PHYSICAL REVIEW D VOLUME 23, NUMBER 10 15 MAY 1981

Generalized noninteracting vortices

M. A. Lohe
Department of Mathematical Physics, The University of Adelaide, Adelaide, South Australia, 5000
(Received 8 December 1980)

The Abelian Higgs model allows noninteracting multivortex solutions for a special choice of coupling constant. We show that more general noninteracting vortices are permitted in models with a general scalar interaction $V(\phi)$, provided that the kinetic term for the scalar fields in the Lagrangian is suitably modified. We discuss properties of the general system and show that the static field equations are solved by a single nonlinear partial differential equation.

I. INTRODUCTION

The simplest of the gauge theories exhibiting soliton behavior is the Abelian Higgs model in two space dimensions. The solitons, or vortices, were described by Nielsen and Olesen[1] using ideas known from the Ginzburg-Landau theory of superconductivity. The static solitons are the localized vortex solutions found by Abrikosov[2] and consist (in three dimensions) of filaments which pierce the superconducting fluid, and in which the magnetic flux is concentrated. The vortices are characterized by a localized energy density and corresponding finite mass, and by a nonzero charge of topological origin. The properties of the static vortices are determined by coupled ordinary differential equations which do not allow analytic solutions, except asymptotically, so that investigations of the detailed vortex interactions must be done numerically. The nature of the interaction depends on a dimensionless coupling constant λ; for $\lambda > 1$ (with our normalization) the vortices repel each other and the static multivortex configuration is not stable, but for $\lambda < 1$ the vortices attract each other.[3,4]

We are interested here in the special case $\lambda = 1$ for which the electromagnetic forces cancel the attractive forces due to the Higgs scalar particle, with the result that the vortices do not interact even at finite separations, as has been verified numerically.[4] For $\lambda = 1$ it is possible to obtain some results analytically, principally because the second-order equations can be integrated to a coupled first-order system. This model is one of several for which the Hamiltonian can be arranged as a sum of positive terms plus a surface integral which is fixed, and which therefore provides a lower bound on the energy. This lower bound can be attained by imposing first-order equations, and the total energy can then be evaluated exactly and is equal to the surface integral, which in turn is related to the topological charge. Examples of such systems are (among others) scalar field theory with self-interaction $V(\phi)$ (Ref. 5) in one space dimension, monopole theory with no scalar self-interactions, and pure four-dimensional Yang-Mills theory.[6] Advantages of such models are the following: to obtain static solutions one need only solve a first-order set of coupled equations; the system is dynamically stable because any perturbation can add only positively to the total energy; the dual requirements of nonzero topological charge and finite energy are combined, and the mass can be calculated exactly. The Abelian Higgs model for $\lambda = 1$ possesses such properties as Bogomol'nyi[5] has shown. The first-order system in this case cannot be solved in closed form but has been investigated[7] assuming rotational symmetry, when all vortices are located at the origin. The general solution has been shown[8] by deformation theory to belong to a $2n$-parameter class, where n is the topological charge, as one would expect if the $2n$ parameters were the arbitrary positions of the n vortices.

We describe here a generalization of the Higgs model which possesses similar properties. The decomposition of the Hamiltonian for the Higgs model as a sum of positive terms plus a surface integral appears to rely heavily on the ϕ^4 interaction, but classically there is no reason why ϕ^4 should be distinguished in this way. There is ample motivation for finding a generalization which allows an arbitrary interaction $V(\phi)$. One might hope to find a model with properties resembling those of the sine-Gordon model, or at least a model for which some exact solutions are possible. In such models there is also the possibility that perturbations about the static vortices might allow analytic description; in the quantum theory this would constitute a description of vortex-meson interactions. At present, phenomena in the Higgs model such as possible vortex-meson bound states must be investigated numerically.

