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# Relationship between lattice and continuum definitions of the gauge-theory coupling

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We generalize the background-field method to lattice gauge theories. By evaluating the lattice partition function in a weak background lattice field for weak coupling we determine the relationship between the lattice definition of the coupling and that of dimensional regularization. This corresponds to a determination of the ratio of the renormalization scale parameters for these definitions. We find, for an SU($N$) lattice gauge theory with the Wilson action, that $(\Lambda_{\overline{\rm MS}}/\Lambda_L)_{{\rm SU}(N)} = 38.852\,704 \exp(-3\pi^2/11N^2)$.

## I. INTRODUCTION

In recent months significant progress has been made in understanding the dynamics of the simplest approximation to quantum chromodynamics (QCD)—namely, quarkless non-Abelian gauge theories. Calculations performed using a variety of different methods, including semiclassical techniques,[1] direct Monte Carlo integration for a Euclidean lattice theory,[2,3] and strong-coupling lattice Hamiltonian expansions,[3] all yield an extremely simple picture of the transition from weak coupling at short distances to strong coupling at large distances. These calculations strongly support the contention that QCD confines color and yield a relation between the slope of the linear heavy-quark potential (the "string tension") and the renormalization scale parameter $\Lambda$ of the theory. If not for the fact that quarks are absent in these calculations this relation would provide a crucial quantitative test of QCD.

The strongest evidence for confinement comes from the Monte Carlo calculations of Creutz and Wilson.[2,3] Here one directly evaluates, using Monte Carlo integration, the expectation value of Wilson loops for a Euclidean lattice gauge theory on a periodic lattice of spacing $a$. One then looks for an area-law behavior of large Wilson loops and determines the value of the string tension $\sigma$, in units of $a^{-2}$, as a function of the lattice coupling $g$. In order to establish that the continuum theory has linear confinement one must show that $\sigma$ remains constant as $g^2$ approaches zero (proportional to $1/\ln a\Lambda$) as it must in the continuum limit. The relation between the bare coupling $g$ and the lattice spacing $a$ is determined by asymptotically free perturbation theory,[5-7]

$$x = 8\pi^2/g^2N = \tfrac{11}{6}\ln[1/(\Lambda a)^2] + \tfrac{17}{22}\ln\{\ln[1/(\Lambda a)^2]\}\,. \tag{1.1}$$

This formula can be regarded as specifying the meaning of the renormalization-group scale parameter $\Lambda$. A change in the value of $\Lambda$ shifts $x$ by a finite amount to this order and different renormalization schemes may yield different values of this parameter. We shall refer to the renormalization scale parameter that appears in the lattice theory as $\Lambda_L$.

Now since, for small coupling, $g$ satisfies Eq. (1.1) then $\sigma a^2$ must behave, for weak coupling, as

$$\begin{aligned}\sigma a^2 &= \sigma/\Lambda_L{}^2(a\Lambda_L)^2\\ &= (\sigma/\Lambda_L{}^2)\left(\frac{48\pi^2}{11g^2N}\right)^{51/121}\exp(-24\pi^2/11g^2N)\,.\end{aligned} \tag{1.2}$$

If one can show that $\sigma a^2$ as determined by Monte Carlo integration has the above dependence on $g$ for sufficiently small $g$, then this is an indication of linear confinement. Furthermore, one can calculate the numerical value of $\sigma/\Lambda_L{}^2$.

The calculations of Creutz and Wilson show that, for SU(2), (a) the behavior of large Wilson loops is consistent with an area law, (b) the behavior of $\sigma a^2$ exhibits an abrupt change from strong-coupling behavior ($\sigma a^2 = \ln g^2$) to the weak-coupling behavior of Eq. (1.2) at $g^2 \approx 2$, and (c) the value of $\sigma/\Lambda_L{}^2$ is $80 \pm 20$.[8]

A similar abrupt transition from strong-coupling to weak-coupling behavior at $g^2 \approx 2$ was found by Kogut, Pearson, and Shigemitsu, who calculated the string tension in a strong-coupling expansion for a Hamiltonian lattice gauge theory and extrapolated their results, using Padé approximants to weak coupling.[4]

The semiclassical results of Ref. 1 are based on a totally different approach. Here one employs semiclassical techniques to construct an effective lattice gauge theory. The basic assumption is that one can obtain an adequate representation of the physics of certain observables (e.g., planar Wilson loops) in terms of a simple Wilson action

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characterized by one coupling constant $g(a)$. The coupling-constant renormalization which yields $g(a)$ is calculated by considering the response of the theory to weakly varying external fields, and consists of the perturbative renormalization given in Eq. (1.2) and a further renormalization due to instantons. One then tests whether the $a$ dependence of the coupling is such as to yield a constant string tension $\sigma \equiv \ln g^2(a)/a^2$ once $g^2$ is large enough. If the basic assumption is valid this should occur once $g^2 \gtrsim 2$, where strong coupling sets in.

The results of this calculation were as follows: (a) An abrupt transition from weak-coupling to strong-coupling behavior occurred at $g^2 \approx 2$ [for SU(2)] at which point the instanton renormalization suddenly turned on. (b) Over a range of distances for which $2 \leq g^2 \leq 16$, where the semiclassical approximation was reasonable, the string tension was independent of $g^2$ or $a$, lending support to the basic assumption. (c) The value of $\sigma$ was determined to be $\sqrt{\sigma} \approx (70 \pm 30)\Lambda_L$,[9] in good agreement with Ref. 2 considering the uncertainties in both calculations.

