After some resummations,

$$E\to E+2g^2\sum_{\vec{p}}l_{\vec{p}}\{-\tfrac{1}{4}(\vec{\nabla}\times\vec{n}^0)_{\vec{p}}+\kappa(s_{\vec{p}})+\tfrac{1}{4}\nabla^2[s+\kappa(s)]_{\vec{p}}\}$$

and therefore, because of the consistency equation, the quantity in curly brackets vanishes and the energy is unchanged.

For the vacuum the inhomogeneous terms on the right-hand side of the consistency equation (3.9) are missing, the solution is $s=0$, and α is the lowest eigenvalue of a periodic solution of Mathieu's equation. The energy

$$E_0=2g^2\sum_{\vec{p}}\left[\frac{1}{2g^4}+\alpha(0)\right] \quad (3.18)$$

diverges as the volume of the system. In the presence of charges, s is nonzero around the string. A flux tube is developed inside to which the electric flux is confined. The radius of the flux tube is finite, narrow at strong coupling, and growing with a tunneling-type exponential dependence on $1/g^2$ for weak coupling. The difference $\Delta E=E-E_0$ has no infrared (volume) divergence. The details of the solution for strong and weak coupling will be carried out in the next two sections.

IV. SOLUTION FOR STRONG COUPLING

In the strong-coupling limit, $g^2\gg 1$, the parameter $\delta=1/2g^4$ is very small and we can treat the term $\delta\cos x$ in Mathieu's equation

$$\left(-\frac{d^2}{dx^2}-\delta\cos x\right)\phi(x)=\alpha(s)\phi(x)\,,\quad \phi(x+2\pi)=e^{2\pi i s}\phi(x) \quad (4.1)$$

perturbatively. The unperturbed states and eigenvalues are

$$\phi_n^{(0)}(x)=\frac{1}{\sqrt{2\pi}}e^{i(s+n)x}\,,\quad \alpha_n^{(0)}=(s+n)^2\,,\quad n\ \text{an integer}. \quad (4.2)$$

The ground state is the one for which $|s+n|\le\frac{1}{2}$. For $s\ne m+\frac{1}{2}$ no states are degenerate and we use Schrödinger-Rayleigh perturbation theory to find

$$\phi_n(x)=\frac{1}{\sqrt{2\pi}}e^{i(s+n)x}+\frac{\delta/2}{\sqrt{2\pi}}\left[\frac{e^{i(s+n+1)x}}{2(s+n)+1}+\frac{e^{i(s+n-1)x}}{-2(s+n)+1}\right], \quad (4.3)$$

$$\alpha_n(s)=(s+n)^2+\frac{\delta^2}{2}\frac{1}{[4(s+n)^2-1]}\,, \quad (4.4)$$

$$\kappa_n(s)=\frac{\langle\phi_n|id/dx|\phi_n\rangle}{\langle\phi_n|\phi_n\rangle}=-(s+n)+2\delta^2\frac{(s+n)}{[4(s+n)^2-1]^2}\,, \quad (4.5)$$

up to higher orders in δ.

In Appendix A we will diagonalize $-(d^2/dx^2+\delta\cos x)$ in the subspace $n=-m,-(m+1)$, which is degenerate for $s=m+\frac{1}{2}$, and show that the state with $|s+n|\le\frac{1}{2}$ really is the ground state. $\alpha(s)$ and $\kappa(s)$ are then periodic in s, as we anticipated, and satisfy the relation (3.12): $(d/ds)\alpha(s) = -2\kappa(s)$. Now we take $\delta=1/2g^4$ and display the dependence on plaquette $\vec{p}$, to solve the consistency condition (3.9). We are interested in the case of two opposite unit charges, the positive at $\vec{p}=(-M/2,0)$, the negative at $\vec{p}=(M/2,0)$. Then

$$n_{\vec{p}}^{0a}=\begin{cases}1 & \text{for } -\frac{M}{2}\le p_1\le\frac{M}{2}\,, p_2=0, a=1\,,\\ 0\,, & \text{otherwise}\end{cases} \tag{4.6}$$

$$(\vec{\nabla}\times\vec{n}^0)_{\vec{p}}=\begin{cases}1\,, & \text{in plaquettes just above the string}\\ -1\,, & \text{in plaquettes just below the string}\\ 0\,, & \text{otherwise}\,.\end{cases}$$

We then can solve equation (3.9) perturbatively, using

$$S_{\vec{p}}=S_{\vec{p}}^{(0)}+\frac{1}{g^8}S_{\vec{p}}^{(1)}\,, \tag{4.7}$$

and find

$$S_{\vec{p}}^{(0)}=\begin{cases}-\frac{1}{4}\,, & \text{in plaquettes just above the string}\\ +\frac{1}{4}\,, & \text{in plaquettes just below the string}\\ 0\,, & \text{otherwise}\,.\end{cases} \tag{4.8}$$

$S_{\vec{p}}^{(1)}$ is nonzero in plaquettes that are second-nearest neighbors to the string, corresponding to the fact that as one goes along in the strong-coupling series, longer deviations from a straight flux path come in. It turns out that only $S_{\vec{p}}^{(0)}$ is needed to calculate ΔE up to order $1/g^6$. We find, from (3.15),

$$\Delta E=E-E_0=M\left(\frac{g^2}{2}-\frac{1}{6g^6}\right)\,. \tag{4.9}$$

M is the length of the string between the two charges, measured in units of the lattice spacing. These two terms agree exactly with the strong-coupling expansion through this order.

