Systematic framework for generating multimonopole solutions

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We describe a systematic framework for the construction of monopole solutions, as static self-dual gauge fields with appropriate boundary conditions, of arbitrary topological charge. This procedure is based on the Atiyah-Ward *Ansatz*, which is explicitly constructed and has some parameters. The solution, in general, is complex and has singularities, though it is static and has appropriate asymptotic behavior for a monopole solution. The conditions for the solution to be nonsingular and gauge transformable to a real form are given in simple algebraic form. We can then, in principle, check by explicit calculation if they are satisfied for some values of the parameters, or prove the existence of a choice of these parameters. However, we have not yet succeeded in determining these parameters besides the already known cases of one- and two-monopole solutions. We give explicit expressions for the gauge transformations and the real potentials, when these parameters can be chosen to satisfy the smoothness and reality conditions.

I. INTRODUCTION

Among the many rich and beautiful mathematical structures of the SU(2) Yang-Mills[1] equations there is the so-called one-monopole solution which is self-dual, smooth, real, with finite energy depending only on three coordinates in four-dimensional Euclidean space. The nomenclature comes from the introduction by 't Hooft and Polyakov[2] of magnetic monopoles as a classical solution in a Yang-Mills-Higgs theory.[3] In the limit of vanishing Higgs potential, an analytic form of a one-monopole solution was found[4] and in this limit the monopole solution can be reinterpreted as a "static" (i.e., independent of the Euclidean time) self-dual Yang-Mills field in four-dimensional Euclidean space, satisfying appropriate boundary conditions. These monopoles are characterized by an integer-valued topological charge,[5] which is just the magnetic charge in suitable units. The monopole solution of Ref. 4 has topological charge one.

There have been a large number of attempts to find solutions with higher topological or magnetic charge. One approach to this problem is to begin with an *Ansatz* incorporating some symmetry to reduce the number of independent field components. However, it is known[6] that the assumption of spherical symmetry leads uniquely to the one-monopole solution. As a result one must begin with an *Ansatz* with weaker symmetries, which usually leads to a rather complicated system of differential equations, which is hard to analyze. We shall not use this approach here.

Another possible approach that we shall use is to apply techniques developed for the construction of instanton[7] solutions. Instantons[8] are the self-dual (or anti-self-dual) Yang-Mills field in four-dimensional Euclidean space with finite action and are again characterized by a topological charge, known as the Chern number, or the Pontriaign number, or the instanton number. The first successful application of this approach is due to Manton.[9] He rederived the one-monopole solution from the Corrigan-Fairlie-'t Hooft-Wilczek[10] (CFtHW) *Ansatz*. However, this solution is in a complex form, i.e., the gauge potentials are complex, which can be made real by an explicitly constructed complex gauge transformation. However, Manton did not find any new solution that can be made real by a complex gauge transformation.

Based on the "twistor" approach to self-dual Yang-Mills fields developed by Ward,[11] Atiyah and Ward[12] (AW) proposed a hierarchy of *Ansätze* $\mathcal{A}_n$, $n=1, 2, \ldots$, for the construction of all instanton solutions. The first *Ansatz* $\mathcal{A}_1$ is the CFtHW *Ansatz* which is given in terms of a spin-zero massless free field, i.e., a solution of the four-dimensional Laplace equation. The *Ansatz* $\mathcal{A}_n$ can be described by certain spin-$(n-1)$ massless fields. Atiyah and Ward use the language and techniques of analytic and algebraic geometry. Moreover, they did not give any explicit forms beyond the $\mathcal{A}_2$ *Ansatz*. Corrigan, Fairlie, Goddard, and Yates[13] (CFGY) gave an explicit construction of all the *Ansätze* $\mathcal{A}_l$, $l=1,2,\ldots$, which takes a particularly simple form in Yang's[14] R gauge. In the

R gauge, the self-dual potentials are described by three functions satisfying a system of second-order (nonlinear) differential equations. Moreover, Corrigan *et al.* have shown that in the R gauge, successive AW *Ansätze* are related by a Bäcklund transformation,[15] which we call the BI (the transformation α of CFGY) for Yang's equations. For our purpose, a Bäcklund transformation (BT) is a transformation, usually given by a system of first-order differential equations, which generates *locally* "new" solutions of self-duality equations from "old" ones. Corrigan *et al.* then integrated this BT to give an independent "elementary" proof that the AW *Ansätze* do indeed give solutions of the self-duality equations and, of course, this gives an explicit construction of the AW *Ansätze*. In fact, we use this result to give a simple *definition* of the AW *Ansätze*. However, this definition is local in nature and does not simplify the discussion of global problems. As a result of the singularity problems[16] this has not led to the explicit construction of any new instanton solutions. However, as the global requirements of the monopole problem are very different from the instanton problem, we view the AW *Ansätze* as generating solutions of the self-duality equation, and then try to satisfy the remaining requirements for an acceptable solution.

Bäcklund transformations are known[17] to be quite useful in generating solutions in two-dimensional models. It is, therefore, natural to try this method in higher dimensions also. Lohe[18] applied the Bäcklund transformation BI twice on the one-monopole solution to construct a three-monopole solution, which turned out to be singular.[19] Towards similar goals, we developed[20] a different two-parameter BT in a manifestly gauge-invariant formulation[21] of self-dual gauge fields. However, so far this has not been used to produce any new finite-energy or finite-action solutions. It has recently been shown by Forgacs, Horvath, and Palla[22] that it is possible to obtain the one-monopole solution by applying yet another Bäcklund transformation on a "vacuumlike" solution. In this approach we can view the solutions as being obtained by application of BT's on a specifically chosen "initial" solution. However, this initial solution is neither obvious nor natural, and appears to give only one acceptable solution. One of the major advantages of this method in two-dimensions is that the BT can be applied repeatedly to get more and more new solutions. This property appears to be absent in the application to the monopole problem.

A more fruitful approach to the instanton problem is the Atiyah-Drinfeld-Hitchin-Manin[23] construction of all instanton solutions. This method has been used by Nahm[24] to construct the one-monopole solution. Again, so far no new solution has been found this way.

This spell was finally broken by Ward's[25] explicit construction of an exact monopole solution of topological charge two. This follows the recent numerical results[26] for the existence of multimonopole solutions and the proof by Taubes[27] of the existence of multimonopole solutions of arbitrary charge when their centers are separated from each other. The Ward solution is based on the $\mathcal{A}_2$ *Ansatz*, and is axially symmetric. However, the solution is in a complex gauge. Ward gave an existence proof of a gauge where the solution is real.

In this paper we generalize the work of Ward to a systematic procedure for construction of monopole solutions with arbitrary topological charge. We also explicitly construct a complex gauge transformation which makes Ward's solution, and any other solution obtained from this procedure, real. In Sec. II we review some necessary results on self-dual gauge fields and give theorem 2.1, a necessary and sufficient condition for the existence of a gauge transformation which makes a complex self-dual gauge field real. Section III is a review of 't Hooft-Polyakov monopoles in the vanishing-Higgs-potential limit. In Sec. IV we formulate the problem in the R gauge and describe a superposition formula for the energy density. This immediately leads to the result that any monopole solution, i.e., static solutions satisfying conditions of smoothness and reality, derived from the $\mathcal{A}_n$ *Ansatz* in general has topological charge n. In Sec. V we describe a systematic procedure for the construction of axially symmetric multimonopole solutions. This construction is given in terms of a single function Λ_0 of a specific form, but containing some free parameters. The resulting solution is also in complex form. *A necessary and sufficient condition for this solution to be gauge transformable to a real form is given in a simple algebraic form. For cases satisfying this condition, we give a real form for the potentials.* For general values of the parameters in function Λ_0, the solution will have singularities and will not satisfy the reality condition. Therefore, specific values of these parameters have to be chosen to obtain real nonsingular multimonopole solutions. The known one- and two-monopole solutions are discussed in Sec. VI. We conclude in Sec. VII with a summary.

II. SELF-DUAL GAUGE FIELDS

In this section we review some of the results from the theory of self-dual gauge fields in four-dimensional Euclidean space. We consider only the results we shall use in the following sections. This section also fixes our notation and conventions.[28]

In this paper we restrict ourselves to SU(2) gauge theory, and we use matrix notation for gauge potentials, etc., defined as

$$A_\mu = g\frac{\sigma^a}{2i}A_\mu^a, \quad \mu=1,2,3,4, \quad a=1,2,3, \tag{2.1}$$

where σ^a are the usual Pauli matrices and g is the coupling constant. Then,

$$F_{\mu\nu} \equiv g\frac{\sigma^a}{2i}F^a_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,. \tag{2.2}$$

For SU(2) gauge theory the gauge potentials A_μ^a are real, i.e., the matrix A_μ is traceless and anti-Hermitian. However, we need to use complex gauge potentials A_μ^a, i.e., the matrix A_μ is traceless but not anti-Hermitian, in some intermediate stages. Then we also need to complexify the gauge transformations, i.e., the gauge group becomes the complexification of SU(2), i.e., $SL(2,C)$ = complex 2×2 matrices of unit determinant.