In all of these calculations it is essential to know the value of $\Lambda_L$ in terms of more familiar renormalization-scheme scale parameters. For example, if the lattice gauge theorists are to compare their calculation of $\sigma$ with experiment (ignoring the absence of quarks) it is necessary to know the value of $\Lambda_L$ in MeV units. The standard asymptotic-freedom predictions, however, are performed using dimensional regularization and there is some evidence from scaling deviations that $\Lambda_{\overline{\text{MS}}}$, for the modified-minimal-subtraction scheme, is in the range of 300–600 MeV. The string tension can also be estimated (again ignoring light quarks) from the slope of the Regge trajectory or the heavy-quark potential to be of order (450 MeV)$^2$. To compare these it is necessary to know the ratio $\Lambda_L/\Lambda_{\overline{\text{MS}}}$.

In the semiclassical calculation it is even more essential to know $\Lambda_L$, since the calculation rests on 't Hooft's evaluation of the density of instantons using a continuum Pauli-Villars regularization scheme. In order to evaluate the effective lattice coupling it is necessary to convert to the lattice coupling—and this can (and does) have a big effect on the instanton density.

In Ref. 1 the relation between $\Lambda_L$ and $\Lambda_{\overline{\text{MS}}}$ was estimated to be $\approx 6.8$. This estimate was based on a comparison of a one-loop calculation involving scalar particles in the continuum theory (using Pauli-Villars or dimensional regularization) with the analogous lattice calculation. Equating these one-loop calculations yields the above value of $\Lambda_L/\Lambda_{\overline{\text{MS}}}$, which differs from one due to the difference between lattice and continuum propagators. The effects of the change in the structure of vertices were not taken into account.

Recently Hasenfratz and Hasenfratz have performed a more careful calculation of $\Lambda_L$.[10] They explicitly evaluated the two- and three-point functions to the one-loop level for the weak-coupling SU($N$) lattice gauge theory, thereby precisely determining the relation between $\Lambda_L$ and $\Lambda_{\overline{\text{MS}}}$. Having heard of their calculation we undertook to independently calculate this ratio. This calculation is presented below. Our results agree with those of Ref. 10.[11] In any case we thought it useful to present them since our method of calculation is quite different. We generalize the well-known background-field method[12] to the lattice and calculate the effective Lagrangian for weak coupling in the presence of a slowly varying background lattice field. The actual calculations required to relate the lattice and the continuum theories are then quite simple. In fact a simple intuitive explanation of the difference between lattice and continuum perturbation theory can be given which yields, with almost no work, the ratio of $\Lambda_L/\Lambda_{\overline{\text{MS}}}$ to an accuracy of 25%.

## II. THE LATTICE BACKGROUND-FIELD METHOD

In this section we shall adapt the familiar background-field technique[12] to lattice gauge theories, and show how it can be used to evaluate $\Lambda_L$. First we shall define our notation and then we shall outline the method. The explicit calculation will be presented in Sec. III.

### A. Notation

Consider Wilson's[13] formulation of the SU($N$) Euclidean lattice gauge theory.[14] We label the sites of the lattice by an integer-valued four-vector $x$. Links are then labeled by pairs $(x, x+\mu)$, $\mu = 1, \ldots, 4$, where the four vector $\mu_\alpha$ is equal to $\delta_{\mu\alpha}$. The basic variables of the lattice gauge theory are $N\times N$ unitary matrices defined on the links and denoted by $U_{x,x+\mu} \equiv U^\dagger_{x+\mu,x}$. The lattice action involves the plaquette variables, defined on elementary squares. At the point $x$ in the $\mu$-$\nu$ plane,

$$W_{\mu\nu}(x) \equiv U_{x,x+\mu}U_{x+\mu,x+\mu+\nu}U^\dagger_{x+\nu,x+\mu+\nu}U^\dagger_{x,x+\nu} = W^\dagger_{\nu\mu}(x)$$
$$\equiv \tfrac{1}{2}[G_{\mu\nu}(x) + iH_{\mu\nu}(x)], \tag{2.1}$$

where $G_{\mu\nu}(x)$ and $H_{\mu\nu}(x)$ are Hermitian. If $\lambda^\alpha$, $\alpha = 1, \ldots, N^2-1$, are the $N\times N$ matrix representations of the generators of SU($N$), normalized so that $\mathrm{Tr}\,\lambda^\alpha\lambda^\beta = 2\delta_{\alpha\beta}$, then we define $H^\alpha_{\mu\nu}$ by

$$H_{\mu\nu}=H^{\alpha}_{\mu\nu}\tfrac{1}{2}\lambda^{\alpha}\ . \tag{2.2}$$

In the continuum limit, i.e., as the lattice spacing $a\to 0$,

$$U_{x,x+\mu}\approx P\exp\left(i\int_{xa}^{(x+\mu)a}A\cdot dx\right),$$

$$W_{\mu\nu}(x)\approx I+iF_{\mu\nu}(x)a^2,$$

and

$$H^{\alpha}_{\mu\nu}=2a^2F^{\alpha}_{\mu\nu},$$

where $F_{\mu\nu}=F^{\alpha}_{\mu\nu}\lambda_{\alpha}/2$ is the continuum field strength.

The Wilson action is

$$S\equiv\frac{1}{g^2(a)}\sum_x\sum_{\nu>\mu}\mathcal{L}_{\mu\nu}(x)\ , \tag{2.3}$$

$$\begin{aligned}\mathcal{L}_{\mu\nu}(x)&\equiv\mathrm{tr}[1-W_{\mu\nu}(x)]+\text{H.c.}\\&=\mathrm{tr}[2-G_{\mu\nu}(x)],\end{aligned} \tag{2.4}$$

and reduces in the continuum limit to the Yang-Mills action

$$S=\int d^4x\,\tfrac{1}{4}\sum_{\alpha,\mu,\nu}(F^{\alpha}_{\mu\nu})^2\ . \tag{2.5}$$

Finally we shall define lattice covariant derivatives. Given a set of link variables $U$ (gauge fields) and a matrix-valued function defined on lattice sites (i.e., matter fields) the lattice covariant derivatives are defined as follows:

$$\begin{aligned}D_\mu f(x)&\equiv U_{x,x+\mu}f(x+\mu)U^{\dagger}_{x,x+\mu}-f(x)\ ,\\ \overline{D}_\mu f(x)&\equiv U^{\dagger}_{x,x+\mu}f(x-\mu)U_{x+\mu,x}-f(x)\ .\end{aligned} \tag{2.6}$$