The expectation value of the electric field is, from (2.31) and (3.11),

$$\langle E_{\vec{p}}^a\rangle=\frac{\langle\tilde{\Psi}|E_{\vec{p}}^a|\tilde{\Psi}\rangle}{\langle\tilde{\Psi}|\tilde{\Psi}\rangle}$$

$$=n_{\vec{p}}^{0a}+\sum_b\epsilon_{ab}[s_{\vec{p}}+\kappa(s_{\vec{p}})-s_{\vec{p}-\hat{b}}-\kappa(s_{\vec{p}-\hat{b}})] \tag{4.10}$$

and we can easily see that, through the order considered, the flux tube is only one plaquette in radius. Of course, our product wave function only allows nearest-neighbor correlations, in the sense that

$$\langle E_{\vec{p}}^aE_{\vec{p}'}^{a'}\rangle=\langle E_{\vec{p}}^a\rangle\langle E_{\vec{p}'}^{a'}\rangle+\text{nothing else}\,, \tag{4.11}$$

except when the electric fields lie on links bounding the same plaquette. So eventually we depart from the strong-coupling series.

V. SOLUTION FOR WEAK COUPLING

For weak coupling, $g^2\ll 1$, the parameter $\delta=1/2g^4$ is large. Then in Mathieu's equation

$$\left(-\frac{d^2}{dx^2}-\delta\cos x\right)\phi(x)=\alpha(s)\phi(x)\,,\quad \phi(x+2\pi)=e^{2\pi i s}\phi(x)\,, \tag{5.1}$$

the periodic potential $-\delta\cos x$ is very deep; see Fig. 1. In this case we can solve Mathieu's equation with the WKB approximation. The lowest energy levels lie near the bottom of the well, with just enough tunneling to satisfy the boundary condition given in Eq. (5.1). Because of this boundary condition the wave function is complex. In the classically allowed region and near the turning points (x_1,x_2) the WKB approximation does not apply for the ground state. Instead we expand the potential about the minima at $x=0$ and 2π, which leads to a differential equation whose solutions are the parabolic cylinder functions. They are valid well into the forbidden region. There then they can be matched onto the WKB solutions, which are valid deep in the forbidden region. Implementing finally the boundary condition $\phi(x+2\pi)=e^{2\pi is}\phi(x)$ we find the ground-state eigenvalue

$$\alpha(s)=-\frac{1}{2g^4}+\frac{1}{2g^2}-\frac{1}{16}+O(g^2)-\frac{\lambda}{\pi}\cos(2\pi s)\,, \tag{5.2}$$

where λ contains the tunneling dependence on g,

$$\lambda=\frac{4\sqrt{2\pi}}{g^3}e^{-4/g^2}\,. \tag{5.3}$$

The first four terms come from the "naive" per-

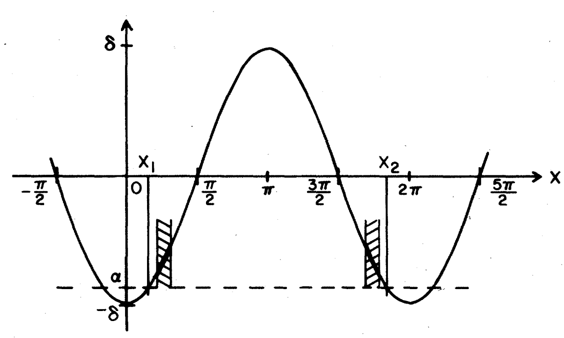

FIG. 1. Potential and ground-state energy for weak coupling. x_1 and x_2 are the classical turning points. Matching of the two solutions is done in the shaded regions.

turbation series for the ground-state energy in a potential $-\delta\cos x$ ignoring the possibility of tunneling. It cancels out in $\alpha(s)-\alpha(0)$ which contributes to the energy difference $\Delta E = E - E_0$ that we are interested in. The calculation leading to the result [(2.2) and (2.3)] is presented in greater detail in Appendix B. $\kappa(s)$ can now be obtained by using the relation (3.12). On the other hand, it can be calculated explicitly. The ground-state wave function can be written in the form

$$\phi(x) = E(x) + iO(x)\,, \tag{5.4}$$

where E and O are, respectively, even and odd real functions of x. To calculate κ we need the integral

$$\begin{aligned} I &= \int_{-\pi}^{\pi} dx\, \phi^*(x)\frac{id\,\phi(x)}{dx} \\ &= \int_{-\pi}^{\pi} dx\left[O(x)\frac{dE(x)}{dx} - E(x)\frac{dO(x)}{dx}\right]. \end{aligned} \tag{5.5}$$

The integral is simply a Wronskian of two linearly independent solutions E and O of a differential equation of the form $(d^2/dx^2)f(x) + g(x)f(x) = 0$, from which it follows that

$$O(x)\frac{dE(x)}{dx} - E(x)\frac{dO(x)}{dx} = \text{const.} \tag{5.6}$$

From our explicit solution we can calculate the value of this constant at, say, $x=0$. κ is then 2π times this constant divided by the norm, which we calculate in Appendix B.

Both methods lead to the result

$$\kappa(s) = -\lambda\,\sin(2\pi s)\,. \tag{5.7}$$

In the weak-coupling limit, $|\kappa(s)| \ll |s|$ and $\kappa(s_{\vec{p}})$ do not change much between neighboring plaquettes. Thus the consistency equation (3.9) is well approximated by

$$(\nabla^2 s)_{\vec{p}} - 4\lambda\sin(2\pi s_{\vec{p}}) = (\vec{\nabla}\times\vec{n}^0)_{\vec{p}}\,. \tag{5.8}$$

The flux is now spread out over many plaquettes, so we approximate this difference equation by the differential equation obtained in the continuum limit: the lattice spacing $a \to 0$. Setting $\vec{x} = a\vec{p}$, $D = aM$, we find from (4.6)

$$(\vec{\nabla}\times\vec{n}^0)_{\vec{p}} \underset{a\to 0}{\sim} -a^2\vartheta\left(\frac{D^2}{4} - x^2\right)\frac{d}{dy}\delta(y)\,. \tag{5.9}$$

Then, using the relation (2.9) between difference and differential operators, Eq. (5.8) becomes in the continuum limit

$$\nabla^2 s(\vec{x}) - \frac{4\lambda}{a^2}\sin(2\pi s) = -\vartheta\left(\frac{D^2}{4} - x^2\right)\frac{d}{dy}\delta(y)\,. \tag{5.10}$$