Following Yang[14] we now consider an analytic continuation of A_μ into *complex space* where x_1, x_2, x_3, and x_4 are complex. The self-duality equations

$$F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{2.3}$$

are then valid also in complex space, in a region containing *real space* where the x_μ are real. Now consider four new complex variables defined by

$$\begin{aligned} \sqrt{2}\,y &= x_1 + ix_2, \quad \sqrt{2}\,\bar{y} = x_1 - ix_2\,, \\ \sqrt{2}\,z &= x_3 - ix_4, \quad \sqrt{2}\,\bar{z} = x_3 + ix_4\,. \end{aligned} \tag{2.4}$$

It is simple to check that the self-duality equations (2.4) reduce to

$$F_{yz} = F_{\bar{y}\bar{z}} = 0\,, \tag{2.5a}$$

$$F_{y\bar{y}} + F_{z\bar{z}} = 0\,. \tag{2.5b}$$

Equation (2.5a) implies that the potentials A_y, A_z ($A_{\bar{y}}, A_{\bar{z}}$) are pure gauges for fixed $\bar{y},\bar{z}$ (y,z), i.e., we can find two 2×2 complex matrices D and $\bar{D}$ such that

$$\begin{aligned} A_y &= D^{-1}D_{,y}, \quad A_z = D^{-1}D_{,z}\,, \\ A_{\bar{y}} &= \bar{D}^{-1}\bar{D}_{,\bar{y}}, \quad A_{\bar{z}} = \bar{D}^{-1}\bar{D}_{,\bar{z}}\,, \end{aligned} \tag{2.6}$$

where $D_{,y} \equiv \partial_y D$, etc. The matrices D ($\bar{D}$) represent the phase factor in complex two-dimensional space of y and z ($\bar{y}$ and $\bar{z}$) and can be written as path-ordered exponentials. The path of integration must lie in the plane $\bar{y},\bar{z}$ = constants (y,z = constants) and is independent of the path chosen in the plane. Since fixing $\bar{y},\bar{z}$ for real x_μ also fixes y and z, we must complexify the space. From $\mathrm{tr}A_\mu = 0$, we have

$$\det D = \det\bar{D} = 1\,. \tag{2.7}$$

We now define a matrix J by[21]

$$J \equiv D\bar{D}^{-1}\,. \tag{2.8}$$

Clearly $\det J = 1$. The remaining self-duality equation (2.5b) can be written as

$$(J^{-1}J_{,y})_{,\bar{y}} + (J^{-1}J_{,z})_{,\bar{z}} = 0\,. \tag{2.9}$$

It is clear from Eq. (2.6) that the definition of D and $\bar{D}$ involves a choice of gauge. Gauge transformations are given by

$$D \to DG, \quad \bar{D} \to \bar{D}G\,, \tag{2.10a}$$

$$A_\mu \to G^{-1}A_\mu G + G^{-1}G_{,\mu}\,, \tag{2.10b}$$

where G is an $SL(2,C)$ matrix. The matrices D and $\bar{D}$ are determined up to the transformation

$$D \to \bar{V}(\bar{y},\bar{z})D, \quad \bar{D} \to V(y,z)\bar{D}\,, \tag{2.11}$$

where $\bar{V}$, V are arbitrary $SL(2,C)$ matrix functions of the variables indicated, which leave the gauge potentials unchanged. Clearly, J defined by (2.8) is *gauge invariant* under $SL(2,C)$ gauge transformations. Under the transformation (2.11), J transforms as

$$J \to \bar{V}(\bar{y},\bar{z})JV^{-1}(y,z)\,. \tag{2.12}$$

Note that the gauge potentials A_μ can be obtained from J by factoring J as in Eq. (2.8). The resulting A_μ are not unique but related to each other by a complex gauge transformation. Furthermore, J' and J related by $J' = \bar{V}(\bar{y},\bar{z})JV(y,z)$, for arbitrary V and $\bar{V}$ with $\det\bar{V} = \det V = 1$, have gauge equivalent potentials.

For the construction of instanton solutions, it is customary to require the gauge potentials A_μ^a to be real and the gauge transformation G to be unitary in the *real space.* In this case we require $D \doteq (D^\dagger)^{-1}$ and $V \doteq (\bar{V}^\dagger)^{-1}$ (the symbol $\doteq$ is used for equations valid only for real values of x_1, x_2, x_3, x_4) which implies $J = D\bar{D}^{-1} \doteq DD^\dagger$ = a positive-definite Hermitian matrix. For the monopole problem it appears to be necessary to allow complex potentials and gauge transformations even in real space. This leads to the following question: Given D and $\bar{D}$, or equivalently J, when is it possible to choose a gauge so that the gauge fields are real? To this end we begin by assuming that there is an $SL(2,C)$ gauge transformation G such that the transformed potential satisfies $A_y'^{\dagger} \doteq -A_{\bar{y}}'$; then

$$A_y'^{\dagger}+A_{\bar{y}}' \doteq [(DG)^{-1}(DG)_{,y}]^{\dagger}+(\bar{D}G)^{-1}(\bar{D}G)_{,\bar{y}}$$
$$\doteq(\bar{D}G)^{-1}(\bar{D}GG^{\dagger}D^{\dagger})_{,\bar{y}}(DG)^{\dagger-1}$$
$$=0.$$

Similarly,

$$A_z'^{\dagger}+A_{\bar{z}}' \doteq (\bar{D}G)^{-1}(\bar{D}GG^{\dagger}D^{\dagger})_{,\bar{z}}(DG)^{\dagger-1}=0.$$

Thus

$$\bar{D}GG^{\dagger}D^{\dagger} \doteq V(y,z). \tag{2.13}$$

Note that (2.13) implies $\det V=1$. Then $J=D\bar{D}^{-1}$ $=DG(\bar{D}G)^{-1} \doteq DGG^{\dagger}D^{\dagger}V^{-1}(y,z)$, i.e.,

$$JV(y,z) \doteq DGG^{\dagger}D^{\dagger} \equiv D'D'^{\dagger}$$
$$=\text{positive-definite Hermitian matrix}. \tag{2.14}$$

On the other hand, if there is an SL(2, C) matrix $V(y,z)$, functions of y and z only, so that JV is a positive-definite Hermitian matrix, then we *can* factorize J as in Eq. (2.14) to obtain real gauge potentials. Therefore, we have the following.

Theorem 2.1: Given J, or D and $\bar{D}$, a necessary and sufficient condition that the gauge potentials are real in some gauge is the existence of an SL(2, C) matrix $V(y,z)$, depending on y and z only, such that

$$JV \doteq \text{positive-definite Hermitian matrix}. \tag{2.15}$$

Furthermore, if V exists then the gauge transformation is given by

$$GG^{\dagger}=\bar{D}^{-1}V(y,z)D^{\dagger-1}. \tag{2.16}$$

This condition is very general and, as will be seen below, becomes more definite for the monopole problem. Note that the gauge transformation G in (2.16) is determined up to an SU(2) gauge transformation.

So far we have not chosen any particular gauge. In the rest of the paper we work exclusively in Yang's R gauge,[14] which is defined by choosing the matrices D and $\bar{D}$ to be lower and upper triangular, respectively, i.e., by

$$D \equiv R \equiv \frac{1}{\sqrt{\phi}}\begin{pmatrix}1 & 0\\ \rho & \phi\end{pmatrix}, \tag{2.17a}$$

$$\bar{D} \equiv \bar{R} \equiv \frac{1}{\sqrt{\phi}}\begin{pmatrix}\phi & -\bar{\rho}\\ 0 & 1\end{pmatrix}, \tag{2.17b}$$

where ϕ, ρ, and $\bar{\rho}$ are independent complex functions of y, z, $\bar{y}$, and $\bar{z}$. From the definition (2.8) we have

$$J \equiv R\bar{R}^{-1}=\frac{1}{\phi}\begin{pmatrix}1 & \bar{\rho}\\ \rho & \phi^2+\rho\bar{\rho}\end{pmatrix}. \tag{2.18}$$

Substitution of (2.18) in (2.9) gives

$$(\partial_y\partial_{\bar{y}}+\partial_z\partial_{\bar{z}})\ln\phi+\frac{\rho_{,y}\bar{\rho}_{,\bar{y}}+\rho_{,z}\bar{\rho}_{,\bar{z}}}{\phi^2}=0, \tag{2.19a}$$

$$\left(\frac{\rho_{,y}}{\phi^2}\right)_{,\bar{y}}+\left(\frac{\rho_{,z}}{\phi^2}\right)_{,\bar{z}}=0, \tag{2.19b}$$

$$\left(\frac{\bar{\rho}_{,\bar{y}}}{\phi^2}\right)_{,y}+\left(\frac{\bar{\rho}_{,\bar{z}}}{\phi^2}\right)_{,z}=0, \tag{2.19c}$$

the self-duality equations in terms of ϕ, ρ, and $\bar{\rho}$. Since J is a gauge-invariant 2×2 matrix with unit determinant, we can *always* parametrize J as in Eq. (2.18), even though we arrived at this via Yang's R gauge. We choose a gauge when we factorize J to obtain D and $\bar{D}$ as in Eq. (2.17). The R-gauge potentials are given by

$$A_u=-\frac{1}{2\phi}\begin{pmatrix}\phi_{,u} & 0\\ 2\rho_{,u} & -\phi_{,u}\end{pmatrix}, \tag{2.20a}$$

$$A_{\bar{u}}=\frac{1}{2\phi}\begin{pmatrix}\phi_{,\bar{u}} & 2\bar{\rho}_{,\bar{u}}\\ 0 & -\phi_{,\bar{u}}\end{pmatrix}, \tag{2.20b}$$

where $u=y,z$. Observe that we have the freedom to add arbitrary functions of $\bar{y},\bar{z}$ (y,z) to ρ $(\bar{\rho})$. In the sequel, we always choose ρ $(\bar{\rho})$ such that ρ $(\bar{\rho})$ does not have any additive function of $\bar{y}$ and $\bar{z}$ (y and z) only.

For the definition and construction of the Atiyah-Ward *Ansatz*, we need the following results.