Note that under gauge transformations

$$\begin{aligned}U_{x,x+\mu}&\to V(x)U_{x,x+\mu}V^{\dagger}(x+\mu)\ ,\\ f(x)&\to V(x)f(x)V^{\dagger}(x)\ ,\\ D_\mu f(x)&\to V(x)D_\mu f(x)V^{\dagger}(x)\ .\end{aligned} \tag{2.7}$$

Also when all $U=1$ (vanishing gauge field), $D_\mu$ ($\overline{D}_\mu$) reduces to the ordinary lattice derivative $\Delta_\mu$ ($\overline{\Delta}_\mu$), where

$$\begin{aligned}\Delta_\mu f(x)&=f(x+\mu)-f(x),\\ \overline{\Delta}_\mu f(x)&=f(x-\mu)-f(x)\ .\end{aligned} \tag{2.8}$$

The analog of integration by parts on the lattice is

$$\begin{aligned}&\sum_x\mathrm{tr}[g(x)(D_\mu f(x))]=\sum_x\mathrm{tr}[(\overline{D}_\mu g(x))f(x)],\\ &\sum_{x,\mu}\mathrm{tr}[(D_\mu g(x))(D^\mu f(x))]\\ &\qquad=\sum_{x,\mu}\mathrm{tr}[(\overline{D}^\mu g(x))(\overline{D}_\mu f(x))],\end{aligned} \tag{2.9}$$

as can easily be verified.

### B. The strategy

Our strategy will be as follows. Let $U^0$ denote a set of link variables which represent a solution of the classical lattice equations of motion on a lattice of spacing $a$.[15] Then the lattice action can be expanded about $U=U^0$, which is an extremum of the action

$$S=S_0+S_2+\text{cubic terms}+\cdots\ , \tag{2.10}$$

where $S_0=S(U^0)$ and $S_2$ is quadratic in the fluctuations of $U$ about $U^0$. We now perform a saddle-point approximation to the functional integral representation of the partition function (the Euclidean vacuum-to-vacuum transition amplitude)

$$Z_a\equiv\int[dU]e^{-S(U)}\ , \tag{2.11}$$

where $[dU]=\prod_x\prod_\mu dU_{x,x+\mu}$ and $dU_{x,x+\mu}$ is the Haar measure on SU($N$). It is convenient to introduce, as in the continuum theory, gauge-fixing terms $S_{\mathrm{gf}}$, and the compensating ghost terms $S_{\mathrm{gh}}$. The precise form of these terms will be given in the following section. We can then expand about $U=U^0$ as follows:

$$Z_a\approx e^{-S^0}\int[dU]e^{-(S_2+S_{\mathrm{gf}}+S_{\mathrm{gh}})}[1+O(g(a)^2)], \tag{2.12}$$

where $g(a)$ is the (bare) coupling appropriate for a lattice of spacing $a$. Note that $S^0$, which is simply the vacuum energy in the background field $U^0$, is of order $1/g^2(a)$ and since we will calculate $\ln Z_a$ up to order $(g^2)^0$ the $O(g^2(a))$ terms in (2.12) can be ignored.

A similar calculation of $Z$ can be performed in the continuum theory, and compared with the lattice calculation. This is most easily done for weak, slowly varying fields, i.e., $aA^0_\mu\ll 1$ and $a\partial_\nu A^0_\mu\ll A^0_\mu$, or $U_{x,x+\mu}\approx 1+i\int A^0\cdot dx$. To perform the continuum calculation we must regularize the theory in one way or another. We shall choose a Pauli-Villars regularization scheme, but dimensional regularization would be equally good. If one compares these two calculations, for weak slowly varying background fields one should find that

$$-\ln\left(\frac{Z_a}{Z_m}\right)=\left[\frac{1}{4g^2(a)}-\frac{1}{4g^2(m)}+d(ma)\right]\int d^4x(F^{\alpha}_{\mu\nu})^2\ , \tag{2.13}$$

where $m$ is the Pauli-Villars regulator mass, $g(m)$ the bare coupling for the continuum theory, and $d(ma)$ is an (infrared finite) number. Given our knowledge of the Yang-Mills $\beta$ function we know that $g^2(a)$ (for $a\to 0$) is given by (1.1) and that analogously $g^2(m)$ is given (for $m\to\infty$) by

$$x_{\rm PV}(m)=\frac{8\pi^2}{g^2(m)N}=\tfrac{11}{6}\ln\left(\frac{m}{\Lambda_{\rm PV}}\right)^2+\tfrac{17}{22}\ln\left[\ln\left(\frac{m}{\Lambda_{\rm PV}}\right)^2\right], \tag{2.14}$$

thus defining the Pauli-Villars renormalization scale parameter $\Lambda_{\rm PV}$. The difference between $x_{\rm PV}/N$ and $x_L/N$, when $a\to 0$ and $m\to\infty$, with $ma$ kept fixed, is thus

$$x_L-x_{\rm PV}=\tfrac{11}{3}\ln\left(ma\,\frac{\Lambda_{\rm PV}}{\Lambda_L}\right)+O\left(\frac{1}{\ln m}\right). \tag{2.15}$$

One expects (and this expectation is upheld as we shall see by explicit calculation) that the difference between the lattice and continuum calculation is simply a finite renormalization (or redefinition) of the coupling and that therefore $\ln(Z_a/Z_m)$ is independent of $m$ or $1/a$ as $m\to\infty$. Therefore it must be that

$$d(ma)=\frac{11N}{96\pi^2}(\ln ma+C(N)), \tag{2.16}$$

where $C(N)$ is a pure number (perhaps $N$ dependent). In that case

$$\ln\frac{Z_a}{Z_m}=\frac{11N}{96\pi^2}\left(\ln\frac{\Lambda_L}{\Lambda_{\rm PV}}-C(N)\right)\int d^4x\,(F_{\mu\nu})^2, \tag{2.17}$$

and if we require that the two calculations yield the same physics, i.e., that $Z_a/Z_m$, then $\Lambda_L/\Lambda_{\rm PV}$ is determined to be

$$\frac{\Lambda_L}{\Lambda_{\rm PV}}=\exp(C(N)). \tag{2.18}$$

Let us note some important points:

(i) The ratio $\Lambda_L/\Lambda_{\rm PV}$ can be computed by performing a one-loop calculation in the presence of a background field.