Because λ is so small, the tunneling term is only significant when s (and hence $\nabla^2 s$) is small. Therefore, we linearize the equation

$$\nabla^2 s(\vec{x}) - \frac{8\pi\lambda}{a^2} s(\vec{x}) = -\vartheta\left(\frac{D^2}{4} - x^2\right)\frac{d}{dy}\delta(y)\,. \tag{5.11}$$

The Green's function for the differential equation (5.11) is $(1/2\pi)K_0([(8\pi\lambda)^{1/2}/a]|\vec{x}|)$ and the solution s is thus

$$s(x,y) = -\frac{(8\pi\lambda)^{1/2}}{2\pi a}\,y\int_{-D/2}^{D/2}\frac{dx'}{[(x-x')^2+y^2]^{1/2}}K_1\left(\frac{(8\pi\lambda)^{1/2}}{a}[(x-x')^2+y^2]^{1/2}\right). \tag{5.12}$$

Here $K_n(z)$ are modified Bessel functions of order n. They fall off, for large z, like

$$K_n(z) \to \left(\frac{\pi}{2}\right)^{1/2} e^{-z} \quad \text{for } z \to \infty\,. \tag{5.13}$$

We see that the tunneling term screens s, with a screening length

$$r_0 = \frac{a}{(8\pi\lambda)^{1/2}} = \frac{ag^{3/2}}{4(2\pi)^{3/4}}\,e^{2/g^2}\,. \tag{5.14}$$

Far from the charges, s depends only on y. Then the integral for s can be evaluated and we find[11]

$$s(y) = -\tfrac{1}{2}(\text{sgn}y)e^{-|y|/r_0}\,. \tag{5.15}$$

Alternatively, if the x dependence is ignored, we can avoid the linearization approximation in (5.11) and simply solve the nonlinear ordinary differential equation

$$\frac{d^2 s}{dy^2} - \frac{4\lambda}{a^2}\sin(2\pi s) = -\frac{d}{dy}\delta(y) \tag{5.16}$$

for x lying between the charges, i.e., $|x| < D/2$. The solution is

$$s(y) = -\frac{2}{\pi}(\text{sgn}y)\arctan(e^{-|y|/r_0}) \quad \text{for } |x| < D/2\,. \tag{5.17}$$

For x values outside the charges, i.e., $|x| > D/2$, we set $s=0$. In this solution we make some errors in the vicinity of the charges. This gives rise to an error in the energy which is independent of the separation of the charges. But we are interested only in the part of the energy which grows with the separation, $E\sim MT$, where T is the string tension.

Using the solution (5.17) we can find the energy of unit length of the flux tube from the charges, which is just the string tension T. From the above, in weak coupling ($p_2=y/a$, $q_0=r_0/a$),

$$s_{\vec{p}}=-\frac{2}{\pi}(\operatorname{sgn}p_2)\arctan(e^{-|p_2|/q_0}),$$

$$\kappa(s_{\vec{p}})=4\lambda(\operatorname{sgn}p_2)\frac{e^{-|p_2|/q_0}(1-e^{-2|p_2|/q_0})}{(1+e^{-2|p_2|/q_0})^2}, \qquad (5.18)$$

$$\alpha(s_{\vec{p}})-\alpha(0)=\frac{8\lambda}{\pi}\frac{e^{-2|p_2|/q_0}}{(1+e^{-2|p_2|/q_0})^2}.$$

The energy $\Delta E=E-E_0$ is now, from (3.15) and with (4.6),

$$\Delta E=\frac{g^2}{2}\sum_{p_1=-M/2}^{M/2}[1+s_{(p_1,0)}+\kappa(s_{(p_1,0)})-s_{(p_1,-1)}-\kappa(s_{(p_1,-1)})]+2g^2\sum_{p_1=-M/2}^{M/2}\sum_{p_2=-\infty}^{\infty}[\alpha(s_{\vec{p}})-\alpha(0)+s_{\vec{p}}\kappa(s_{\vec{p}})]. \qquad (5.19)$$

Note that $s_{p_2=0}-s_{p_2=-1}=-1+1/\pi q_0$ and $\kappa(s_0)-\kappa(s_{-1})=O(\lambda/q_0)$ which is negligible relative to the first term. Thus the first sum gives a contribution to the string tension T,

$$T_1=\frac{g^2}{2\pi q_0}. \qquad (5.20)$$

In the second term, the sum over p_2 can be limited to $p_2>0$ by inserting a factor 2. We approximate the sum by an integral and change variables from p_2 to $u=e^{-p_2/q_0}$. Thus the contribution of the second term to the string tension is

$$T_2=2\times 2g^2\int_0^1\frac{q_0du}{u}\left[\frac{8\lambda}{\pi}\frac{u^2}{(1+u^2)^2}-\frac{8\lambda}{\pi}\frac{u(1-u^2)}{(1+u^2)^2}\arctan u\right]. \qquad (5.21)$$

The integrals are

$$\int_0^1\frac{du\,u}{(1+u^2)^2}=\frac{1}{4}, \quad \int_0^1\frac{du(1-u^2)}{(1+u^2)^2}\arctan u=\frac{\pi}{8}-\frac{1}{4}.$$

Thus, using (5.14) and $q_0=r_0/a$,

$$T_2=\frac{2g^2}{\pi^2q_0}-\frac{g^2}{2\pi q_0}. \qquad (5.22)$$

The total string tension is then

$$T=\frac{2g^2}{\pi^2q_0}=\frac{32g^{1/2}}{(2\pi)^{5/4}}e^{-2/g^2}. \qquad (5.23)$$

The string tension is exponentially small but finite for nonzero couplings, and vanishes with an essential singularity at zero coupling.