Lemma 2.1: Let $(\phi,\rho,\bar{\rho})$ be a solution of Eq. (2.19). Then $(\phi^I,\rho^I,\bar{\rho}^I)$ defined by

$$\phi^I=\frac{\phi}{\phi^2+\rho\bar{\rho}},\quad \rho^I=\frac{\bar{\rho}}{\phi^2+\rho\bar{\rho}},\quad \bar{\rho}^I=\frac{\rho}{\phi^2+\rho\bar{\rho}} \tag{2.21}$$

is also a solution of (2.19). Furthermore, the corresponding potentials are related by a gauge transformation.

Proof: This follows from

$$J^I \equiv \begin{pmatrix}\dfrac{1}{\phi^I} & \dfrac{\bar{\rho}^I}{\phi^I}\\ \dfrac{\rho^I}{\phi^I} & \dfrac{(\phi^I)^2+\rho^I\bar{\rho}^I}{\phi^I}\end{pmatrix}=(i\sigma_1)J(-i\sigma_1)$$

and the remarks following Eq. (2.12). The gauge transformation G^I is given by $G^I=R^{-1}(-i\sigma_1)R^I$.

Lemma 2.2: Let $(\phi, \rho, \bar{\rho})$ be a solution of Eq. (2.19). Then $(\phi^B, \rho^B, \bar{\rho}^B)$ defined by

$$\phi^B = \frac{1}{\phi}\ , \tag{2.22a}$$

$$\rho^B_{,y} = \frac{-\bar{\rho}_{,\bar{z}}}{\phi^2}\ ,\quad \rho^B_{,z} = \frac{\bar{\rho}_{,\bar{y}}}{\phi^2}\ , \tag{2.22b}$$

$$\bar{\rho}^B_{,\bar{y}} = \frac{\rho_{,z}}{\phi^2}\ ,\quad \bar{\rho}^B_{,\bar{z}} = -\frac{\rho_{,y}}{\phi^2} \tag{2.22c}$$

is also a solution of Eq. (2.19).

Proof: We write (2.22b) and (2.22c) as

$$\frac{\rho^B_{,y}}{(\phi^B)^2} = -\bar{\rho}_{,\bar{z}}\ ,\quad \frac{\bar{\rho}^B_{,z}}{(\phi^B)^2} = \bar{\rho}_{,\bar{y}},\ \text{etc.}$$

Then the result follows from $\bar{\rho}_{,\bar{y},\bar{z}} = \bar{\rho}_{,\bar{z},\bar{y}}$, etc.

Note that transformation I, $(\phi, \rho, \bar{\rho}) \xrightarrow{I} (\phi^I, \rho^I, \bar{\rho}^I)$ defined by Eq. (2.21), when operated twice gives an identify (i.e., $\phi^{II} = \phi$, $\rho^{II} = \rho$, etc.). Similarly, acting with the operator B, $(\phi, \rho, \bar{\rho}) \xrightarrow{B} (\phi^B, \rho^B, \bar{\rho}^B)$ as in Eq. (2.22), twice is also a trivial operation (i.e., $\phi^{BB} = \phi$, $\rho^{BB}_{,y} = \rho_{,y} \cdots$, etc.) in that it does not change the gauge potentials. Therefore, in order to use B more than once, we must interpose the I transformation between two B's. The transformation BI,

$$(\phi, \rho, \bar{\rho}) \xrightarrow{I} (\phi^I, \rho^I, \bar{\rho}^I) \xrightarrow{B} (\phi^{BI}, \rho^{BI}, \bar{\rho}^{BI})\ ,$$

i.e., I followed by B, is a Bäcklund transformation. It produces *locally* new solutions of the self-duality equations from old ones.

Lemma 2.3: A solution of Eq. (2.19) is given by

$$\rho_{,y} = \phi_{,\bar{z}},\quad \rho_{,z} = -\phi_{,\bar{y}}\ , \tag{2.23a}$$

$$\rho_{,\bar{y}} = +\phi_{,z},\quad \bar{\rho}_{,\bar{z}} = -\phi_{,y}\ , \tag{2.23b}$$

$$\phi_{,y\bar{y}} + \phi_{,z\bar{z}} = 0\ . \tag{2.23c}$$

Proof: Follows from the substitution of (2.23) in (2.19).

The solution given in this lemma, i.e. Eq. (2.23), is the well-known Corrigan-Fairlie-'t Hooft-Wilczek *Ansatz*. Note that Eq. (2.23c) is the integrability condition of Eqs. (2.23a) and (2.23b).

We now define the Atiyah-Ward *Ansatz* $\mathcal{A}_n$, $n = 1, 2, \ldots$. The $\mathcal{A}_1$ is defined to be the 't Hooft-Corrigan-Fairlie-Wilczek *Ansatz* given by Eq. (2.23). We define $\mathcal{A}_n$, $n \geq 2$, by

$$\mathcal{A}_1 \xrightarrow{BI} \mathcal{A}_2 \xrightarrow{BI} \mathcal{A}_3 \xrightarrow{BI} \cdots \xrightarrow{BI} \mathcal{A}_n. \tag{2.24}$$

Let us denote the functions $\phi, \rho, \bar{\rho}$ of the $\mathcal{A}_n$ *Ansatz* by $\phi_n, \rho_n, \bar{\rho}_n$. Therefore, $(\phi_1, \rho_1, \bar{\rho}_1)$ is a solution of Eq. (2.23), and $(\phi_n, \rho_n, \bar{\rho}_n)$ is given by

$$(\phi_1, \rho_1, \bar{\rho}_1) \xrightarrow{(BI)^{n-1}} (\phi_n, \rho_n, \bar{\rho}_n),\quad n \geq 2.$$

Corrigan, Fairlie, Goddard, and Yates[13] have given an explicit construction of $\phi_n, \rho_n, \bar{\rho}_n$, $n \geq 2$, in terms of a "spin-$(n-1)$ massless anti-self-dual linear field." Their solution begins by defining $(2n+1)$ functions Δ_k, $-n \leq k \leq n$, which satisfy the following equations:

$$\partial_y \Delta_k = -\partial_{\bar{z}} \Delta_{k+1}\ , \tag{2.25a}$$

$$\partial_z \Delta_k = \partial_{\bar{y}} \Delta_{k+1}\ . \tag{2.25b}$$

It follows from (2.25) that Δ_k satisfies the free field equation

$$(\partial_y \partial_{\bar{y}} + \partial_z \partial_{\bar{z}}) \Delta_k = 0\ . \tag{2.26}$$

Let us also define, for $n \geq 1$,

$$\mathfrak{D}_n = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-n+1} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-n+2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{vmatrix}\ , \tag{2.27}$$

where $|\cdots|$ denotes determinant. We now state the result of Corrigan *et al.*

Theorem 2.2: The $\mathcal{A}_n$ *Ansatz*, $n \geq 2$, is given by

$$\phi_n = \frac{\mathfrak{D}_n}{\mathfrak{D}_{n-1}}\ , \tag{2.28a}$$

$$\rho_n = \frac{(-1)^n}{\mathfrak{D}_{n-1}} \begin{vmatrix} \Delta_{-1} & \Delta_{-2} & \cdots & \Delta_{-n} \\ \Delta_0 & \Delta_{-1} & & \Delta_{-n+1} \\ \Delta_1 & \Delta_0 & & \Delta_{-n+2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \Delta_{n-2} & \Delta_{n-3} & \cdots & \Delta_{-1} \end{vmatrix}\ , \tag{2.28b}$$

$$\bar{\rho}_n = \frac{(-1)^{n-1}}{\mathfrak{D}_{n-1}} \begin{vmatrix} \Delta_1 & \Delta_0 & \cdots & \Delta_{-n+2} \\ \Delta_2 & \Delta_1 & & \Delta_{-n+3} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \Delta_n & \Delta_{n-1} & \cdots & \Delta_1 \end{vmatrix}\ . \tag{2.28c}$$

Proof of this theorem is given in Ref. 13. It should be noted that our definition of *Ansatz* $\mathcal{A}_n$ is related to the R_K *Ansatz* of Ref. 13 by $\mathcal{A}_n \equiv IR_{n+1}$. An extremely useful relation which can be proven by means of Jacobi's theorem on the determinant of adjugate matrices is the following:

$$\phi_n{}^2+\rho_n\bar{\rho}_n=\frac{\mathfrak{D}_{n+1}}{\mathfrak{D}_{n-1}}\ . \tag{2.29}$$

The potentials for the $\mathfrak{A}_n$ *Ansatz* are then given by the substitution of ϕ_n, ρ_n, and $\bar{\rho}_n$ in Eq. (2.20). It is easily seen that we can write these potentials in terms of ϕ_{n-1}, ρ_{n-1}, $\bar{\rho}_{n-1}$ by using the defining Eqs. (2.22) and (2.21) for B and I. In other words, we do not need to integrate the differential equations (2.22b) and (2.22c) in the final BI transformation. Therefore, the potentials of the $\mathfrak{A}_n$ *Ansatz* are given by the $(2n-1)$ functions Δ_k, $-(n-1)\leqslant k\leqslant(n-1)$, i.e., by a spin-$(n-1)$ anti-self-dual linear field [Eqs. (2.25) and (2.26)].

Note that in the R gauge the usual reality condition $D\doteq(D^\dagger)^{-1}$ becomes $\phi\doteq$ real and $\bar{\rho}\doteq\rho^*$. However, this reality condition is not preserved by the transformation BI. Therefore, the reality condition in the sequence (2.24) alternates between those of an SU(2) and SU(1,1) gauge theory. However, this is irrelevant for us, since we are allowing complex potentials and infer reality by applying theorem 2.1. Note that both SU(2) and SU(1,1) have the same complexification, SL(2, C).