(ii) By considering only weak fields that are slowly varying we can greatly simplify our calculation. Any contribution to $S_2$ that produces, when we carry out the Gaussian lattice integrals, terms such as $(F_{\mu\nu})^4$ or $(D_\lambda F_{\mu\nu})^2$ that are small compared to $(F_{\mu\nu})^2$ can be dropped.

(iii) The calculation can be performed in a way that is manifestly infrared finite. Thus, in evaluating $\ln(Z_a/Z_m)$, all integrals that occur can be evaluated at zero external momentum. This enormously simplifies the numerical labor.

## III. THE CALCULATION

### A. The measure

Let us parametrize the fluctuations of the link variables $U$ by four traceless Hermitian matrices $\alpha^\mu(x)$ defined by

$$U_{x,x+\mu}\equiv e^{ig\alpha^\mu(x)}U^0_{x,x+\mu}\,. \tag{3.1}$$

Owing to the invariance of the measure $[dU]$ under right (or left) multiplication,

$$[dU]=[d(e^{ig\alpha}U^0)]$$
$$=[de^{ig\alpha}]=[d\alpha][1+O(g^2\alpha^2)]\,. \tag{3.2}$$

For our purposes, terms of order $g^2\alpha^2$ can be ignored and thus we can take

$$[dU]=[d\alpha]=\prod_{x,\mu}d\alpha^\mu(x)\,, \tag{3.3}$$

and, for the same reason, allow $\alpha^\mu(x)$ to range over an infinite range.

### B. The quadratic action

Next we need to compute $S_2$. We begin with the plaquette variable

$$W_{\mu\nu}(x)=\exp[ig\alpha^\mu(x)]U^0_{x,x+\mu}\exp[ig\alpha_\nu(x+\mu)]U^0_{x+\mu,x+\mu+\nu}U^{0\dagger}_{x+\mu+\nu,x+\nu}\exp[-ig\alpha^\mu(x+\nu)]U^\dagger_{x,x+\nu}\exp[-ig\alpha^\nu(x)]\,, \tag{3.4}$$

whose trace can be written using $Ve^{i\alpha}=\exp(iV\alpha V^{-1})\,V$ as

$$\operatorname{tr}W_{\mu\nu}(x)=\operatorname{tr}\{\exp[-ig(D^0_\nu\alpha^\mu(x)+\alpha^\mu(x))]\exp[ig\alpha^\nu(x)]\exp[ig\alpha^\mu(x)]\exp[ig(D^0_\mu\alpha^\nu(x)+\alpha^\nu(x))]W^0_{\mu\nu}\}\,, \tag{3.5}$$

where $D_\mu^0$ ($W_{\mu\nu}^0$) is the lattice covariant derivative (plaquette variable) with $U=U^0$. Next we combine terms using

$$e^{gA}e^{gB}=\exp\{g(A+B)+(g^2/2)[A,B]\}(1+O(g^3))$$

to obtain

$$\mathrm{tr}\,W_{\mu\nu}(x)=\mathrm{tr}\{[\exp(igD_\mu^0\alpha^\nu(x)-igD_\nu^0\alpha^\mu(x))+(g^2E_{\mu\nu}(x))]W_{\mu\nu}^0(x)\}+O(g^3)\ , \tag{3.6}$$

where

$$E_{\mu\nu}(x)=[\alpha^\nu(x),\ \alpha^\mu(x)]+\tfrac{1}{2}[\alpha^\nu(x),D_\mu^0\alpha^\nu(x)]+\tfrac{1}{2}[D_\nu^0\alpha^\mu(x),\ \alpha^\mu(x)]+\tfrac{1}{2}[D_\nu^0\alpha^\mu(x),\ D_\mu^0\alpha^\nu(x)]\ . \tag{3.7}$$

Now we expand and keep only the quadratic terms in $\alpha$ to obtain

$$2\,\mathrm{Re}\,\mathrm{tr}[W_{\mu\nu}(x)-W_{\mu\nu}^0(x)]=-\tfrac{1}{2}g^2\mathrm{tr}[(D_\mu^0\alpha^\nu(x)-D_\nu^0\alpha^\mu(x))^2G_{\mu\nu}^0(x)]+ig^2\,\mathrm{tr}[E_{\mu\nu}(x),H_{\mu\nu}^0(x)]+O(g^3)\ , \tag{3.8}$$

where $G_{\mu\nu}$ and $H_{\mu\nu}$ and defined in Eq. (2.1).

For weak background fields $G_{\mu\nu}^0=2+O((F_{\mu\nu})^2)$, so that the first term can be simplified to equal

$$-\mathrm{tr}[(D_\mu^0\alpha^\nu(x)-D_\nu^0\alpha^\mu(x))^2]-\tfrac{1}{2}\mathrm{tr}[(\Delta_\mu\alpha^\nu(x)-\Delta_\nu\alpha^\mu(x))^2]\,\frac{\mathrm{tr}(G_{\mu\nu}^0-2)}{N}\ , \tag{3.9}$$

where we have thrown away terms of higher order than $O(F^2)$ and used the fact that when the $\alpha$'s are integrated out only $\mathrm{tr}G_{\mu\nu}^0$ will appear so that $G_{\mu\nu}^0$ can be replaced by $\mathrm{tr}G_{\mu\nu}^0/N$.