From the solution for $s_{\vec{p}}$ (5.18) and equation (4.10) we find the electric field between the two charges, approximating the difference in (4.10) by a derivative,

$$\langle E_{\vec{p}}^{\,1}\rangle=\frac{2}{\pi q_0}\frac{e^{-|p_2|/q_0}}{(e^{-2|p|/q_0}+1)}+O\left(\frac{\lambda}{q_0}\right) \text{ for } |p_1|<\frac{M}{2}. \qquad (5.24)$$

Thus the electric flux is confined to a tube of radius q_0 lattice spacings. The tunneling between neighboring wells in the periodic potential, $-\delta\cos x$, causes this confining of the electric flux and thus the linear confinement of the two static external charges.

Without tunneling, λ in (5.3) would be zero. Then the solution of the consistency condition (5.8) would be, calculated in the continuum approximation,

$$S_{\vec{p}}\simeq\frac{1}{2\pi}\left[\arctan\left(\frac{p_2}{p_1+M/2}\right)-\arctan\left(\frac{p_2}{p_1-M/2}\right)\right]. \qquad (5.25)$$

The corresponding electric field would be the usual Coulomb field for two space dimensions, and for the energy (5.19) we would find, apart from the infinite self-energy of the charges, the usual Coulomb energy for two space dimensions:

$$E=E_{\text{Coulomb}}\simeq\frac{g^2}{2\pi}\ln M. \qquad (5.26)$$

The development of an electric flux tube of radius q_0 and thus the linear confining energy with string tension (5.23) occurs only for charges separated much farther than the screening length q_0. For charges that are close together, the screening is too weak. They then develop in their vicinity a Coulomb-type electric field.

VI. CONCLUSIONS

We have performed a variational calculation with an ansatz separable in plaquette space. This led to Mathieu's equation and a self-consistency constraint. We were able to solve this analytically for strong and weak coupling. For intermediate couplings Mathieu's equation cannot be solved analytically, but a numerical solution can be obtained. We are working at solving Mathieu's equation for intermediate couplings on a computer. Having completed this we can evaluate the string tension all the way from the strong-coupling down to the weak-coupling region, always having used the same calculational method, our variational calculation. We then can see whether the string tension varies smoothly from the strong-coupling to the weak-coupling behavior, as suggested by Monte Carlo calculations,[12] or in a somewhat more abrupt fashion that would indicate a roughening transition. Such a roughening transition is indicated by a stable singularity in the logarithmic Padé approximants at $2/g^4 \approx 1.02$.[13]

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APPENDIX A: DEGENERATE PERTURBATION THEORY

For $s=m+\frac{1}{2}$ the unperturbed states (4.2) for $n=-m$ and $n=-(m+1)$ are degenerate. Here we will diagonalize $-(d^2/dx+\delta\cos x)=A$ for $s=m+\hat{s}$, $0\leq\hat{s}\leq 1$ exactly. We consider, for $n,n'=-m,\,-m-1$,

$$A_{n,n'}=\int_{-\pi}^{\pi}dx\,\phi_n^{*(0)}(x)A\phi_{n'}^{(0)}(x)=\begin{pmatrix}\hat{s}^2 & -\delta/2\\ -\delta/2 & (\hat{s}-1)^2\end{pmatrix} \tag{A1}$$

and find the eigenvalues

$$\alpha_{\mp}=\tfrac{1}{4}+(\hat{s}-\tfrac{1}{2})^2\mp[(\hat{s}-\tfrac{1}{2})^2+\delta^2/4]^{1/2}\,. \tag{A2}$$

The minus (–) refers to the ground state, the plus (+) to the first excited state. The orthonormalized eigenstates

$$\phi_{\mp}(x)=\frac{1}{\sqrt{2\pi}}(\beta_{\mp}e^{i\hat{s}x}+\gamma_{\mp}e^{i(\hat{s}-1)x}) \tag{A3}$$

have the coefficients

$$\begin{aligned}\beta_{\mp}&=\frac{\{(\hat{s}-\frac{1}{2})^2+\delta^2/4\mp(\hat{s}-\frac{1}{2})[(\hat{s}-\frac{1}{2})^2+\delta^2/4]^{1/2}\}^{1/2}}{\sqrt{2}\,[(\hat{s}-\frac{1}{2})^2+\delta^2/4]^{1/2}}\,,\\ \gamma_{\mp}&=\pm\frac{\{(\hat{s}-\frac{1}{2})^2+\delta^2/4\pm(\hat{s}-\frac{1}{2})[(\hat{s}-\frac{1}{2})^2+\delta^2/4]^{1/2}\}^{1/2}}{\sqrt{2}\,[(\hat{s}-\frac{1}{2})^2+\delta^2/4]^{1/2}}\,.\end{aligned} \tag{A4}$$

Doing usual perturbation theory with the other unperturbed states included, we find the ground state

$$\phi_0(x)=\phi_-(x)+\frac{\delta/2}{\sqrt{2\pi}}\left\{\frac{\beta_-e^{i(\hat{s}+1)x}}{3\hat{s}+\frac{1}{2}+[(\hat{s}-\frac{1}{2})^2+\delta^2/4]^{1/2}}+\frac{\gamma_-e^{i(\hat{s}-2)x}}{-3\hat{s}+\frac{7}{2}+[(\hat{s}-\frac{1}{2})^2+\delta^2/4]^{1/2}}\right\} \tag{A5}$$

as well as

$$\alpha_0(\hat{s})=\tfrac{1}{4}+(\hat{s}-\tfrac{1}{2})^2-[(\hat{s}-\tfrac{1}{2})^2+\delta^2/4]^{1/2}-\frac{\delta^2}{4}\left\{\frac{\beta_-^2}{3\hat{s}+\frac{1}{2}+[(\hat{s}-\frac{1}{2})^2+\delta^2/4]^{1/2}}+\frac{\gamma_-^2}{-3\hat{s}+\frac{7}{2}+[(\hat{s}-\frac{1}{2})^2+\delta^2/4]^{1/2}}\right\}, \tag{A6}$$

$$\kappa_0(\hat{s})=-(\hat{s}-1)-\beta_-^2-\frac{\delta^2}{4}\left\{\frac{\beta_-^2(2-\beta_-^2)}{\{3\hat{s}+\frac{1}{2}+[(\hat{s}-\frac{1}{2})^2+\delta^2/4]^{1/2}\}^2}+\frac{\gamma_-^2(-2+\gamma_-^2)}{\{-3\hat{s}+\frac{7}{2}+[(\hat{s}-\frac{1}{2})^2+\delta^2/4]^{1/2}\}^2}\right\}. \tag{A7}$$