III. THE 't HOOFT-POLYAKOV MONOPOLE SOLUTION IN THE LIMIT OF VANISHING HIGGS POTENTIAL

The 't Hooft-Polyakov monopole[2] arises as classical soliton, i.e., static, localized, nonsingular, finite-energy solution of SU(2) gauge theory with triplet of Higgs field in four-dimensional Minkowski space. The Lagrangian density for the model,[3,28] is

$$\mathcal{L}=-\tfrac{1}{4}F^{a\alpha\beta}F^a_{\alpha\beta}-\tfrac{1}{2}D_\alpha Q^aD^\alpha Q^a+\tfrac{1}{2}\mu^2Q^aQ^a-\tfrac{1}{4}\lambda(Q^aQ^a)^2\,, \tag{3.1}$$

where

$$F^a_{\alpha\beta}=\partial_\alpha A^a_\beta-\partial_\beta A^a_\alpha+g\epsilon_{abc}A^b_\alpha A^c_\beta \tag{3.2}$$

and

$$D_\alpha Q^a=\partial_\alpha Q^a+g\epsilon_{abc}A^b_\alpha Q^c\,. \tag{3.3}$$

The field equations are

$$D_\alpha F^{a\alpha\beta}=g\epsilon_{abc}Q^bD^\beta Q^c\,, \tag{3.4a}$$

$$D_\alpha D^\alpha Q^a=-\mu^2Q^a+\lambda(Q^bQ^b)Q^a\,. \tag{3.4b}$$

In this section we consider only static fields, i.e., all time derivatives are zero. This also means that we do not make any time-dependent gauge transformation. 't Hooft and Polyakov considered a solution of (3.4) of the form

$$A^a_i=\frac{1}{g}\epsilon_{aij}\frac{x_j}{r^2}\,[1-K(r)]\,, \tag{3.5a}$$

$$A^a_0=0\,, \tag{3.5b}$$

$$Q^a=\frac{1}{g}\frac{x_a}{r^2}H(r)\,, \tag{3.5c}$$

where

$$r^2=x_ix_i\,. \tag{3.6}$$

The energy of this solution, defined from the usual canonical energy-momentum tensor $T_{\alpha\beta}$, reduces to the form

$$E=\int d^3x[\tfrac{1}{2}B^a_iB^a_i+\tfrac{1}{2}D_iQ^aD_iQ^a-\tfrac{1}{2}\mu^2Q^aQ^a+\tfrac{1}{4}\lambda(Q^aQ^a)^2]\,, \tag{3.7}$$

where

$$B^a_i=\tfrac{1}{2}\epsilon_{ijk}F^a_{jk}\,. \tag{3.8}$$

The finiteness of energy requires for the 't Hooft-Polyakov solution

$$\frac{H}{r}\xrightarrow[r\to\infty]{}C=\frac{\mu g}{\sqrt{\lambda}}$$

or more generally

$$Q^2\equiv Q^aQ^a\xrightarrow[r\to\infty]{}\frac{C^2}{g^2}\,. \tag{3.9}$$

The Abelian electromagnetic field has been identified by 't Hooft as

$$\mathfrak{F}_{\alpha\beta}\equiv\frac{Q^a}{Q}F^a_{\alpha\beta}-\frac{1}{gQ^3}\epsilon_{abc}Q^aD_\alpha Q^bD_\beta Q^c \tag{3.10a}$$

$$\equiv\partial_\alpha(\hat{Q}^aA^a_\beta)-\partial_\beta(\hat{Q}^aA^a_\alpha)-\frac{1}{g}\epsilon_{abc}\hat{Q}^a\partial_\alpha\hat{Q}^b\partial_\beta\hat{Q}^c\,, \tag{3.10b}$$

where $\hat{Q}^a\equiv Q^a/Q\equiv Q^a/(Q^aQ^a)^{1/2}$. Note that $\mathfrak{F}_{\alpha\beta}$ is gauge invariant and reduces to the usual definition $\mathfrak{F}_{\alpha\beta}=\partial_\alpha W_\beta^3-\partial_\beta W_\alpha{}^3$, when $Q^a\sim\delta^a_3$. It then follows that the 't Hooft-Polyakov solution has magnetic charge q given by $4\pi q=4\pi/g$, and the electric charge is zero.

So far we have considered solutions with $A^a_0=0$. Julia and Zee[29] found a solution with nonzero A^a_0, which is a dyon, i.e., has both electric and magnetic charge. It was shown by Arafune, Freund, and Goebel[5] that the magnetic charge q, in units of $1/g$, is "quantized" and conserved from the topology of Higgs fields, and is not of dynamical origin. The electric charge, however, remains unquantized in the classical theory. In this paper, we do not consider solutions with electric charge, and we shall take $A^a_0=0$.

An exact solution was obtained[4] by considering the limit μ, $\lambda\to0$ with C fixed,

$$A^a_i=\frac{1}{g}\epsilon_{aij}\frac{x_j}{r^2}\left[1-\frac{Cr}{\sinh(Cr)}\right], \tag{3.11a}$$

$$Q^a = \frac{1}{g}\frac{x_a}{r^2}\left[1-(Cr)\coth(Cr)\right]. \tag{3.11b}$$

Note that in this limit the boundary condition (3.9) does not follow from the finiteness of the energy; in fact, the solution is unstable against changes in C. But C is the only mass parameter and sets the length scale, and we assume (3.9) to hold.

It was shown by Bogomol'nyi[4] that in this limit, with Eq. (3.9) and $A_0^a = 0$, the energy is bounded below by

$$E \geq \frac{4\pi C}{g^2} n, \tag{3.12}$$

where $n = 0, 1, 2, \ldots,$ is the magnetic charge. Moreover, the equality in (3.12) is satisfied if and only if the Bogomol'nyi equations

$$B_i^a = D_i Q^a \tag{3.13}$$

are satisfied. Note that these equations, together with the Bianchi identity $D_k B_k^a = 0$ for static gauge fields, implies the field equation (3.4), for static fields and λ, $\mu = 0$. Bogomol'nyi obtained the solution (3.4) by solving Eq. (3.12) and, therefore the energy is given by $4\pi C/g^2$. Note that the existence of solutions of the Bogomol'nyi equation (3.13) is a "dynamical" problem. However, if there are solutions to (3.13) with appropriate boundary conditions, then they are topologically and energetically stable. The small fluctuation equation about a solution of (3.13) does, however, have zero modes, i.e., solutions with zero eigenvalue. These zero modes correspond to the parameters of the solution, and the general solution with magnetic charge, n, has $(4n-1)$ parameters.[30]

When the Bogomol'nyi equations (3.13) are satisfied then the energy is given by

$$E = \tfrac{1}{2}\int d^3x (B_i^a B_i^a + D_i Q^a D_i Q^a) = \int d^3x (B_i^a D_i Q^a).$$

Now, $B_i^a D_i Q^a = \partial_i (B_i^a Q^a) - Q^a D_i B_i^a = \partial_i (Q^a \partial_i Q^a) = \frac{1}{2}\partial_i \partial_i (Q^a Q^a)$, where we have used the Bianchi identity. Thus for a solution of the Bogomol'nyi equation, the energy is given in terms of the Higgs field by

$$E = \tfrac{1}{2}\int d^3x (\partial_i \partial_i Q^2). \tag{3.14}$$

So far we considered only "positive" magnetic charge. For negative magnetic charge, (3.12) holds with $q \to -q$ and (3.13) becomes $B_i^a = -D_i Q^a$. The monopole solution of charge $-q$ can be obtained from a solution with charge $q > 0$ by $Q^a \to -Q^a$ and $A_k^a \to A_k^a$.

IV. MONOPOLES AS SELF-DUAL FIELDS

In the last two sections we discussed self-dual gauge fields (in R^4) and monopoles separately. In this section we connect these two and discuss some general results of the application of the R-gauge technique for the monopole problem.

We begin with the simple observation that the self-duality equations (2.3) in Euclidean space

$$\begin{aligned}\tfrac{1}{2}\epsilon_{ijk4}F_{ij} \equiv \tfrac{1}{2}\epsilon_{ijk}F_{ij} &= F_{k4} \\ &= \partial_k A_4 + [A_k, A_4] - \partial_4 A_k,\end{aligned}$$

become identical with the Bogomol'nyi equations (3.13) when $\partial_4 A_\mu = 0$ and the Higgs field Q^a is identified with A_4^a. Since x_4 can be thought of as the Euclidean "time," we shall use the work "static" in this context to mean "independent of the Euclidean time x_4." In what follows we shall restrict the gauge transformations to static both in Euclidean and Minkowski space. Therefore, with the above restriction, the monopole problem becomes identical with the self-dual Yang-Mills fields, provided we require suitable boundary conditions.

We can therefore obtain a (multi)monopole solution as a solution of the self-duality equations (2.3) which satisfies the following.

(M1) The potentials are static: $\partial_4 A_\mu = 0$.

(M2) In some gauge the potentials A_μ^a are real and smooth, i.e., A_μ^a and its derivatives are nonsingular.

(M3) The gauge-invariant quantity, the square of the Higgs field $Q^2 \equiv Q^a Q^a \equiv A_4^a A_4^a$ has the asymptotic behavior

$$Q^2 \underset{r\to\infty}{\sim} \frac{C^2}{g^2} - \frac{2nC}{g^2}\frac{1}{r} + O(r^{-2}), \tag{4.1}$$

and $n = 0, 1, 2, \ldots$.