### C. Gauge fixing

In perturbing about a background field it is useful to work in a background-field gauge. The lattice analog of this gauge condition is

$$\sum_{\mu=1}^4(U_{x,x+\mu}U_{x,x+\mu}^{0\dagger}-U_{x-\mu,x}^\dagger U_{x-\mu,x}^0)=0\ , \tag{3.10}$$

which is trivially satisfied when $U^0=U$. This, when expanded, yields the condition on the $\alpha$'s

$$\sum_{\mu=1}^4\overline{D}_\mu^0\alpha^\mu(x)=0\ . \tag{3.11}$$

We shall work (for convenience) in the background-field Feynman gauge, by adding to the action the gauge-fixing term

$$S_{gf}=\frac{1}{g^2}\sum_x\mathrm{tr}\left[\left(\sum_{\mu=1}^4D^\mu\alpha_\mu(x)\right)^2\right]\ . \tag{3.12}$$

A straightforward computation shows that the corresponding Faddeev-Popov determinant is $\mathrm{Det}[\sum_{\mu=1}^4\overline{D}_\mu^0D_\mu]$. In the usual way it can be represented by a Gaussian integral over a set of complex matrix ghost fields $\phi(x)$ whose action is

$$S_{gh}=\frac{1}{g^2}\sum_x\sum_{\mu=1}^4\mathrm{tr}[(D^\mu\phi(x))^\dagger(D^\mu\phi(x))]\ . \tag{3.13}$$

### D. The full quadratic action

The full quadratic action involving the gauge fields is given by $S=S_2+S_{gf}$. We find it useful to collect the various terms into four groups. The $(D_\mu^0\alpha^\nu-D_\nu^0\alpha^\mu)^2$ term in $S_2$ combines with $S_{gf}$ to yield

$$S_{sc}+\frac{1}{2g^2}\sum_x\sum_{\nu>\mu}\mathrm{tr}\{\alpha^\nu([\overline{D}_\mu^0,D_\nu^0]\alpha^\mu)+\alpha^\mu([\overline{D}_\nu^0,D_\mu^0]\alpha^\nu)\}\ , \tag{3.14}$$

where $S_{sc}$ is the action of a collection of scalar matter fields $\alpha^\lambda$ ($\lambda=1,\ldots,4$),

$$S_{sc}=\tfrac{1}{2}\sum_x\sum_{\lambda,\mu}\mathrm{tr}[(D_\mu^0\alpha^\lambda)(D_\mu^0\alpha^\lambda)]\ . \tag{3.15}$$

The second term in Eq. (3.9) can be written as

$$S_T=-\frac{a^4}{48Ng^2}\sum_{x,\alpha,\beta}(F^{\alpha\beta}(x))^2\sum_{\nu>\mu}\mathrm{tr}(\Delta_\mu\alpha^\nu(x)-\Delta_\nu\alpha^\mu(x))^2\ , \tag{3.16}$$

where we have used the fact that when the $\alpha$'s are integrated out $\langle(F_{\alpha\beta})^2\rangle$ will be independent of $\alpha$ and $\beta$ and have replaced $(F_{\alpha\beta})^2$ by $\frac{1}{6}\sum_{\alpha>\beta}(F_{\alpha\beta})^2$.

Finally the two terms $S_A$ and $S_B$ are combinations of the last term in Eq. (2.26) and the last term in Eq. (3.14),

$$S_A=\frac{a^2}{g^2}\sum_x\sum_{\nu>\mu}\mathrm{tr}[A_{\mu\nu}(x)F_{\mu\nu}(x)]\ ,$$
$$A_{\mu\nu}(x)=-2i\{2[\alpha^\nu,\alpha^\mu]+[\alpha^\nu,D_\nu^0\alpha^\nu]+[D_\mu^0\alpha^\nu,\alpha^\mu]-\tfrac{1}{2}[D_\nu^0\alpha^\mu,D_\mu^0\alpha^\nu]\}\ , \tag{3.17}$$

and

$$S_B=\frac{a^2}{g^2}\sum_x\sum_{\nu>\mu}\mathrm{tr}[B_{\mu\nu}(x)F_{\mu\nu}]\ ,$$
$$B_{\mu\nu}(x)=-i\{[\alpha^\nu(x),D_\mu^0\alpha^\nu(x)]+[D_\nu^0\alpha^\mu(x),\alpha^\mu(x)]\}\ . \tag{3.18}$$

Collecting everything together we have

$$S_2+S_{gf}=S_{sc}+S_T+S_A+S_B\ . \tag{3.19}$$

In the continuum limit only $S_{sc}$ and $S_A$ survive and these become

$$S_{sc}=\tfrac{1}{2}\int d^4x\,\mathrm{Tr}(D_\mu^0\delta A^\lambda)^2 \tag{3.20}$$

and

$$S_A = -2i\int d^4x\, \mathrm{tr}([\delta A^\nu, \delta A^\mu]F^{\mu\nu}) \tag{3.21}$$

when we expand $A_\mu$ about the background gauge field $A^0_\mu, A_\mu = A^0_\mu + \delta A_\mu$.

In doing perturbation theory about the background field the free piece of the action will determine the propagator of $\alpha^\mu$ or $\delta A^\mu$. For the lattice theory this is

$$S_f = \tfrac{1}{2}\sum_x \sum_{\lambda,\mu} \mathrm{tr}[(\Delta_\mu \alpha_\lambda(x))^2]\,, \tag{3.22}$$

whereas for the continuum theory it is simply

$$S_f = \tfrac{1}{2}\int d^4x\, \mathrm{tr}(\partial_\mu \delta A_\lambda)^2\,, \tag{3.23}$$

corresponding to the momentum-space propagators in the Feynman gauge,

$$\frac{\delta_{\alpha\beta}}{\sum_{\mu=1}^4 \Delta_\mu \bar{\Delta}_\mu} = \frac{\delta_{\alpha\beta}a^2}{\sum_{\mu=1}^4 4\sin^2(p_\mu a/2)}$$

for the lattice theory, and $\delta_{\alpha\beta}/\partial^2 = \delta_{\alpha\beta}a^2/p^2$ for the continuum theory.