For $\hat{s}\approx 0$, i.e., $s\approx m$ (up to order δ^2),

$$\begin{aligned}&\beta_-=1-\frac{\delta^2}{8(2\hat{s}-1)^2}\,,\quad \gamma_-=\frac{\delta}{2(1-2\hat{s})}\,,\\ &\alpha(s)=(s-m)^2+\frac{\delta^2}{2}\frac{1}{[4(s-m)^2-1]}\,,\\ &\kappa(s)=-(s-m)+2\delta^2\frac{(s-m)}{[4(s-m)^2-1]^2}\,,\end{aligned} \tag{A8}$$

which is exactly (4.4) and (4.5) for $n=-m$. For $\hat{s}\approx 1$, i.e., $s\approx m+1$ (up to order δ^2),

$$\begin{aligned}&\beta_-=\frac{\delta}{2(2\hat{s}-1)}\,,\quad \gamma_-=1-\frac{\delta^2}{8(2\hat{s}-1)^2}\,,\\ &\alpha(s)=[s-(m+1)]^2+\frac{\delta^2}{2}\frac{1}{\{4[s-(m+1)]^2-1\}}\,,\\ &\kappa(s)=-[s-(m+1)]+2\delta^2\frac{[s-(m+1)]}{\{4[s-(m+1)]^2-1\}^2}\,,\end{aligned} \tag{A9}$$

which is the same as (4.4) and (4.5) for $n=-(m+1)$. Thus we see that the ground state always has $|s+n|\leq\frac{1}{2}$.

For $\hat{s}=\frac{1}{2}$,

$$\beta_\mp=\frac{1}{\sqrt{2}}\,,\quad \gamma_\mp=\pm\frac{1}{\sqrt{2}}\,,$$
$$\alpha_\mp=\frac{1}{4}\mp\frac{\delta}{2}-\frac{\delta^2}{8}\,, \tag{A10}$$
$$\kappa_\mp=0\,.$$

For $\delta\neq 0$, the ground state and first excited state never cross. The second equation is a consequence of the fact that for half-integer s, the solutions of Mathieu's equation have definite parity.

APPENDIX B: DETAILS OF THE WEAK-COUPLING SOLUTION

In the weak-coupling limit, where δ is large, we expand the potential $-\delta\cos x$ near the minima $2\pi n$,

$$x=2\pi n+\xi\,,\quad \delta\cos x=\delta\left[1-\frac{\xi^2}{2}+\frac{\xi^4}{24}+O(\xi^6)\right]. \tag{B1}$$

Then we rescale

$$\alpha+\delta=(2\delta)^{1/2}(\nu+\tfrac{1}{2})\,,\quad (2\delta)^{1/4}\xi=y \tag{B2}$$

and Mathieu's equation (5.1) now reads

$$(2\delta)^{1/2}\left[\frac{d^2}{dy^2}+\nu+\frac{1}{2}-\frac{y^2}{4}+\frac{1}{(2\delta)^{1/2}}\frac{y^4}{48}+O\left(\frac{1}{2\delta}\right)\right]\phi(y)=0\,. \tag{B3}$$

We expand ν and ϕ in a power series in $1/(2\delta)^{1/2}$,

$$\nu=\nu_0+\frac{1}{(2\delta)^{1/2}}\nu_1+O(\delta^{-1})\,,$$
$$\phi=\phi_0+\frac{1}{(2\delta)^{1/2}}\phi_1+O(\delta^{-1})\,, \tag{B4}$$

and compare equal powers of $1/(2\delta)^{1/2}$. This gives us a series of coupled differential equations, the first of which is

$$\left(\frac{d^2}{dy^2}+\nu_0+\frac{1}{2}-\frac{y^2}{4}\right)\phi_0(y)=0\,. \tag{B5}$$

This is the differential equation for the parabolic cylinder functions (see Bateman[14]) and has the general solution

$$\phi_0(y)=\beta D_{\nu_0}(y)+\gamma D_{-\nu_0-1}(iy)\,. \tag{B6}$$

The second of the coupled differential equations determines ν_1 and ϕ_1. We only quote the result for ν_1:

$$\nu_1=-\tfrac{1}{8}(\nu_0^2+\nu_0+\tfrac{1}{2})\,. \tag{B7}$$

Thus near the bottom of the potential wells, near $x=0$ and $x=2\pi$, we have the wave functions

$$\phi_{PC}(x)=\beta_0 D_{\nu_0}((2\delta)^{1/4}x)+\gamma_0 D_{-\nu_0-1}(i(2\delta)^{1/4}x)\quad\text{near } x=0\,, \tag{B8}$$

$$\phi_{PC}(x)=\beta_1 D_{\nu_0}((2\delta)^{1/4}(x-2\pi))+\gamma_1 D_{-\nu_0-1}(i(2\delta)^{1/4}(x-2\pi))\quad\text{near } x=2\pi\,. \tag{B9}$$

From the condition $\phi(x+2\pi)=e^{2\pi i s}\phi(x)$ and the fact that D_{ν_0} and $D_{-\nu_0-1}$ are linearly independent, we find the relation between the coefficients,

$$\beta_1=e^{2\pi i s}\beta_0\,,\quad \gamma_1=e^{2\pi i s}\gamma_0\,. \tag{B10}$$

The classical turning points are at

$$\alpha+\delta\cos x_t=0\,. \tag{B11}$$

Using $\alpha=-\delta+(2\delta)^{1/2}(\nu_0+\frac{1}{2})$, we find them to be

$$x_t=\arccos\left(-\frac{\alpha}{\delta}\right)\simeq 2\pi n\pm\left(\frac{2}{\delta}\right)^{1/4}(2\nu_0+1)^{1/2}+O(\delta^{-3/4})\,. \tag{B12}$$

The WKB solution of Mathieu's equation (5.1) is, for x well in the forbidden region,

$$\phi_{WKB}(x)=(-\alpha-\delta\cos x)^{-1/4}\left\{C_1\exp\left[\int_{x_1}^{x}dx'(-\alpha-\delta\cos x')^{1/2}\right]+C_2\exp\left[-\int_{x_1}^{x}dx'(-\alpha-\delta\cos x')^{1/2}\right]\right\}, \tag{B13}$$

where x_1 is the turning point between 0 and π.