We obtain the generalization we require by modifying the kinetic term for the Higgs field [Eqs. (1) and (2)]. We do not discuss the quantum theory or whether such models could be renormalizable. The decomposition of the Hamiltonian into positive terms plus a surface integral is given in Eq. (11) and the first-order system which determines static solutions is given in Eqs. (12). These equations reduce to a single elliptic partial differential equation shown in Eq. (21). Finally we follow the analysis of Weinberg[8] to show that the general n-vortex solution of the first-order equations depends on $2n$ parameters.

II. GENERALIZED VORTICES

The Lagrangian we use is

$$\mathcal{L} = -\tfrac{1}{4}(F_{\mu\nu})^2 + \tfrac{1}{2}F(\phi)(D_\mu\phi^a)^2 - V(\phi)\,. \tag{1}$$

Here $\mu, \nu = 0, 1, 2$ and $a = 1, 2$, and we have a two-component real Higgs field ϕ, with

$$D_\mu\phi^a = \partial_\mu\phi^a - \epsilon^{ab}A_\mu\phi^b\,.$$

F and V are real functions, depending only on $|\phi|$ so as to ensure the gauge invariance of $\mathcal{L}$, and being at least continuous and finite everywhere for finite $|\phi|$. We also require that F and V be non-negative in order that the Hamiltonian H be non-negative, and V should have a minimum at a nonzero value of $|\phi|$ so as to obtain symmetry breaking. We now define V, for each F, according to

$$V = \tfrac{1}{2}w^2\,, \tag{2}$$

where

$$w(|\phi|) = -\int_v^{|\phi|} ds\, sF(s)\,.$$

We will frequently use $V' = ww' = -|\phi|Fw$, where $V'(\phi) = dV(\phi)/d|\phi|$. Evidently V is non-negative and has a single minimum, at $|\phi| = v$. As a special case of Eq. (2) we can take $F \equiv 1$ which implies that $V = \frac{1}{8}(|\phi|^2 - v^2)^2$, and we regain the Higgs model with the special coupling constant $\lambda = 1$ mentioned above. Equation (2) ensures, if we may think in quantum-mechanical terms, that the masses of the Higgs meson and the massive photon are equal, implying that the corresponding forces due to each are of equal range.

The mass m of the scalar and vector particles is given by

$$m^2 = F(v)v^2 = \frac{V''(v)}{F(v)}\,. \tag{3}$$

The field equations which follow from the Lagrangian (1) are

$$\begin{aligned} \partial^\mu F_{\mu\nu} &= -F(\phi)\epsilon^{ab}\phi^a D_\nu\phi^b\,, \\ D_\mu[F(\phi)D^\mu\phi^a] &= -\frac{\partial V}{\partial\phi^a} + \frac{1}{2}\frac{\partial F}{\partial\phi^a}(D_\mu\phi^b)^2\,. \end{aligned} \tag{4}$$

The static field equations are $(i, j = 1, 2)$

$$\partial_i F_{ij} = F(\phi)\epsilon^{ab}\phi^a D_j\phi^b\,, \tag{5a}$$

$$F(\phi)[D_iD_i\phi^a + w(\phi)\phi^a] = \frac{F'(\phi)}{2|\phi|}[\phi^a(D_i\phi^b)^2 - 2D_i\phi^a(\phi^bD_i\phi^b)]\,. \tag{5b}$$

We wish to find a first-order system which solves these equations, and to look for smooth solutions with a nontrivial winding number n. Here n must be an integer and is proportional to the charge corresponding to the conserved current

$$j^\mu = \epsilon^{\mu\nu\rho}\epsilon^{ab}\partial_\nu\phi^a\partial_\rho\phi^b\,. \tag{6}$$

Provided V has a minimum at some nonzero $|\phi| = v$, the nontrivial topological structure at spatial infinity ensures the existence of the winding number n, which guarantees topological stability for the system.