We now wish to calculate the quantity $d(ma)$ defined in Eq. (2.13). Denote by $\langle\theta\rangle_a$ the expectation value of an observable $\theta$ in the lattice theory using the free action $S_f$:

$$\langle\theta\rangle_a = \mathrm{const}\times\int[d\alpha]e^{-S_f}\theta\,, \tag{3.24}$$

where the normalization is fixed by $\langle 1\rangle = 1$. Let $\langle\theta\rangle_m$ denote the corresponding average in the continuum theory using Pauli-Villars regulators ($m$ denotes the Pauli-Villars mass). Then it is easy to see that

$$-d(ma)\int d^4x (F^\alpha_{\mu\nu})^2 = \tfrac{1}{2}\ln(Z^{\mathrm{sc}}_a/Z^{\mathrm{sc}}_m) + I_1 + I_2 + I_3\,, \tag{3.25}$$

where $Z^{\mathrm{sc}}_a$ or $Z^{\mathrm{sc}}_m$ are the respective partition functions computed using the action $S_{\mathrm{sc}}$ [Eqs. (3.15) and (3.20)], with no additional ghost terms. The complex ghost fields simply cancel half of the contribution of the four scalar particles in $S_{\mathrm{sc}}$ and thus account for the factor of $\frac{1}{2}$ in the above equation. Finally

$$\begin{aligned} I_1 &= -\langle S_T\rangle_a\,,\\ I_2 &= \tfrac{1}{2}\langle (S_B)^2\rangle_a\,,\\ I_3 &= \tfrac{1}{2}[\langle (S_A)^2\rangle_a - \langle (S_A)^2\rangle_m]\,. \end{aligned} \tag{3.26}$$

$S_T$ appears only to lowest order since it is already of order $(F_{\mu\nu})^2$. It is trivial to see that there can be no linear terms arising from $S_A$ or $S_B$. The only nontrivial point is that the cross terms between $S_A$, $S_B$, or $S_{\mathrm{sc}}$ is a straightforward consequence of the fact that $S_A$ is odd in the vector indices of $\alpha_\mu$ while $S_B$ and $S_{\mathrm{sc}}$ are even. There is probably an equally simple reason for the vanishing of the $S_B$ and $S_{\mathrm{sc}}$ cross terms; however, we have been unable to find it, and can only rely on explicit calculation. The numerical values of these four terms are calculated in the next section.

## IV. THE RESULTS

We now proceed to evaluate the separate contributions to the coupling-constant renormalization. First we consider $I_1$, which equals $-\langle S_T\rangle_a$. This term, which is responsible in factor for 75% of the ratio $\Lambda_L/\Lambda_{\mathrm{PV}}$, has a simple origin and can be easily analytically calculated. It arises because the Wilson action is, unlike the continuum action, a bounded function of the gauge field. Thus when we expand $\mathcal{L}_{\mu\nu}$ in powers of $F_{\mu\nu}$ we obtain $\mathcal{L}_{\mu\nu} = \frac{1}{2}(F_{\mu\nu})^2 - \mathrm{const}\times(F_{\mu\nu})^4$. The quartic term is of course negative, thus effectively reducing the value of $1/g^2(a)$. This yields a negative contribution to $C(N)$ thus reducing the value of $\Lambda_L/\Lambda_{\mathrm{PV}}$. Clearly such a term has no classical analog. Indeed $S_T$ is a dimension-eight (irrelevant) operator, which is multiplied by $a^4$ and whose only contribution as $a\to 0$ is in its contribution to dimension-four operators which diverge as $a^{-4}$. Thus its only effect is to yield a finite coupling renormalization. It is trivial to evaluate this term since the only Feynman graph that contributes to $\langle S_T\rangle_a$ is a tadpole graph [see Fig. 1(a)] with one propagator, and a vertex which equals the inverse propagator. These therefore cancel yielding

$$\begin{aligned} I_1 = -\langle S_T\rangle_a &= \frac{N^2-1}{32N}\int d^4x (F^\alpha_{\mu\nu})^2 \int (dp)_a \frac{D(p)}{D(p)}\\ &= \frac{N^2-1}{32N}\int d^4x (F^\alpha_{\mu\nu})^2\,, \end{aligned} \tag{4.1}$$

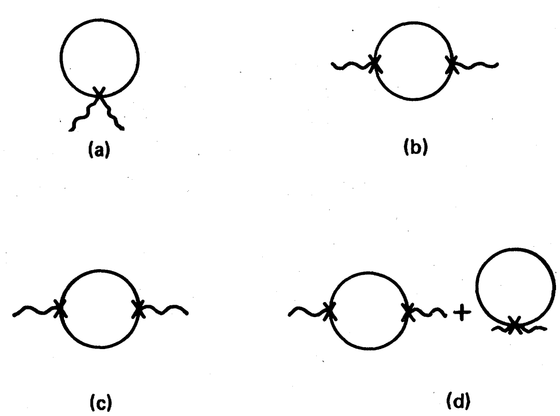


FIG. 1. An illustration of the graphs that contribute to $\ln Z$ in the presence of a background field. Solid lines represent lattice propagators, $x^0$ refers to the vertex at which the external field appears.

where we denote the lattice momentum-space integral by

$$\int (dp)_a \equiv \prod_{\mu=1}^{4} \int_{-\pi/a}^{+\pi/a} \frac{dp_\mu a}{(2\pi)^4}. \tag{4.2}$$

Note that this term, unlike all the other terms that we encounter, is not proportional to $N$ (the factor of $N^2-1$ is simply the number of gauge modes in the loop), and is solely responsible for the $N$ dependence of $C(N)$ and $\Lambda_L$.