Left matching region. In the left shaded region, Fig. 1, for which we assume $1\gg x\gg x_1$ we have to match the parabolic cylinder solution (B8) onto the WKB solution (B13). In this region, $(2\delta)^{1/4}x\gg(2\delta)^{1/4}x_1\simeq 2(\nu_0+\frac{1}{2})^{1/2}=O(1)$ and we can therefore use the asymptotic expressions for the parabolic cylinder functions, which can be found in Bateman's book.[14] Thus

$$\phi_{PC}\simeq\beta_0 e^{-\sqrt{2\delta}x^2/4}(2\delta)^{\nu_0/4}x^{\nu_0}+\gamma_0 e^{\sqrt{2\delta}x^2/4}(2\delta)^{-(\nu_0+1)/4}e^{-i(\pi/2)(\nu_0+1)}x^{-\nu_0-1}\,. \tag{B14}$$

For the integral appearing in the WKB solution we find

$$\int_{x_1}^{x}dx'(-\alpha-\delta\cos x')^{1/2}\simeq\int_{x_1}^{x}dx'\left[\frac{\delta}{2}x'^2-(2\delta)^{1/2}(\nu_0+1)\right]^{1/2}$$
$$=\left(\frac{\delta}{2}\right)^{1/2}\left\{\tfrac{1}{2}x(x^2-x_1^2)^{1/2}-\tfrac{1}{2}x_1^2\ln\left[\frac{(x^2-x_1^2)^{1/2}+x}{x_1}\right]\right\}$$
$$\simeq(2\delta)^{1/2}x^2/4-\tfrac{1}{2}(\nu_0+\tfrac{1}{2})-(\nu_0+\tfrac{1}{2})\{\ln x-\ln(2/\delta)^{1/4}-\ln[\tfrac{1}{2}(\nu_0+\tfrac{1}{2})]^{1/2}\}\,,$$

where we took $\frac{1}{2}\delta x_1^2=(2\delta)^{1/2}(\nu_0+1)$ in the first line and we use $x \gg x_1=(2/\delta)^{1/4}(2\nu_0+1)^{1/2}$ in the last line. With $[-\alpha-\delta\cos x]^{1/4}\simeq(\delta/2)^{-1/4}x^{-1/2}$, we find the WKB solution in the matching region

$$\phi_{\mathrm{WKB}}\simeq C_1 e^{\sqrt{2\delta}x^2/4}x^{-\nu_0-1}\left(\frac{2}{\delta}\right)^{1/4+(\nu_0+1/2)/4}\exp\{-\tfrac{1}{2}(\nu_0+\tfrac{1}{2})[1-\ln\tfrac{1}{2}(\nu_0+\tfrac{1}{2})]\}$$
$$+C_2 e^{-\sqrt{2\delta}x^2/4}x^{\nu_0}\left(\frac{2}{\delta}\right)^{1/4-(\nu_0+1/2)/4}\exp\{\tfrac{1}{2}(\nu_0+\tfrac{1}{2})[1-\ln\tfrac{1}{2}(\nu_0+\tfrac{1}{2})]\}. \quad \text{(B15)}$$

Comparing (B14) and (B15) we find the relation among the coefficients:

$$\beta_0(2\delta)^{\nu_0/4}=C_2\left(\frac{2}{\delta}\right)^{1/4-(\nu_0+1/2)/4}\exp\{\tfrac{1}{2}(\nu_0+\tfrac{1}{2})[1-\ln\tfrac{1}{2}(\nu_0+\tfrac{1}{2})]\},$$
$$\gamma_0(2\delta)^{-(\nu_0+1)/4}e^{-i(\pi/2)(\nu_0+1)}=C_1\left(\frac{2}{\delta}\right)^{1/4+(\nu_0+1/2)/4}\exp\{-\tfrac{1}{2}(\nu_0+\tfrac{1}{2})[1-\ln\tfrac{1}{2}(\nu_0+\tfrac{1}{2})]\}. \quad \text{(B16)}$$

Right matching region. In the right shaded region, Fig. 1, we can proceed similarly when we define

$$\xi=x-2\pi\,,$$
$$\xi_2=x_2-2\pi\simeq-\left(\frac{2}{\delta}\right)^{1/4}(2\nu_0+1)^{1/2}. \quad \text{(B17)}$$

The difference being that ξ and ξ_2 are negative, and that we have to include in the WKB solution the integral over the whole forbidden region

$$J=\int_{x_1}^{x_2}dx(-\alpha-\delta\cos x)^{1/2} \quad \text{(B18)}$$

plus the integral starting from the right turning point x_2. The latter is analogous to the integral encountered in the left matching region.