For static solutions with $A_0 = 0$ the Hamiltonian density can be arranged as follows:

$$\begin{aligned} \mathcal{H} &= \tfrac{1}{4}(F_{ij})^2 + \tfrac{1}{2}F(\phi)(D_i\phi^a)^2 + V(\phi) \\ &= \tfrac{1}{4}(F_{ij} + \epsilon_{ij}w)^2 + \frac{F(\phi)}{4}(\epsilon_{ij}D_j\phi^a + \epsilon^{ab}D_i\phi^b)^2 \\ &\quad - \epsilon_{ij}\partial_i(A_jw) + \frac{F(\phi)}{2}\epsilon_{ij}\epsilon^{ab}\partial_i\phi^a\partial_j\phi^b\,, \end{aligned}$$

where we have had to use the relation (2). The term $\epsilon_{ij}\partial_i(A_jw)$ when integrated can be converted to a surface integral which vanishes, since $w \to 0$ at spatial infinity. Using the identity

$$\epsilon_{ij}\epsilon^{ab}\phi^c\phi^a\partial_i\phi^b\partial_j\phi^c = -\frac{|\phi|^2}{2}\epsilon^{ab}\epsilon_{ij}\partial_i\phi^a\partial_j\phi^b\,, \tag{7}$$

we also obtain

$$\frac{F(\phi)}{2}\epsilon_{ij}\epsilon^{ab}\partial_i\phi^a\partial_j\phi^b = \partial_i[(C - w)|\phi|^{-2}\epsilon_{ij}\epsilon^{ab}\phi^a\partial_j\phi^b]\,, \tag{8}$$

where C is some constant. This expression is checked by formally performing the differentiation on the right-hand side. Since this manipulation is valid only if $[C - w(\phi)]|\phi|^{-2}$ is nonsingular for all $|\phi|$ we see that we must choose

$$C = w(0)\,. \tag{9}$$

The factor $[w(0) - w(\phi)]|\phi|^{-2}$ is then nonsingular at $|\phi| = 0$ provided the same is true of $F(\phi)$, as follows from the relation (2). The expression (8) therefore contributes a surface integral which is easily evaluated in terms of the winding number n, where

$$n = \frac{1}{2\pi v^2}\int d^2x\, \epsilon_{ij}\epsilon^{ab}\partial_i\phi^a\partial_j\phi^b\,. \tag{10}$$

The Hamiltonian is now

$$H=\int d^2x\left[\tfrac{1}{4}(F_{ij}+\epsilon_{ij}w)^2+\frac{F(\phi)}{4}(\epsilon_{ij}D_j\phi^a+\epsilon^{ab}D_i\phi^b)^2\right]+2\pi n w(0)\,, \tag{11}$$

showing that the energy of any solution with winding number n has a lower bound of $2\pi n w(0)$. Here we have assumed that n is positive, but by suitable sign changes negative values of n can also be accommodated, leading in general to a lower bound of $2\pi|n|w(0)$. We can attain this bound by imposing the first-order equations

$$\begin{aligned}F_{ij}&=-\epsilon_{ij}w(\phi)\,,\\ D_i\phi^a&=\epsilon_{ij}\epsilon^{ab}D_j\phi^b\,,\end{aligned} \tag{12}$$

which are a simple generalization of those obtained by Bogomol'nyi[5] for a ϕ^4 interaction. From Eq. (11) we see that any solution of these equations must provide a local minimum of the Hamiltonian and will therefore satisfy the field equations. This can also be checked directly by substitution into Eqs. (5). It happens then that both sides of Eq. (5b) are equal to zero. Equations (12) also imply the equipartition of energy between electromagnetic and potential terms, as is implied by Derrick's theorem[9]:

$$\int d^2x\,\tfrac{1}{4}(F_{ij})^2=\int d^2x\,V(\phi)\,. \tag{13}$$

The mass M of vortices which are solutions of Eqs. (12) is given by [from Eq. (11)]

$$M=2\pi n w(0)\,. \tag{14}$$

Since there is the possibility of rescaling $F(\phi)$ and $w(\phi)$ this mass should be compared to the squared meson mass, giving

$$\frac{M}{m^2}=\frac{2\pi n w(0)}{F(v)v^2}\,. \tag{15}$$

The vortex mass is related to topological properties of the theory. In fact we could define a current

$$j^\mu=\epsilon^{\mu\nu\rho}\frac{F(\phi)}{2}\epsilon^{ab}\partial_\nu\phi^a\partial_\rho\phi^b \tag{16}$$

which is also identically conserved and for which the charge is equal to the vortex mass (14).