$I_2$ also has no classical limit, since it arises from $S_B$ a dimension-five operator. The only graph that contributes is depicted in Fig. 1(b) and yields

$$I_2 = \tfrac{1}{2}\langle (S_B)^2\rangle_a = \int d^4x (F^\alpha_{\mu\nu})^2 N \int \frac{(dp)_a 16\pi^2 \sin^2 p_1 a}{D^2(p)}. \tag{4.3}$$

Here we have used the fact that we can replace $D_\mu$, in Eq. (3.18), by $\Delta_\mu$ since we are working to $O(F^2)$ only, and that in momentum space

$$\Delta_\mu(p) = e^{ip_\mu a} - 1, \quad \overline{\Delta}_\mu(p) = e^{-ip_\mu a} - 1. \tag{4.4}$$

This is an example of a nontrivial lattice momentum-space integral which must be evaluated numerically. However, all such one-loop integrals can be reduced to one-dimensional integrals by expressing $D^{-k}(p)$ as

$$D^{-k}(p) = \int_0^\infty d\beta \frac{\beta^{k-1}}{\Gamma(k)} \prod_{\mu=1}^{4} e^{-2\beta(1-\cos p_\mu a)}. \tag{4.5}$$

Then all $\int_{-\pi/a}^{+\pi/a} dp_\mu$ integrals can be evaluated in terms of Bessel functions. After such an exercise we find

$$I_1 = \int d^4x (F^\alpha_{\mu\nu})^2 \frac{N}{8} \int_0^\infty d\beta\, e^{-8\beta} I_0^3(2\beta) I_1(2\beta) = \int d^4x (F^\alpha_{\mu\nu})^2 \frac{11N}{96\pi^2}(0.322\,288), \tag{4.6}$$

where the number in the parentheses is the contribution to $-C(N) = \ln(\Lambda_{PV}/\Lambda_L)$. All numerical integrations were performed with an accuracy of one part in $10^{-8}$.

The term $I_3$ does have a corresponding continuum limit and contains logarithmic divergences as $a \to 0$. The corresponding continuum calculation is done with a Pauli-Villars regulator for the $\delta A_\mu$ field. Again, in Eq. (3.17), the $D_\mu$'s can be replaced by $\Delta_\mu$'s to this order, and the only graphs which contribute are depicted in Fig. 1(c). They yield

$$I_3 = \int d^4x (F^\alpha_{\mu\nu})^2 N \left\{ \int (dp)_a \frac{[(1+\cos p_1 a)/2][(1+\cos p_2 a)/2]}{D^2(p)} - \int \frac{d^4p}{(2\pi)^4}\left[\frac{1}{p^4} - \frac{1}{(p^2+m^2)^2}\right]\right\}$$

$$= \int d^4x (F^\alpha_{\mu\nu})^2 \frac{N}{4} \int_0^\infty d\beta \left[\beta e^{-8\beta} I_0^2(2\beta)(I_0(2\beta)+I_1(2\beta))^2 - \frac{(1-e^{-\beta m^2 a^2})}{4\pi^2\beta}\right] \tag{4.7}$$

$$= \int d^4x (F^\alpha_{\mu\nu})^2 \frac{11N}{96\pi^2}(0.678\,249 - \tfrac{12}{11}\ln ma). \tag{4.8}$$

Note that both integrals in Eq. (4.7) have infrared divergences; however, their difference is finite. In subtracting these two integrals we have been careful to first introduce infrared regulators (e.g., a gluon mass) in order to avoid error. This term is essentially the only one considered in Ref. 1. However, there we only took into account the effects of the lattice propagator, which yields a positive contribution since $1/D(p) > 1/p^2$, and not the effect of the lattice derivatives which produce the term $(1+\cos p_1 a)/2$ which suppresses the integrand on the boundary of the Brillouin zone ($p_1 a = \pm\pi$).

Finally we consider the scalar loops contribution $\frac{1}{2}\ln(Z_a^{sc}/Z_m^{sc})$. There is simply the difference of the vacuum energy of a scalar particle (times two) in the external field for the lattice and continuum theories. The diagrams that contribute are depicted in Fig. 1(d) and, after some work, yield

$$\tfrac{1}{2}\ln\left(\frac{Z_a^{sc}}{Z_m^{sc}}\right) = \int d^4x (F^\alpha_{\mu\nu})^2 \frac{N}{12}\left\{-\int (dp)_a \frac{\cos p_1 a \cos p_2 a}{D^2(p)} + \int \frac{d^4p}{(2\pi)^4}\left[\frac{1}{p^4} - \frac{m^4}{(p^2+m^2)^4}\right]\right\}$$

$$= \int d^4x (F^\alpha_{\mu\nu})^2 \frac{N}{192}\left\{-\int_0^\infty d\beta \left[16\beta e^{-8\beta} I_0^2(2\beta) I_1^2(2\beta) + \frac{1-e^{-\beta m^2 a^2}}{\beta\pi^2}\right] + \frac{5}{6}\right\}$$

$$= \int d^4x (F^\alpha_{\mu\nu})^2 \frac{11N}{96\pi^2}(0.050\,864 + \tfrac{1}{12}\ln ma). \tag{4.9}$$

We now combine all the terms to yield

$$d(ma) = \frac{11N}{96\pi^2}\left(\ln ma - 3.743\,111 + \frac{3\pi^2}{11N^2}\right). \tag{4.10}$$

Thus

$$C(N) = \frac{3\pi^2}{11N^2} - 3.743\,111 \tag{4.11}$$

and

$$\left(\frac{\Lambda_L}{\Lambda_{PV}}\right)_{SU(N)} = 0.023\,680 e^{3\pi^2/11N^2}.$$