The integral (B18) is symmetric about $x=\pi$. We can express it as complete elliptic integrals. First we replace $-\alpha$ by $\delta\cos x_1$, according to (B11); then we change the integration variables to $\phi=x/2$, using $\cos x=1-2\sin^2\phi$. Thus

$$J=4(2\delta)^{1/2}\sin\frac{x_1}{2}\int_{x_1/2}^{\pi/2}d\phi\left(\frac{1}{\sin^2(x_1/2)}\sin^2\phi-1\right)^{1/2}$$
$$=4(2\delta)^{1/2}\sin\frac{x_1}{2}\left[\sin\frac{x_1}{2}F(\epsilon,k)-\frac{1}{\sin(x_1/2)}E(\epsilon,k)\right]\Bigg|_{x_1/2}^{\pi/2},$$

where $k=\cos(x_1/2)$, $\epsilon=\arcsin[\cos\phi/\cos(x_1/2)]$. The last step we found in Gradstein and Ryzhik, formula 2.599.2. E and F are elliptic integrals. Their properties are given in Ref. 15, Chap. 8.1. For $\phi=\pi/2$:

$$\epsilon=0\,,\quad F(0,k)=E(0,k)=0\,.$$

For $\phi=x_1/2$:

$$\epsilon=\frac{\pi}{2}\,,\quad F\left(\frac{\pi}{2},k\right)=K(k)\,,\quad E\left(\frac{\pi}{2},k\right)=E(k)\,.$$

Another parameter of the elliptic integrals is $k'=(1-k^2)^{1/2}$. For us $k'=\sin(x_1/2)\simeq[1/(2\delta)^{1/4}](\nu_0+\frac{1}{2})^{1/2}$ is much smaller than 1. In this case Gradstein and Ryzhik give an expansion of $E(k)$ and $K(k)$ and we find

$$J=4(2\delta)^{1/2}[E(k)-k'^2K(k)]$$
$$\simeq 4\sqrt{2\delta}-2(\nu_0+\tfrac{1}{2})[\ln 4+\tfrac{1}{4}\ln(2\delta)-\tfrac{1}{2}\ln(\nu_0+\tfrac{1}{2})+\tfrac{1}{2}]+O(\delta^{-1/2}\ln\delta)\,. \quad \text{(B19)}$$

Matching the WKB and parabolic-cylinder-function solutions then gives the analogous relation to (B16),

$$(2\delta)^{\nu_0/4}\left[\beta_1-\gamma_1\frac{\sqrt{2\pi}}{\Gamma(\nu_0+1)}e^{i(\pi/2)(3\nu_0+2)}\right]=C_1\left(\frac{2}{\delta}\right)^{1/4-(\nu_0+1/2)/4}(2\delta)^{-(\nu_0+1/2)/2}4^{-(2\nu_0+1)}2^{\nu_0+1/2}$$
$$\times\exp[4\sqrt{2\delta}-\tfrac{1}{2}(\nu_0+\tfrac{1}{2})+\tfrac{1}{2}(\nu_0+\tfrac{1}{2})\ln\tfrac{1}{2}(\nu_0+\tfrac{1}{2})]\,,$$
$$(2\delta)^{-(\nu_0+1)/4}e^{-i(\pi/2)(\nu_0+1)}\left[\gamma_1-\beta_1\frac{\sqrt{2\pi}}{\Gamma(-\nu_0)}e^{i(\pi/2)(3\nu_0+1)}\right]=C_2\left(\frac{2}{\delta}\right)^{1/4+(\nu_0+1/2)/4}(2\delta)^{(\nu_0+1/2)/2}4^{2\nu_0+1}2^{-(\nu_0+1/2)}$$
$$\times\exp[-4\sqrt{2\delta}+\tfrac{1}{2}(\nu_0+\tfrac{1}{2})-\tfrac{1}{2}(\nu_0+\tfrac{1}{2})\ln\tfrac{1}{2}(\nu_0+\tfrac{1}{2})]\,. \quad \text{(B20)}$$

Now we implement the boundary condition (B10). Then from (B16) and (B20) we find the equations

$$e^{2\pi i s}\left[\beta_0-\gamma_0\frac{\sqrt{2\pi}}{\Gamma(\nu_0+1)}e^{i(\pi/2)(3\nu_0+2)}\right]=\gamma_0(2\delta)^{-(\nu_0+1/2)/2}\,4^{-(2\nu_0+1)}e^{i(\pi/2)(\nu_0-1)}e^{4\sqrt{2\delta}}\,,$$
$$e^{2\pi i s}\left[\gamma_0-\beta_0\frac{\sqrt{2\pi}}{\Gamma(-\nu_0)}e^{i(\pi/2)(3\nu_0+1)}\right]=\beta_0(2\delta)^{(\nu_0+1/2)/2}\,4^{2\nu_0+1}e^{-i(\pi/2)(\nu_0+1)}e^{-4\sqrt{2\delta}}\,. \tag{B21}$$

These should give us ν_0 as a function of s.

In the forbidden region the wave functions should be small and we therefore expect from (B14) γ_0 to be much smaller than β_0. We therefore neglect it on the left-hand side of the first equation in (B21) (remember that δ is large). To be able to satisfy the second equation $\Gamma(-\nu_0)$ must be large. This requires

$$\nu_0=n+\eta\,,\quad |\eta|\ll 1\,,\ n \text{ an integer} \tag{B22}$$

in which case we find $1/\Gamma(-\nu_0)=(-1)^{n+1}n!\,\eta$. We neglect η everywhere but here and find

$$\gamma_0\simeq\beta_0(2\delta)^{(n+1/2)/2}\,4^{2n+1}e^{-i(\pi/2)(n-1)}e^{2\pi i s}e^{-4\sqrt{2\delta}} \tag{B23}$$

and, from the second equation,

$$\beta_0 e^{2\pi i s}[(2\delta)^{(n+1/2)/2}\,4^{2n+1}e^{-i(\pi/2)(n-1)}e^{2\pi i s}e^{-4\sqrt{2\delta}}-(-1)^{n+1}\sqrt{2\pi}\,n!\,\eta e^{i(\pi/2)(3n+1)}]=\beta_0(2\delta)^{(n+1/2)/2}\,4^{2n+1}e^{-i(\pi/2)(n+1)}e^{-4\sqrt{2\delta}}\,.$$