Note that in (M1) and (M3) we have assumed that we consider static gauge transformations only. We do, however, include complex gauge transformations. It is immediately clear from Eq. (3.14) that (M2) and (M3) imply the finiteness of energy and that the magnetic charge (in units of $1/g$) or the topological charge is n. To see this, we have, using Gauss's theorem,

$$\begin{aligned}E = \tfrac{1}{2}\int d^3x (\partial_i \partial_i Q^2) &= \lim_{r\to\infty} \tfrac{1}{2}\int r^2 d\Omega \left(\frac{\partial}{\partial r} Q^2\right) \\ &= \frac{4\pi C}{g^2} n.\end{aligned} \tag{4.2}$$

We now turn to the construction of the self-dual gauge field in Yang's R gauge satisfying (M1–3). To satisfy (M1) we begin by noting that the R-gauge potentials (2.20) are given by the ratios such as $\phi_{,u}/\phi$, $\rho_{,u}/\phi$, $\bar{\rho}_{,\bar{u}}/\phi$. This suggests that the x_4

dependence of $(\phi, \rho, \bar{\rho})$ be factorizable with the same factor, say $f(x_4)$. Now, $\partial_4 A_{\bar{y}}=0$ implies

$$\frac{\partial_4 f(x_4)}{f(x_4)} = \text{constant}.$$

Now, this constant has the dimension of mass. This problem has only one-dimensional parameter C. We therefore take[31] $f(x_4)=e^{iCx_4}$. Therefore, to satisfy (M1) we require

$$\phi = e^{iCx_4}\phi_s\,, \tag{4.3a}$$

$$\rho = e^{iCx_4}\rho_s\,, \tag{4.3b}$$

$$\bar{\rho} = e^{iCx_4}\bar{\rho}_s\,, \tag{4.3c}$$

where $\phi_s, \rho_s, \bar{\rho}_s$ are functions of x_1, x_2, and x_3 only. Note that we can consistently require (4.3) for all $\mathfrak{A}_n$ *Ansätze.*

The implementation of (M2) and (M3) is considerably simplified by the following superposition formula.[16]

Theorem 4.1: Suppose that we require (4.3) for $(\phi_k, \rho_k, \bar{\rho}_k)$ corresponding to the *Ansatz* $\mathfrak{A}_k$, $k=1, 2, \ldots$. Then the Higgs field for the nth *Ansatz* satisfies

$$Q_n{}^2 \equiv (A_4^a A_4^a)_n = \frac{C^2}{g^2} - \frac{1}{g^2}\sum_{k=1}^{n} \nabla^2 \ln\phi_k\,. \tag{4.4}$$

Proof: Using $A_4 = (-i/\sqrt{2}\,)(A_z - A_{\bar{z}})$ and Eq. (2.20) we have

$$Q^2 = \frac{-2}{g^2}\,\mathrm{tr} A_4 A_4 = \frac{1}{g^2}\left[\frac{(\partial_3\phi)^2 + 2\rho_{,z}\bar{\rho}_{,\bar{z}}}{\phi^2}\right]. \tag{4.5}$$

We now investigate the effect of the transformations I and B on (4.5). From lemma (2.1), I is the gauge transformation given by

$$G^I = -iR^{-1}\sigma_1 R^I = \frac{-i}{(\phi^2+\rho\bar{\rho})^{1/2}}\begin{pmatrix}\bar{\rho} & \phi \\ \phi & -\rho\end{pmatrix}. \tag{4.6}$$

If (4.3) is satisfied, then G^I is static and, therefore Q^2 is invariant under I. Now from the definition $(\phi_{k+1}, \rho_{k+1}, \bar{\rho}_{k+1}) \equiv BI(\phi_k, \rho_k, \bar{\rho}_k) \equiv (\phi_k^{BI}, \rho_k^{BI}, \bar{\rho}_k^{BI})$, we have

$$\begin{aligned}(Q_{k+1})^2 &= \frac{1}{g^2}\left[\frac{(\partial_3\phi_k^{BI})^2 + 2\rho_{k,z}^{BI}\bar{\rho}_{k,\bar{z}}^{BI}}{(\phi_{k+1})^2}\right] = \frac{1}{g^2}\left[\frac{(\partial_3\phi_k^I)^2 - 2\rho_{k,y}^I\bar{\rho}_{k,\bar{y}}^I}{(\phi_k^I)^2}\right] \\ &= \frac{1}{g^2}\left[\frac{(\partial_3\phi_k^I)^2 + 2\rho_{k,z}^I\bar{\rho}_{k,\bar{z}}^I}{(\phi_k^I)^2}\right] + \frac{2}{g^2}(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\ln\phi_k^I \\ &= (Q_k^I)^2 - \frac{1}{g^2}\nabla^2\ln\phi_k = (Q_k)^2 - \frac{1}{g^2}\nabla^2\ln\phi_k\,,\end{aligned} \tag{4.7}$$

where we have used Eqs. (2.22), (2.19), (4.3), and the invariance of Q^2 under I, successively. Now for the $\mathfrak{A}_1$ *Ansatz*,

$$\begin{aligned}Q_1{}^2 &= \frac{1}{g^2}\left(\frac{(\partial_3\phi_1)^2 + 2\rho_{1,z}\bar{\rho}_{1,\bar{z}}}{\phi^2}\right) \\ &= \frac{1}{g^2}\left(\frac{2\phi_{1,z}\phi_{1,\bar{z}} + 2\rho_{1,z}\bar{\rho}_{1,\bar{z}}}{\phi_1{}^2} + C^2\right)\end{aligned}$$

using $\phi_{1,z} = (1/\sqrt{2}\,)(\partial_3 - C)\phi_1$, $\phi_{1,\bar{z}} = (1/\sqrt{2}\,)(\partial_3 + C)\phi_1$ from Eq. (4.3). Now using Eqs. (2.23) and (2.19) we get

$$\begin{aligned}Q_1{}^2 &= -\frac{2}{g^2}(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\ln\phi_1 + \frac{C^2}{g^2} \\ &= \frac{C^2}{g^2} - \frac{1}{g^2}\nabla^2\ln\phi_1\,.\end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8) we immediately get (4.4).

This theorem leads to some immediate results. First, following Manton[9] and Ward[25] we choose

$$\phi_1 = e^{iCx_4}\Lambda_0\,, \tag{4.9}$$

where Λ_0 is a function of x_1, x_2, and x_3. Then from Eq. (2.23c), Λ_0 satisfies the Helmholtz equation

$$\nabla^2\Lambda_0 = C^2\Lambda_0\,. \tag{4.10}$$

Therefore, if we choose the class of solutions such that the asymptotic behavior of Λ_0 is given by

$$\Lambda_0 \underset{r\to\infty}{\sim} \frac{e^{Cr}}{r}\,, \tag{4.11}$$

we therefore have

$$\ln\phi_1 \underset{r\to\infty}{\sim} (Cr + iCx_4) + O(\ln r)\,.$$

Now this asymptotic behavior for $r\to\infty$ is preserved by the transformation BI given by Eqs. (2.21) and (2.22). Thus,

$$\ln\phi_k \underset{r\to\infty}{\sim} (Cr + iCx_4)\,.$$

Then from Eq. (4.4), we immediately see that (4.9) and (4.11) imply for the $\mathfrak{A}_n$ *Ansatz*

$$Q_n{}^2 \underset{r\to\infty}{\sim} \frac{C^2}{g^2} - \frac{2nC}{g^2}\frac{1}{r} + O(r^2)\,. \tag{4.12}$$

Therefore, under these assumptions, i.e., Eqs. (4.3), (4.9), and (4.11), the $\mathcal{A}_n$ *Ansatz* gives a monopole solution of topological charge n, *provided* the condition (M2) can be satisfied.

From Eqs. (3.14) and (4.4), the energy density $\mathcal{E}_n$ of the $\mathcal{A}_n$ *Ansatz* is given by

$$\mathcal{E}_n = -\frac{1}{g^2}\nabla^2\nabla^2 \ln\phi_1\phi_2\cdots\phi_n \tag{4.13}$$

$$= -\frac{1}{g^2}\nabla^2\nabla^2 \ln \mathfrak{D}_n , \tag{4.14}$$

where we have used Eq. (2.28a) to get Eq. (4.14). *Therefore, we require that $\mathfrak{D}_n$ is never zero and has no singularities except for $r\to\infty$ for the energy density to be nonsingular.* Similarly from Eqs. (2.28) and (2.20), the gauge potentials are nonsingular if $\mathfrak{D}_{n-1}$ is nonvanishing. However, the singularities in the gauge potentials are acceptable as long as they can be removed by a gauge transformation.

V. AXIALLY SYMMETRIC MULTIMONOPOLES

In this section we discuss axially symmetric monopole solutions. It is known[6] that the only solution with spherical symmetry, i.e., any rotation is equivalent to a gauge transformation, is the one-monopole solution. It is therefore natural to look for multimonopole solutions which are axially symmetric. The minimal six-function axially symmetric *Ansatz* has been discussed by Jang, Park, and Wali[26] and by Manton,[9] and numerically analyzed by Rebbi and Rossi and by Adler and Piran.[26] The recent two-monopole solution of Ward is axially symmetric.

As we have seen in Sec. IV, we have for the $\mathcal{A}_1$ *Ansatz*

$$\phi_1 = e^{iCx_4}\Lambda_0 , \tag{5.1}$$

where Λ_0 is a function of x_1, x_2, and x_3 only and satisfies the Helmholtz equation

$$\nabla^2\Lambda_0 = C^2\Lambda_0 . \tag{5.2}$$

Moreover, to obtain a multimonopole of topological charge n we must use the $\mathcal{A}_n$ *Ansatz* and have the asymptotic behavior $\Lambda_0 \sim_{r\to\infty} e^{Cr}/r$. With this and cylindrical symmetry in mind, imitating Ward, we choose

$$\Lambda_0 = \sum_{i=1}^{n} \alpha_i \frac{\sinh CR_i}{R_i} , \tag{5.3}$$

where

$$R_i^{\,2} = x_1^{\,2} + x_2^{\,2} + (x_3 - c_i)^2 \tag{5.4}$$

and α_i, c_i can be complex, subject to the restriction that Λ_0 is real. It is convenient to define

$$\sqrt{2}\, y = x_1 + ix_2 \equiv \xi \doteq l e^{i\theta} , \tag{5.5a}$$

$$\sqrt{2}\, \bar{y} = x_1 - ix_2 \equiv \bar{\xi} \doteq l e^{-i\theta} , \tag{5.5b}$$

i.e., $l^2 = x_1^{\,2} + x_2^{\,2}$, etc.