This yields

$$\left(\frac{\Lambda_L}{\Lambda_{PV}}\right)_{SU(2)} = 0.046\,413, \quad \left(\frac{\Lambda_{PV}}{\Lambda_L}\right)_{SU(2)} = 21.545\,854,$$

$$\left(\frac{\Lambda_L}{\Lambda_{PV}}\right)_{SU(3)} = 0.031\,936, \quad \left(\frac{\Lambda_{PV}}{\Lambda_L}\right)_{SU(3)} = 31.321\,2960,$$

$$\left(\frac{\Lambda_L}{\Lambda_{PV}}\right)_{SU(\infty)} = 0.023\,68, \quad \left(\frac{\Lambda_{PV}}{\Lambda_L}\right)_{SU(\infty)} = 42.229\,161.$$

We can now compare the lattice definition of the coupling to other more familiar continuum definitions, i.e., dimensional regularization, $\overline{\mathrm{MS}}$, or momentum-space regularization. To do this it suffices to establish the relation between $\Lambda_{PV}$ and $\Lambda$'s corresponding to these definitions. This can be done quite easily in the continuum theory using, for example, the background-field method described above.

The relation of $\Lambda_{PV}$ to $\Lambda_{DR}$ is determined by comparing calculations using Pauli-Villars regulators to those using dimensional regularization and minimal subtraction. This calculation was performed by 't Hooft[16] with the result ($\gamma$ = Euler's constant)

$$\ln\left(\frac{\Lambda_{PV}}{\Lambda_{DR}}\right) = \tfrac{1}{2}\ln 4\pi - \gamma/2 + \tfrac{1}{12} = 1.060\,238, \qquad (4.12)$$

$$\left(\frac{\Lambda_{PV}}{\Lambda_{DR}}\right) = 2.887\,057.$$

In the so-called $\overline{\mathrm{MS}}$ scheme[17] the coupling is defined with a finite renormalization which corresponds to removing the factor $\frac{1}{2}(\ln 4\pi - \gamma)$ from the above equation. Thus the Pauli-Villars and $\overline{\mathrm{MS}}$ $\Lambda$'s are almost identical,

$$\left(\frac{\Lambda_{PV}}{\Lambda_{\overline{MS}}}\right) = e^{1/12} = 1.086\,904. \qquad (4.13)$$

Finally we can use the relation, established by Celmaster and Gonsalves,[18] between $\Lambda_{DR}$ and $\Lambda_{MOM}$, where $\Lambda_{MOM}$ corresponds to a (symmetric) momentum-space subtraction procedure. This yields a gauge-dependent definition of $g^2$ and $\Lambda_{MOM}$, which is, in the absence of matter fields,

$$\left(\frac{\Lambda_{MOM}}{\Lambda_{DR}}\right)_{\text{Landau gauge}} = 8.86,$$

$$\left(\frac{\Lambda_{MOM}}{\Lambda_{DR}}\right)_{\text{Feynman gauge}} = 7.69, \qquad (4.14)$$

$$\left(\frac{\Lambda_{MOM}}{\Lambda_{PV}}\right)_{\text{Feynman gauge}} = 2.66.$$

Using these relations we can compare $\Lambda_L$ to $\Lambda_{DR}$, $\Lambda_{\overline{MS}}$, or $\Lambda_{MOM}$. Thus, for example,

$$\left(\frac{\Lambda_{\overline{MS}}}{\Lambda_L}\right)_{SU(N)} = 38.852\,704\, e^{-3\pi^2/11N^2}, \qquad (4.15)$$

$$\left(\frac{\Lambda_{\text{MOM, Feynman gauge}}}{\Lambda_L}\right)_{SU(N)} = 112.33\, e^{-3\pi^2/11N^2}.$$

We can now compare with Hasenfratz and Hazenfratz's calculation, in which they find (in the Feynman gauge)[10]

$$\left(\frac{\Lambda_{MOM}}{\Lambda_L}\right)_{SU(2)} = 57.5, \quad \left(\frac{\Lambda_{MOM}}{\Lambda_L}\right)_{SU(3)} = 83.5,$$

whereas we find

$$\left(\frac{\Lambda_{MOM}}{\Lambda_L}\right)_{SU(2)} = 57.3, \quad \left(\frac{\Lambda_{MOM}}{\Lambda_L}\right)_{SU(3)} = 83.29. \qquad (4.16)$$

The slight disagreement is probably not significant.

Finally we note that the large value of $\Lambda_{\overline{MS}}/\Lambda_L$ or $\Lambda_{MOM}/\Lambda_L$ implies that the calculated values of $\sqrt{\sigma}/\Lambda_L$ are much closer to those one would expect in the real world. Thus in the case of SU(3) the values determined by semiclassical or Monte Carlo techniques lie in the range $\sqrt{\sigma} = (150-250)\Lambda_L$. When translated to $\Lambda_{\overline{MS}}$ this means that $\sqrt{\sigma} = (7-12)\Lambda_{\overline{MS}}$. In the real world $\sqrt{\sigma} \sim 450$ MeV and $\Lambda_{\overline{MS}} \sim (200-600)$ MeV. Some authors have tried to decrease the remaining discrepancy by using $\Lambda_{MOM}$ to obtain $\sqrt{\sigma} = (1.5-2.5)\Lambda_{MOM}$. However, this is problematic since the value of $\Lambda_{MOM}$ is highly dependent on the number of fermion fields and $\Lambda_{MOM}$ is, in any case, determined by fits to experiment to be larger than $\Lambda_{\overline{MS}}$. We instead ascribe the discrepancy to the absence of quarks in these treatments. In the semiclassical approach it is clear that the introduction of quarks will lead to a decrease in $\sqrt{\sigma}/\Lambda_L$ by an amount which we can crudely estimate to be about 2–5.

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