The linear dependence of the vortex mass on the charge suggests that these vortices are noninteracting and that one ought to be able to find solutions describing vortices located at arbitrary points on the place. If such solutions exist they would be dynamically stable as Eq. (11) shows, since any perturbation can add only positively to the energy. The noninteracting property is evident if we calculate the instantaneous force on a vortex due to any other vortices.[10] The symmetric energy-momentum tensor for static solutions with $A_0=0$ is given by

$$T_{ij}=F_{ik}F_{jk}+F(\phi)D_i\phi^aD_j\phi^a-\delta_{ij}[\tfrac{1}{4}(F_{kl})^2+V(\phi)+\tfrac{1}{2}F(\phi)(D_k\phi^a)^2] \tag{17}$$

and for solutions of Eqs. (12) is identically zero. Let us choose, for a time-dependent interacting vortex system, a configuration in the gauge $A_0=0$ which is initially static and satisfies Eq. (12). Then the force at that initial instant acting on an arbitrary volume Ω is[10]

$$\frac{d}{dt}\int_\Omega d^2x\,T_{0j}=\int_S dS^iT_{ij}\,, \tag{18}$$

where the surface S encloses Ω, and we have used the conservation properties of $T_{\mu\nu}$. Since $T_{ij}=0$ we see that the instantaneous force between vortices is zero.

Next, let us simplify Eqs. (12). This can be done as before[11] with the help of the polar decomposition

$$\phi_1+i\phi_2=Re^{i\alpha}\,.$$

The second set of Eqs. (12) then reduce to

$$A_i=-\partial_i\alpha-\epsilon_{ij}\partial_j(\ln R)\,. \tag{19}$$

The gauge condition $\partial_iA_i=0$ gives

$$\Delta\alpha=0\,, \tag{20}$$

and from Eq. (12),

$$\Delta(\ln R)=-w(R)\,. \tag{21}$$

For the solution of Eq. (20) we choose

$$\alpha=\sum_{i=1}^{n}\arctan\left(\frac{x_2-a_2{}^i}{x_1-a_1{}^i}\right), \tag{22}$$

where the $2n$ parameters $\underline{a}^i=(a_1{}^i,a_2{}^i)$ for $i=1,\ldots,n$ can be taken as the vortex positions. The parameter n is the topological charge as calculated from Eq. (10), and is a positive integer. So far we have ignored singularities. We must have smooth gauge fields, and from Eq. (19) this means that $\ln R$ must be singular as $\underline{x}\to\underline{a}^i$ in order to cancel the singularity arising from α at $\underline{x}=\underline{a}^i$. We require

$$R\sim|\underline{x}-\underline{a}^i|\quad\text{for } \underline{x}\sim\underline{a}^i\ (i=1,\ldots,n)\,. \tag{23}$$

This means in fact that Eq. (21) should be replaced by

$$\Delta(\ln R)+w(R)=2\pi\sum_{i=1}^{n}\delta(\underline{x}-\underline{a}^i) \tag{24}$$

to accommodate the singular behavior of $\ln R$ as R approaches zero. We also impose the boundary condition

$$R \to v \quad \text{as } |\underline{x}| \to \infty . \tag{25}$$

We have now arrived at the point where the Higgs model with a ϕ^4 interaction stands, where we need to solve a single nonlinear elliptic partial differential equation. It is hoped that one can progress further by selecting some suitable $V(R)$ for which Eq. (24) can be solved exactly. We can choose Eq. (24) to be linear by taking $V(R)=[\ln(R/v)]^2$, but this potential behaves badly at $R=0$. A series solution of Eq. (24) can be constructed by perturbing about Laplace's equation. Let $G=\ln(R/v)$ and consider