We can now integrate Eqs. (2.23a) and (2.23b) for ρ_1 and $\bar{\rho}_1$ with ϕ_1 given by Eqs. (5.1) and (5.3). To this end we define a real function of l and x_3 by

$$\Lambda_0 \equiv \bar{y}^{-1}\partial_y\Lambda_1 \equiv y^{-1}\partial_{\bar{y}}\Lambda_1 \equiv l^{-1}\partial_l\Lambda_1 . \tag{5.6}$$

Λ_1 is then given by the indefinite integral

$$\Lambda_1 = \int l\, dl\, \Lambda_0 = \sum_{i=1}^{n} \alpha_i \int R_i dR_i \frac{\sinh CR_i}{R_i} . \tag{5.7}$$

Note that in defining Λ_1 we do not include any integration "constants," which may be any arbitrary functions of x_3. We then have the following.

Lemma 5.1: The functions

$$\phi_1 = e^{iCx_4}\Lambda_0 = e^{iCx_4}\bar{y}^{-1}\partial_y\Lambda_1 = e^{iCx_4}y^{-1}\partial_{\bar{y}}\Lambda_1 , \tag{5.8a}$$

$$\rho_1 = e^{iCx_4}\bar{\xi}^{-1}(\partial_3 + C)\Lambda_1 , \tag{5.8b}$$

$$\bar{\rho}_1 = e^{iCx_4}\xi^{-1}(\partial_3 - C)\Lambda_1 \tag{5.8c}$$

solve the $\mathcal{A}_1$ *Ansatz* Eqs. (2.23).

Proof: We can write

$$\rho_1 = (\bar{y}^{-1}\Lambda_1 e^{C(\bar{z}-z)/\sqrt{2}})_{,\bar{z}} , \tag{5.9a}$$

$$\bar{\rho}_1 = (y^{-1}\Lambda_1 e^{C(\bar{z}-z)/\sqrt{2}})_{,z} , \tag{5.9b}$$

$$\phi_1 = (\bar{y}^{-1}\Lambda_1 e^{C(\bar{z}-z)/\sqrt{2}})_{,y} \tag{5.9c}$$

$$= (\bar{y}^{-1}\Lambda_1 e^{C(\bar{z}-z)/\sqrt{2}})_{,\bar{y}} . \tag{5.9d}$$

The equations $\rho_{1,y} = \phi_{1,\bar{z}}$ and $\bar{\rho}_{1,\bar{y}} = \phi_{1,z}$ are obviously satisfied. The remaining equations of the $\mathcal{A}_1$ *Ansatz* $\rho_{1,z} = -\phi_{1,\bar{y}}$, $\bar{\rho}_{1,\bar{z}} = -\phi_{1,y}$ can be verified by explicit computation using Eq. (5.3).

To construct the $\mathcal{A}_n$ *Ansatz* for $n \geq 2$, we need to solve the equations (2.25) for Δ_k, $-n \leq k \leq n$. To do this, we define for $k \geq 2$ the real functions Λ_k by

$$\Lambda_k \equiv \bar{y}^{-1}\partial_y\Lambda_{k+1} \equiv y^{-1}\partial_{\bar{y}}\Lambda_{k+1} \equiv l^{-1}\partial_l\Lambda_{k+1} . \tag{5.10}$$

Therefore, as in Eq. (5.7), Λ_{k+1} is given by a $(k+1)$-fold indefinite integral

$$\Lambda_{k+1} = \int l\, dl \int \cdots \int l\, dl\, \Lambda_0 \tag{5.11a}$$

$$= \sum \alpha_i \int R_i dR_i \int \cdots \int R_i dR_i \times \frac{\sinh CR_i}{R_i} . \tag{5.11b}$$

Again as in Eq. (5.7), we do not include any constants of integration.

Lemma 5.2: The functions Δ_k, $1 \leq k \leq n$, defined by

$$\Delta_0 = e^{iCx_4}\Lambda_0 , \tag{5.12a}$$

$$\Delta_{-k}=(-1)^k e^{iCx_4}\bar{\xi}^{-k}(\partial_3+C)^k\Lambda_k\,, \qquad (5.12b)$$

$$\Delta_k=e^{iCx_4}\xi^{-k}(\partial_3-C)^k\Lambda_k\,, \qquad (5.12c)$$

are a solution of Eq. (2.25).

Proof: This proof proceeds by induction. First, since $\Delta_0=\phi_1$, $\Delta_1=\bar{\rho}_1$, $\Delta_{-1}=-\rho_1$, the case $n=1$ is just lemma 5.1. For $k\geq 1$, we have

$$\partial_z\Delta_k=e^{iCx_4}\xi^{-k}\frac{(\partial_3-C)^{k+1}}{\sqrt{2}}\Lambda_k\,, \qquad (5.13a)$$

$$\partial_{\bar{y}}\Delta_{k+1}=e^{iCx_4}\xi^{-k}(\partial_3-C)^{k+1}(\xi^{-1}\partial_y\Lambda_{k+1})\,, \qquad (5.13b)$$

$$\partial_y\Delta_k=e^{iCx_4}\xi^{-k-1}(\partial_3-C)^k(\sqrt{2}y\partial_y\Lambda_k-\sqrt{2}k\Lambda_k)\,, \qquad (5.13c)$$

$$\partial_{\bar{z}}\Delta_{k+1}=e^{iCx_4}\xi^{-k-1}(\partial_3-C)^k\frac{(\partial_3{}^2-C^2)}{\sqrt{2}}\Lambda_{k+1}\,. \qquad (5.13d)$$

Then Eq. (2.25b) follows immediately from Eqs. (5.10), (5.13a), and (5.13b). To show that Eq. (2.25a) is satisfied, from Eqs. (5.13c) and (5.13d), $2y\partial_y\Lambda_k=l\partial_l\Lambda_k$, it is sufficient to prove, by induction, that

$$(\partial_3{}^2-C^2)\Lambda_{k+1}+l\partial_l\Lambda_k-2k\Lambda_k=0,\quad k\geq 0\,. \qquad (5.14)$$

The case $k=0$ follows from direct calculation using Eq. (5.3). The induction is then completed by taking $\int l\,dl$ of Eq. (5.14) and using the fact that in the definition of Λ_k, $k\geq 1$, we do not include integration constants. This completes the proof of Eq. (5.12c). The proof of Eq. (5.12b) is similar.

Note that functions Δ_k defined by (5.12) satisfy the Helmholtz equation

$$\partial_j\partial_j\Delta_k=C^2\Delta_k,\quad k=0,\pm1,\pm2,\ldots\,. \qquad (5.15)$$

We can then construct $(\phi_n,\rho_n,\bar{\rho}_n)$ using Eqs. (2.28) and (5.12).

Lemma 5.3: The solution constructed above has the form

$$\rho_n=P_n\bar{\xi}^{-n}e^{iCx_4}\,, \qquad (5.16a)$$

$$\phi_n=\Phi_n l^{-n}e^{iCx_4}\,, \qquad (5.16b)$$

$$\bar{\rho}_n=-\bar{P}_n\xi^{-n}e^{iCx_4}\,, \qquad (5.16c)$$

where P_n, Φ_n, $\bar{P}_n$ are *real* functions of l and x_3 only. Note that P_n, Φ_n, and $\bar{P}_n$ are in fact defined by Eq. (5.16).

Proof: Observe that, from Eq. (5.12),

$$\Delta_k=e^{iCx_4}\times(\text{a real function of } l \text{ and } x_3)\times\xi^{-k}$$

$$=e^{iCx_4}\times(\text{a real function of } l \text{ and } x_3)\times\bar{\xi}^{k}\,.$$

Similarly,

$$\Delta_{-k}=e^{iCx_4}\times(\text{a real function of } l \text{ and } x_3)\times\bar{\xi}^{-k}\,,\ \text{etc.}$$

Now, $\mathfrak{D}_n$ has the form

$$\begin{bmatrix} \bar{\xi}^0 & \bar{\xi}^{-1} & \cdots & \bar{\xi}^{-n+1} \\ \bar{\xi}^1 & \bar{\xi}^0 & \cdots & \bar{\xi}^{-n+2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \bar{\xi}^{n-1} & \bar{\xi}^{n-2} & & \bar{\xi}^0 \end{bmatrix}$$

with every term multiplied by a factor, which is a product e^{iCx_4} and a real function of l and x_3. Now if we factor $\bar{\xi}$ from the second row, $\bar{\xi}^2$ from the third row, etc., then the $\bar{\xi}$ dependence of the rows becomes the same as that of the first row. Now, using the fact that every term in the determinant has one and only one term from each column, we immediately see that $\mathfrak{D}_n=e^{inCx_4}\times$ (a real function of l and x_3). This verifies the form (5.16b). The others follow in the same manner.

We have at this stage constructed a static solution of self-duality equations (2.3) corresponding to the $\mathfrak{A}_n$ *Ansatz*, in terms of the single real function Λ_0 defined by Eq. (5.3). Now, to complete the construction of multimonopole solutions, we need to satisfy the conditions (M2) and (M3). From the discussion following theorem 4.1, (M3) is clearly satisfied. We state this formally as theorem 5.1.