Solving for η we finally find

$$\eta=2(-1)^{n+1}\frac{4^{2n+1}}{\sqrt{2\pi}\,n!}(2\delta)^{(n+1/2)/2}e^{-4\sqrt{2\delta}}\cos(2\pi s)\ll 1\,. \tag{B24}$$

Clearly the ground state has $n=0$ and thus we find the eigenvalue α from (B2), (B4), (B7), (B22), and (B24):

$$\alpha(s)=-\frac{1}{2g^4}+\frac{1}{2g^2}-\frac{1}{16}+O(g^2)$$
$$-\frac{8}{\sqrt{2\pi}\,g^3}e^{-4/g^2}\cos(2\pi s)\,. \tag{B25}$$

We remark that we have matched the two different solutions, WKB and parabolic cylinder functions, over an entire region, thereby ensuring a smooth connection. Furthermore, we have used the WKB solution only where it is valid, for all $\nu_0\geq 0$. The criterion is (see, e.g., Landau and Lifshitz[16])

$$R\equiv\left|\frac{d}{dx}\,\frac{1}{(-\alpha-\delta\cos x)^{1/2}}\right|\ll 1\,.$$

In our matching region, $1\gg x\gg x_1=(2/\delta)^{1/4}(2\nu_0+1)^{1/2}$. Thus,

$$R=\left|\frac{\frac{1}{2}\delta\sin x}{(-\alpha-\delta\cos x)^{3/2}}\right|\simeq\left|\frac{\frac{1}{2}\delta x}{[(\delta/2)x^2-(\delta/2)x_1^{\,2}]^{3/2}}\right|$$
$$\simeq\left(\frac{2}{\delta}\right)^{1/2}\frac{1}{x^2}\ll\left(\frac{2}{\delta}\right)^{1/2}\frac{1}{x_1^{\,2}}\,.$$

Therefore, $R\ll 1/(2\nu_0+1)\leq 1$ and the criterion is satisfied. For the calculation of κ we want to write the wave function in the form

$$\phi(x)=E(x)+iO(x)\,. \tag{B26}$$

We can easily do this for small x, using the linear dependence relation (see Bateman[14])

$$D_{-\nu-1}(iz)=\frac{\Gamma(-\nu)}{\sqrt{2\pi}}e^{i(\pi/2)(\nu+1)}[D_\nu(z)-e^{-\nu\pi i}D_\nu(-z)]$$

for $\nu=\eta$, from (B8) and (B23). We find

$$E(x)=C[D_\eta((2\delta)^{1/4}x)+D_\eta(-(2\delta)^{1/4}x)]\,,$$
$$O(x)=-C\tan(2\pi s)[D_\eta((2\delta)^{1/4}x)-D_\eta(-(2\delta)^{1/4}x)]\,, \tag{B27}$$

where C is some normalization constant. We then can calculate (5.6) from the series expansion of the parabolic cylinder function for small arguments (see Bateman or Gradstein and Ryzhik) at the origin $x=0$, and find

$$I=2\pi\,2C^2\sqrt{2\pi}\,(2\delta)^{1/4}\eta\tan(2\pi s)\,. \tag{B28}$$

The norm is

$$N=\int_{-\pi}^{\pi}dx\,\phi^*(x)\phi(x)=2\int_0^{\pi}dx[E^2(x)+O^2(x)]\,. \tag{B29}$$

For small η one can show that the contribution of $O^2(x)$ to the norm is only of order η. For E we can take $E=2CD_0((2\delta)^{1/4}x)$ for the whole range of integration and furthermore extend the integration from π to infinity. In all these steps we make only errors of the order η. Thus

$$N\simeq 8C^2\int_0^{\infty}dx\,D_0^{\,2}((2\delta)^{1/4}x)$$
$$=8C^2(2\delta)^{-1/4}\int_0^{\infty}dy\,D_0^{\,2}(y)$$
$$=4\sqrt{2\pi}\,C^2(2\delta)^{-1/4}\,. \tag{B30}$$

Then we find for κ,

$$\kappa(s)=\frac{\langle\phi|id/dx|\phi\rangle}{\langle\phi|\phi\rangle}=\frac{I}{N}=\frac{2\pi}{2}(2\delta)^{1/2}\eta\tan(2\pi s)$$
$$=-\frac{4\sqrt{2\pi}}{g^3}e^{-4/g^2}\sin(2\pi s)\,. \tag{B31}$$

[1]K. G. Wilson, Phys. Rev. D 10, 2445 (1974).
[2]J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975).
[3]K. Osterwalder and L. Seiler, Ann. Phys. (N. Y.) 110, 440 (1978).
[4]E. Fradkin and L. Susskind, Phys. Rev. D 17, 2637 (1978).
[5]A. M. Polyakov, Phys. Lett. 59B, 82 (1975); Nucl. Phys. B120, 429 (1977).
[6]A. H. Guth, Phys. Rev. D 21, 2291 (1980).
[7]R. Balian, J. T. Drouffe, and C. Itzykson, Phys. Rev. D 11, 2104 (1975).
[8]J. M. Drouffe and C. Itzykson, Phys. Rep. 38C, 133 (1978).
[9]T. Banks, R. Myerson, and J. Kogut, Nucl. Phys. B129, 493 (1977).
[10]S. D. Drell *et al.*, Phys. Rev. D 19, 619 (1979).
[11]The integral can be found in Bateman (Ref. 14), Vol. 2, p. 55, formula (50), for $\nu=1$, $\mu=-\frac{1}{2}$. Further we use $K_{1/2}(2)=(\pi/2z)^{1/2}e^{-z}$, Ref. 14, Vol. 2, p. 10, formula (40).
[12]G. Bhanot and M. Creutz, Phys. Rev. D 21, 2892 (1980).
[13]A. Hasenfratz, E. Hasenfratz, and P. Hasenfratz, CERN Report No. TH. 2890-CERN, 1980 (unpublished).
[14]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi (McGraw-Hill, New York, 1953).
[15]J. S. Gradstein and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).
[16]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics, Nonrelativistic Theory* (Pergamon, New York, 1958).