$$\Delta G = -w(ve^{\alpha G}). \tag{26}$$

By writing G as a power series in α we obtain a sequence of Poisson equations, except for the first which is Laplace's equation, for which the solution can be chosen to have the correct singularities. However, convergence of the series needs to be demonstrated. The situation is simplified if we choose all vortices to be located at the origin, for then R is a function of the radial distance r alone. Generally, R attains its vacuum value radially at infinity and we find

$$R - v \sim K_0(mr), \quad \text{large } r \tag{27}$$

where K_0 is a Bessel function, and the mass m is defined in Eq. (3).

III. DEFORMATION THEORY

If solutions to Eq. (24) exist with the boundary conditions (23) and (25) then we expect the general solution to depend on at least the $2n$ parameters determining the vortex positions. By deforming a given solution we can count the modes of fluctuation which leave the energy unchanged. Weinberg's analysis,[8] which is recovered for $F(\phi)\equiv 1$, generalizes easily and allows us to show that there is precisely a $2n$-parameter family of solutions.

First we impose the Coulomb gauge $\partial_i A_i = 0$ as before. By expanding Eqs. (12) about a solution and retaining only linear terms, we find

$$\mathfrak{D}\eta = 0, \tag{28}$$

where $\eta = (\delta\phi_1, \delta\phi_2, \delta A_1, \delta A_2)^t$, and

$$\mathfrak{D} = \begin{pmatrix} \partial_1 - A_2 & -\partial_2 - A_1 & -\phi_2 & -\phi_1 \\ \partial_2 + A_1 & \partial_1 - A_2 & \phi_1 & -\phi_2 \\ -F\phi_1 & -F\phi_2 & -\partial_2 & \partial_1 \\ 0 & 0 & \partial_1 & \partial_2 \end{pmatrix} \tag{29}$$

(our fields are defined with opposite sign to those of Weinberg). The index $\mathcal{I}(\mathfrak{D})$ is defined as the number of solutions of Eq. (28) minus the number of solutions of the equation

$$\mathfrak{D}^*\psi = 0, \tag{30}$$

where $\mathfrak{D}^*$ is the adjoint of $\mathfrak{D}$. We can show that Eq. (30) has no solutions, i.e., the kernel of $\mathfrak{D}^*$ vanishes, so that $\mathcal{I}(\mathfrak{D})$ counts the number of infinitesimal deformations of a given solution. Let

$$D = \begin{pmatrix} -\phi_1 & -\phi_2 & -\partial_2 & -\partial_1 \\ -\phi_2 & \phi_1 & \partial_1 & \partial_2 \\ 0 & 0 & -\phi_2 & -\phi_1 \\ 0 & 0 & \phi_1 & -\phi_2 \end{pmatrix}. \tag{31}$$

The equation $D\mathfrak{D}^*\psi = 0$ implies

$$\begin{aligned} &F|\phi|^2\psi_3 - \Delta\psi_3 = 0, \\ &-\Delta\psi_4 = 0, \\ &|\phi|^2\psi_1 + (\phi_1\partial_1 - \phi_2\partial_2)\psi_3 + (\phi_2\partial_1 + \phi_1\partial_2)\psi_4 = 0, \\ &|\phi|^2\psi_2 + (\phi_1\partial_2 + \phi_2\partial_1)\psi_3 + (\phi_2\partial_2 - \phi_1\partial_1)\psi_4 = 0. \end{aligned} \tag{32}$$