Theorem 5.1: For the solution $\phi_n,\rho_n,\bar{\rho}_n$ given by Eq. (5.16), the Higgs field satisfies

$$Q_n{}^2\equiv(A_4^aA_4^a)\to\frac{C^2}{g^2}-\frac{2nC}{g^2r}+O(r^{-2})\,. \qquad (5.17)$$

From Eqs. (5.3), (5.7), and (5.12) we see that

$$\Delta_k\underset{r\to\infty}{\sim}(\text{powers of } x_i)e^{Cr+iCx_4}$$

for all k. Then from Eqs. (2.28), we see that $\ln\phi_k\underset{r\to\infty}{\sim}(Cr+iCx_4)$ and then Eq. (5.17) follows from Eq. (4.4).

We now turn towards the existence of a gauge transformation which makes the solution real, i.e., $A_\mu'^\dagger=-A_\mu'$. From theorem 2.1, we need to find a matrix $V(y,z)$, $\det V=1$ such that JV is a positive-definite Hermitian matrix. In general, it is rather difficult to get such a matrix V. However, the specific form of the $(\phi_n,\rho_n,\bar{\rho}_n)$ given by Eq. (5.16) simplifies the problem considerably. From Eq. (2.18) and (5.16), we have

$$J_n=\begin{pmatrix} \dfrac{l^ne^{-iCx_4}}{\Phi_n} & \dfrac{-l^n\bar{P}_n}{\xi^n\Phi_n} \\ \dfrac{l^nP_n}{\bar{\xi}^n\Phi_n} & \dfrac{e^{iCx_4}(\Phi_n{}^2-P_n\bar{P}_n)}{l^n\Phi_n} \end{pmatrix}. \qquad (5.18)$$

Now, since the necessary gauge transformation for the one-monopole solution is known, we can use Eq. (2.13) to determine the matrix V_1. The matrix V_1 for the one-monopole solution and the form of J_n given in Eq. (5.18) strongly suggests that we take V to be (note that $\xi=\sqrt{2}y$)

$$V_n(y,z)=\begin{pmatrix} 0 & -(\gamma\xi^n)^{-1} \\ \gamma\xi^n & 0 \end{pmatrix}. \tag{5.19}$$

Then

$$J_nV_n=\begin{pmatrix} -\gamma\dfrac{l^n\bar{P}_n}{\Phi_n} & \dfrac{l^n e^{-iCx_4}}{\gamma\xi^n\Phi_n} \\ \dfrac{-\gamma\xi^n e^{iCx_4}(\Phi_n{}^2-P_n\bar{P}_n)}{l^n\Phi_n} & -\dfrac{P_n}{\gamma l^n\Phi_n} \end{pmatrix}.$$

Therefore, J_nV_n is Hermitian if γ is real and

$$\left(\frac{l^n}{\gamma\xi^n\Phi_n}\right)^*=\left(\frac{(P_n\bar{P}_n-\Phi_n{}^2)}{l^n\Phi_n}\gamma\xi^n\right),$$

i.e., $\gamma^2(P_n\bar{P}_n-\Phi^2)=1$. Now, if this condition is satisfied then P_n and $\bar{P}_n$ have the same sign, and therefore we can choose the sign of γ so that the diagonal elements are positive, so that J_nV_n is a positive-definite Hermitian matrix. Then we can find $G_nG_n^\dagger$ from Eq. (2.16). It will be shown in the Appendix that the V_n given by (5.19) is the only possible choice compatible with all our requirements. We then have the following theorem.

Theorem 5.2: The gauge potentials given by Eq. (5.16) can be made real by a gauge transformation if and only if

$$\gamma^2(P_n\bar{P}_n-\Phi_n{}^2)=1, \tag{5.20}$$

where γ is a real constant. Moreover, if Eq. (5.20) is satisfied, the necessary gauge transformation G is given by

$$G_nG_n^\dagger=\gamma\begin{pmatrix} -\bar{P}_n & \Phi_n e^{-in\theta} \\ \Phi_n e^{in\theta} & -P_n \end{pmatrix}, \tag{5.21}$$

where θ is defined by Eq. (5.5) and γ is chosen such that $-\gamma\bar{P}_n$ is positive. If Λ_0 is multiplied by a constant factor (real nonzero) then Δ_k and therefore $(\phi_n,\rho_n,\bar{\rho}_n)$ are multiplied by the same factor, which does not affect the potentials. Using this factor, we can always choose $\gamma=-1$ in Eqs. (5.20) and (5.21), and $P_n,\bar{P}_n$ then become real and positive. So, we then have

$$P_n\bar{P}_n-\Phi_n{}^2=1 \tag{5.22}$$

and

$$G_nG_n^\dagger=\begin{pmatrix} \bar{P}_n & -\Phi_n e^{-in\theta} \\ -\Phi_n e^{in\theta} & P_n \end{pmatrix}. \tag{5.23}$$

Note that $\det(G_nG_n^\dagger)=P_n\bar{P}_n-\Phi_n{}^2=1$.

It is easily seen that (5.23) determines G_n up to an SU(2), i.e., real, gauge transformation. Therefore, to solve G explicitly we must choose a gauge. A possible choice is to require G to be lower triangular. This can always be done and is a "natural" choice since we are working in Yang's R gauge. Then we can solve Eq. (5.23) for G to get

$$G_n^R=\frac{1}{\sqrt{\bar{P}_n}}\begin{pmatrix} \bar{P}_n & 0 \\ -\Phi_n e^{in\theta} & 1 \end{pmatrix}. \tag{5.24}$$

The transformed gauge potentials are given by

$$(A_y')_n=\frac{e^{-i\theta}}{2\sqrt{2}}\begin{pmatrix} -M_n & 0 \\ 2N_n e^{in\theta} & M_n \end{pmatrix}, \tag{5.25a}$$

$$(A_{\bar{y}}')_n=\frac{e^{i\theta}}{2\sqrt{2}}\begin{pmatrix} M_n & 2N_n e^{in\theta} \\ 0 & -M_n \end{pmatrix}, \tag{5.25b}$$

$$(A_3')_n=\frac{1}{2}\begin{pmatrix} 0 & \dfrac{V_n}{\Phi_n}e^{-in\theta} \\ -\dfrac{V_n}{\Phi_n}e^{in\theta} & 0 \end{pmatrix}, \tag{5.25c}$$

$$(A'_4)_n=\frac{1}{2i}\begin{pmatrix} W_n-V_n & V_n e^{-in\theta} \\ V_n e^{in\theta} & V_n-W_n \end{pmatrix}, \tag{5.25d}$$

where

$$M_n=\partial_l\left(\frac{\Phi_n}{l^n\bar{P}_n}\right), \tag{5.26a}$$

$$N_n=\frac{\partial_l\bar{P}_n}{\Phi_n\bar{P}_n}, \tag{5.26b}$$

$$V_n=\frac{(\partial_3+C)\bar{P}_n}{\bar{P}_n}, \tag{5.26c}$$

and

$$W_n=\frac{\partial_3\Phi_n}{\Phi_n}=\partial_3[\ln\Phi_n]. \tag{5.26d}$$

We have used Eq. (5.22) to derive (5.25). Since M_n, N_n, V_n, and W_n are real, the potentials A_μ^a are explicitly real.

It can easily be verified that the one-monopole solution does not take its simplest form (3.11) in the "R gauge" chosen in Eq. (5.24). The one-monopole solution is given by Eq. (3.11) when we choose Yang's K gauge i.e., we take G to be Hermitian. Again, we can solve (5.23) to get

$$G_n^K = \frac{1}{(P_n + \bar{P}_n + 2)^{1/2}} \begin{pmatrix} \bar{P}_n + 1 & -\Phi_n e^{-in\theta} \\ -\Phi_n e^{in\theta} & P_n + 1 \end{pmatrix}. \tag{5.27}$$

The explicitly anti-Hermitian form of A'_μ in this Hermitian gauge can be calculated using Eq. (5.22). The resulting potentials in the Hermitian gauge has a somewhat more complicated form than given in Eqs. (5.25) and (5.26). However, this gives a greater possibility of cancellations, and the multimonopole solutions may also become "simple" in this gauge.

Besides the gauge transformation, we still have yet to discuss the singularity problem, i.e., as mentioned in Sec. IV. That the determinants $\mathfrak{D}_n$ and $\mathfrak{D}_{n-1}$ do not vanish, to ensure that the potentials and the energy density are nonsingular. Then the gauge transformations given by Eqs. (5.24) or (5.27) obviously do not introduce any singularities in the potentials.

It is possible that, as in the discussion of the gauge transformations, the special form of the functions $(\phi_n, \rho_n, \bar{\rho}_n)$ in Eq. (5.16) can be used to simplify the discussion of the singularity structure.

VI. EXPLICIT ONE- AND TWO-MONOPOLE SOLUTIONS

In Sec. V we described a procedure for the construction of the multimonopole solution, which was obtained by generalizing the procedure of Manton[9] and Ward[25] for one- and two-monopole solutions. However, we have not shown the existence of a solution which satisfies the reality condition (5.20) and the requirements of nonsingularity. In this section we describe the one- and two-monopole solution as concrete examples of the procedure described in the previous section. This immediately gives a real form of Ward's two-monopole solution.

We begin with the one-monopole solution. For this we take

$$\Lambda_0 = \frac{\sinh Cr}{r}, \tag{6.1}$$

i.e., we take $n=1$, $\alpha_1 = 1$, and $c_1 = 0$ in Eq. (5.3). We then have

$$\Lambda_1 = \frac{\cosh Cr}{C} \tag{6.2}$$

and ϕ_1, ρ_1, $\bar{\rho}_1$ of the $\mathfrak{A}_1$ *Ansatz* is then given by Eq. (5.16) with $n=1$ and

$$P_1 = (\partial_3 + C)\Lambda_1 = \cosh Cr + \frac{x_3}{r}\sinh Cr, \tag{6.3a}$$

$$\bar{P}_1 = -(\partial_3 - C)\Lambda_1 = \cosh Cr - \frac{x_3}{r}\sinh Cr, \tag{6.3b}$$

$$\Phi_1 = \frac{l}{r}\sinh Cr. \tag{6.3c}$$

The reality condition (5.22) is easily seen to be satisfied and $P_1, \bar{P}_1$ are both positive. Since Λ_0 is never zero the energy density is nonsingular. For the gauge transformation, we have from Eq. (5.23)

$$G_1 G_1^\dagger = \begin{pmatrix} \bar{P}_1 & -\Phi_1 e^{-i\theta} \\ -\Phi_1 e^{-i\theta} & P_1 \end{pmatrix} = e^{-C\sigma_i x_i}. \tag{6.4}$$

In this case we have $G_1^K = e^{-C\sigma_i x_i/2}$, and after this gauge transformation the potentials are given by Eq. (3.11).

For the two-monopole solution of Ward,[25] we have to use the $\mathfrak{A}_2$ *Ansatz*. In this case, we take $c_1 = ic$ and $c_2 = -ic$, with c real, and define

$$R^2 = x_1^2 + x_2^2 + (x_3 - ic)^2, \tag{6.5a}$$

$$\bar{R}^2 = x_1^2 + x_2^2 + (x_3^2 + ic)^2, \tag{6.5b}$$

and we take

$$\Lambda_0 = \frac{C}{2\pi}\left(\frac{\sinh CR}{R} + \frac{\sinh C\bar{R}}{\bar{R}}\right). \tag{6.6}$$

Λ_0 is clearly real. Then from Eq. (5.11), we have

$$2\pi\Lambda_1 = \cosh CR + \cosh C\bar{R} \tag{6.7}$$

and

$$2\pi\Lambda_2 = R\sinh CR - \frac{\cosh CR}{C} + \bar{R}\sinh C\bar{R} - \frac{\cosh C\bar{R}}{C}. \tag{6.8}$$

Then again, we have ϕ_2, ρ_2, and $\bar{\rho}_2$ given by Eq. (5.16) with $n=2$. That is, we have

$$\rho_2 = P_2 \bar{\xi}^2 e^{iCx_4}, \tag{6.9a}$$

$$\phi_2 = \Phi_2 l^{-2} e^{iCx_4}, \tag{6.9b}$$

$$\bar{\rho}_2 = -\bar{P}_2 \xi^{-2} e^{iCx_4}, \tag{6.9c}$$

where

$$P_2 = \frac{1}{\Lambda_0}[(\partial_3 + C)\Lambda_1]^2 - (\partial_3 + C)^2\Lambda_2, \tag{6.10a}$$

$$\bar{P}_2 = \frac{1}{\Lambda_0}[(\partial_3 - C)\Lambda_1]^2 - (\partial_3 - C)^2\Lambda_2, \tag{6.10b}$$

$$\Phi_2=\frac{1}{\Lambda_0}\left[(\partial_l\Lambda_1)^2+(\partial_3\Lambda_1)^2-C^2\Lambda_1{}^2\right] \qquad (6.10c)$$

with Λ_0, Λ_1, and Λ_2 being given by Eqs. (6.6), (6.7), and (6.8), respectively.

With some algebra it is possible to prove that $\phi_1\phi_2\neq 0$, i.e., the energy density is nonsingular, implies that

$$c=\frac{\pi}{2C} \qquad (6.11)$$

For $c=\pi/2C$, it is easily verified that $\Lambda_0\neq 0$.

With a lot of algebra it is possible to verify that Eq. (5.22) holds and P_2 and $\overline{P}_2$ of Eqs. (6.9a) and (6.9b) are positive. Therefore, a real form of the Ward solution is given by Eqs. (5.25) and (5.26) with $n=2$ and P_2, $\overline{P}_2$, and Φ_2 as above.

VII. SUMMARY

In summary, we have given a general framework for generating multimonopole solutions as a static self-dual Yang-Mills field. The solution is given by the Atiyah-Ward *Ansätze*, which are constructed explicitly in terms of a single real function Λ_0 with a specific form given in Eq. (5.3), having the parameters α_i and c_i. The solution corresponding to the $\mathfrak{A}_n$ *Ansatz* has the asymptotic behavior of a monopole solution of topological charge n. Since the function Λ_0 must be real, it is natural to choose α_i to be real and take c_i's in complex-conjugate pairs or real. The condition given in Eq. (5.20) for the existence of a gauge transformation, making the solution real, and the nonsingularity condition (i.e., the determinants $\mathfrak{D}_n$ and $\mathfrak{D}_{n-1}$ do not vanish) has to be verified by explicit calculation. However, in the absence of further insight and/or simplification the required algebra is extremely complicated for topological charge higher than two. Therefore, it remains[32] to verify by explicit calculation, or still better a general proof, that we can choose values of the parameters α_i and c_i, such that the solution satisfies the reality and nonsingularity conditions.

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APPENDIX

In this appendix we complete the proof of theorem 5.2. We have already shown that Eq. (5.20) is a sufficient condition for the existence of a gauge transformation which makes the potentials real. Here we complete the proof that this condition is also necessary. This can be done by "solving" Eq. (2.13) with the $\phi_n, \rho_n, \overline{\rho}_n$ given in Eq. (5.16). Since $GG^\dagger$ is a positive-definite Hermitian matrix, we can always parametrize $GG^\dagger$ as

$$GG^\dagger=\begin{pmatrix}\alpha & \beta\\ \beta^* & \delta\end{pmatrix}, \qquad (A1)$$

where α, δ are real and positive, and $\alpha\delta-\beta\beta^*=1$. Since the gauge transformation is static, α, β, and γ are functions x_1, x_2, and x_3 only. Then, the equation $\overline{D}GG^\dagger D^\dagger=V(y,z)$ becomes

$$V_{11}(y,z)=\alpha\left(\frac{\phi_n}{\phi_n^*}\right)^{1/2}-\frac{\beta^*}{(\phi_n\phi_n^*)^{1/2}}\overline{\rho}_n, \qquad (A2a)$$

$$V_{21}(y,z)=\frac{\beta^*}{(\phi_n\phi_n^*)^{1/2}}, \qquad (A2b)$$

$$V_{12}(y,z)=\alpha\rho_n^*\left(\frac{\phi_n}{\phi_n^*}\right)^{1/2}+\beta(\phi_n\phi_n^*)^{1/2}-\delta\overline{\rho}_n\left(\frac{\phi_n^*}{\phi_n}\right)^{1/2}$$
$$-\frac{\beta^*}{(\phi_n\phi_n^*)^{1/2}}\overline{\rho}_n\rho_n^*, \qquad (A3a)$$

$$V_{22}(y,z)=\delta\left(\frac{\phi_n}{\phi_n^*}\right)^{1/2}+\frac{\beta^*}{(\phi_n\phi_n^*)^{1/2}}\rho_n^*, \qquad (A3b)$$

where

$$V(y,z)=\begin{pmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{pmatrix}$$

and ϕ_n, ρ_n, and $\overline{\rho}_n$ are given by Eq. (5.14). Then, since β^* and $(\phi_n\phi_n^*)^{1/2}$ are functions of x_1, x_2, and x_3, V_{21} must be a function of y only, i.e., $V_{21}(y,z)=V_{21}(y)$. Let us first assume that $V_{21}(y)\neq 0$. Now Eq. (A2a) becomes

$$V_{11}(y,z)=\left[\alpha+\frac{V_{21}(y)\overline{P}_n}{(\sqrt{2}y)^n}\right]e^{iCx_4}. \qquad (A4)$$

This implies that $V_{11}(y,z)=f(y)e^{-Cz/\sqrt{2}}$ for some function f of y only. Then,

$$\alpha=f(y)e^{-Cx_3}-\frac{V_{12}(y)}{(\sqrt{2}y)^n}\overline{P}_n. \qquad (A5)$$

Now α must be real, i.e., $\alpha=\alpha^*$. This implies that if $f(y)\neq 0$, then $\overline{P}_n=g(y,\overline{y})e^{-Cx_3}$. Since $\overline{P}_n$ does not have this form we must have $f(y)\equiv 0$. Another way to see this is to observe that, from (A2a) and (A2b) nonzero V_{11} is equivalent to having an additive function of y and z only in the definition of $\overline{\rho}_n$, which we have set equal to zero. Similarly, from (A3b), we must have $V_{22}=0$ also. Then Eq. (A4) reduces to

$$\alpha = -\frac{V_{12}(y)}{(\sqrt{2}y)^n}\,\bar{P}_n .$$

and since both α and $\bar{P}_n$ are real, we must have

$$V_{12} = \gamma(\sqrt{2}y)^n = \gamma\xi^n ,$$

where γ is a real constant. Then from $\det V = 1$, we have

$$V_{21} = -(\gamma\xi^n)^{-1} .$$

Thus we have arrived at Eq. (5.19) for V. To complete the proof we note that if $V_{21}=0$, then from Eqs. (A2b) and (A5) $\beta=0$, $\alpha = ke^{-Cx_3}$, where k = constant, and similarly $\delta = k^{-1}e^{Cx_3}$. It is then easily verified that the resulting V does *not* satisfy $J_n V \doteq$ Hermitian.

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[32]While this manuscript was under preparation, this has been done by M. K. Prasad [Commun. Math. Phys. (to be published)].