Since the ψ functions must be square integrable we find from positivity that $\psi_3=\psi_4=0$ and hence also $\psi_1 = 0 = \psi_2$. For this we use the fact that F is non-negative. From here we may follow Weinberg's analysis exactly. We can write

$$\mathcal{I}(\mathfrak{D}) = \mathrm{Tr}\left(\frac{M^2}{\mathfrak{D}^*\mathfrak{D}+M^2}\right) - \mathrm{Tr}\left(\frac{M^2}{\mathfrak{D}\mathfrak{D}^*+M^2}\right), \tag{33}$$

to be evaluated in the limit $M^2 \to \infty$. We find

$$\mathfrak{D}^*\mathfrak{D} = -\Delta - L_1,$$

$$\mathfrak{D}\mathfrak{D}^* = -\Delta - L_2,$$

where L_1, L_2 are first-order differential operators, and satisfy

$$\begin{aligned} \mathrm{Tr}L_1 &= -2F_{12} - 2|A|^2 - (F^2+2)|\phi|^2, \\ \mathrm{Tr}L_2 &= 2F_{12} - 2|A|^2 - (F^2+2)|\phi|^2, \end{aligned} \tag{34}$$

so that $\mathrm{Tr}L_1 - \mathrm{Tr}L_2 = -4F_{12}$. Finally, using

$$\int d^2x\, F_{12} = -2\pi n$$

we can evaluate $\mathcal{I}(\mathfrak{D})$ to obtain[8] $\mathcal{I}(\mathfrak{D}) = 2n$, as expected.

IV. CONCLUSION

We have described a generalization of the Abelian Higgs model with $\lambda = 1$, and have shown that all the main features of the classical vortices generalize with little modification. We have introduced an arbitrary function $F(\phi)$ into the theory without disturbing the main properties, and hope that a study of the general system will lead to a better understanding of vortex behavior. A similar generalization, of modifying the scalar kinetic term of the Lagrangian, does not seem to exist for an SU(2) monopole theory in the Bogomol'nyi-Prasad-Sommerfield limit[9] without destroying the noninteracting property.

Recently, considerable progress has been made by Taubes[12] in understanding the classical vortex system for the case $F \equiv 1$. The solutions of Eq. (24) with boundary conditions (23) and (25) are shown to exist and to be unique, and the first-order equations (12) are shown to yield all finite-energy vortex solutions for $F \equiv 1$. Existence and uniqueness can also be proved for the general case described here,[13] with mild restrictions on F, and other properties also follow[13] using similar arguments to those of Taubes.

[1] H. B. Nielsen and P. Olesen, Nucl. Phys. B61, 45 (1973).
[2] A. A. Abrikosov, Zh. Eksp. Teor. Fiz. 32, 1442 (1957) [Sov. Phys.—JETP 5, 1174 (1957)].
[3] E. B. Bogomol'nyi and A. I. Vainshtein, Yad. Fiz. 23, 1111 (1976) [Sov. J. Nucl. Phys. 23, 588 (1976)].
[4] L. Jacobs and C. Rebbi, Phys. Rev. B 19, 4486 (1979); C. Rebbi, in *Geometrical and Topological Methods in Gauge Theories*, proceedings of the CMS Summer Research Institute, Montreal, 1979, edited by J. P. Harnad and S. Schnider (Springer, Berlin, 1980), p. 96.
[5] E. B. Bogomol'nyi, Yad. Fiz. 24, 861 (1976) [Sov. J. Nucl. Phys. 24, 449 (1976)].
[6] A. A. Belavin *et al.*, Phys. Lett. 59B, 85 (1975).
[7] H. J. de Vega and F. A. Schaposnik, Phys. Rev. D 14, 1100 (1976).
[8] E. J. Weinberg, Phys. Rev. D 19, 3008 (1979).
[9] P. Goddard and D. Olive, Rep. Prog. Phys. 41, 1357 (1978).
[10] L. O'Raifeartaigh, Soo Yong Park, and K. C. Wali, Phys. Rev. D 20, 1941 (1979).
[11] M. A. Lohe, Phys. Lett. 70B, 325 (1977); J. Hietarinta, Phys. Rev. D 15, 1137 (1977).
[12] C. H. Taubes, Commun. Math. Phys. 72, 277 (1980); 75, 207 (1980).
[13] M. A. Lohe and John van der Hoek, University of Adelaide report (unpublished).