High-temperature Yang-Mills theories and three-dimensional quantum chromodynamics

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We demonstrate that for sufficiently high temperature T the behavior of any four-dimensional gauge theory with small coupling constant α, at distances beyond the electrical Debye screening length $\xi_D \sim 1/\sqrt{\alpha}T$, is determined precisely by the corresponding three-dimensional theory. This is the magnetic sector of the original theory, and in the non-Abelian case it is a Yang-Mills theory like three-dimensional quantum chromodynamics (QCD_3). We study QCD_3 in the loop expansion, which is only valid for distances $\ll 1/\alpha T$, in both covariant and Coulomb gauges. At a finite order in the loop expansion, the presence of logarithmic infrared divergences signals the appearance of new operators in the operator-product expansion. For example, in a covariant gauge, the gauge self-energy develops infrared divergences at two-loop order associated with the operator $\vec{A}^{\,2}$. Infrared divergences in the Wilson loop are also considered and shown to cancel below the order at which gauge-invariant local operators can appear in the operator-product expansion. The infrared structure of QCD_3 at distances $\gtrsim 1/\alpha T$ cannot be directly probed in the loop expansion, however. We present a simpler model which is calculable in this infrared limit, and which might serve as a prototype for QCD_3. The model is massless scalar QED_3, which with N charged scalars is soluble in a $1/N$ expansion as $N \to \infty$. Using the $1/N$ expansion, we demonstrate that infrared softening occurs: the long-range behavior of the photon propagator in massless scalar QED_3 is less singular than that of free fields. Infrared softening might also occur in QCD_3, although it cannot be demonstrated to finite order in the loop expansion. The implications of an assumed infrared softening in QCD_3 for the magnetic sector of Yang-Mills theories at high temperatures are also discussed. In particular, we consider the possibility that, if the softening is sufficiently great, there is screening of hot non-Abelian magnetic fields and possible confinement of primordial magnetic monopoles.

I. INTRODUCTION

The study of field theories at finite temperature is a subject of considerable interest. It is relevant both to the early universe and, more speculatively, to the creation of fireballs in heavy-ion collisions. For any non-Abelian gauge theory, there exists a critical temperature T_c at which a phase transition to a qualitatively different medium occurs.

In gauge theories which undergo spontaneous symmetry breakdown at zero temperature, a phase transition occurs owing essentially to the evaporation of the Higgs effect at finite temperature. By using perturbation theory to one-loop order, it can be shown that the vacuum expectation values of scalar fields which drive symmetry breaking at zero temperature vanish above the critical temperature.[1,2] Thus, in the Weinberg-Salam model the SU(2)×U(1) gauge symmetry is restored for temperatures $T > T_c \sim 250$ GeV.

For gauge theories like quantum chromodynamics (QCD) which are unbroken at zero temperature, the phase transition at finite temperature is one of deconfinement, whereby quarks and gluons are freed. This was first shown in the lattice theory at fixed lattice spacing,[3,4] with deconfinement resulting heuristically from the condensation of electric flux strings at T_c.[5] Monte Carlo simulations with a finite lattice[6] and estimates of instanton effects[7,8] indicate that in QCD $T_c \sim 200$ MeV.[9]

The properties of the phase transition itself depend on understanding the theory near the critical point T_c. The present work will be restricted to the question of what happens to non-Abelian gauge theories as they are driven to temperatures T much greater than T_c. We assume all coupling constants are small, so that at least naively, perturbation theory will be a reasonable approximation. This is true for QCD because of asymptotic freedom, and for most realistic grand unified models at temperatures $T \lesssim 10^{14}$ GeV. It is important to emphasize the elementary point that as we always work above the critical temperature it is immaterial whether or not the gauge symmetry is broken at zero temperature.

To study equilibrium processes at finite temperature, all bosonic (fermionic) fields are taken to be periodic (antiperiodic) with period $\beta = 1/T$ in Euclidean time. In a gauge theory, it is alternatively possible to demand periodicity only up to a gauge transformation, but we shall not avail ourselves of this choice. Instead, we insist that regardless of how the gauge is fixed locally, globally all fields be strictly periodic (antiperiodic).[10]

A general expectation is that thermal fluctuations will act to screen long-range correlations. How this occurs in detail can be understood as an example of the decoupling theorem.[11] In brief, different excitations receive temperature-dependent masses m, and since Euclidean space-time governs at finite temperatures, m is related to an inverse propagator at zero momentum. For the corresponding field ϕ, this mass acts to dampen long-range correlations as

$$\langle\phi(t,\bar{x}),\phi(0,0)\rangle \underset{|\bar{x}|\to\infty}{\sim} e^{-m|\bar{x}|}. \qquad (1.1)$$

The temperature-dependent masses m are of the form $m\sim\alpha^p T$, where $\alpha=g^2$ is a fine-structure constant, and the power p depends on the type of field considered. Consequently, fluctuations in a ϕ field are screened over distances $\xi\gg 1/(\alpha^p T)$. If α is small, a hierarchy of distance scales can then be established. By working out from short distances, the theory will simplify enormously as various massive excitations decouple.

To begin with, correlations over spatial distances $\xi\gg\beta$ ($\beta=1/T$) are those of the static theory. The boundary conditions in time require all energies p_0 to be even multiples of πT for bosons ($p_0=2n\pi T$) and odd multiples of πT for fermions [$p_0=(2n+1)\pi T$]. Thus, since their energy can never vanish, fermions decouple classically for $\xi\gg\beta$, and the only bosonic modes which propagate for $\xi\gg\beta$ are those with zero energy. The static theory consists of the gauge field along with any scalar bosons to which it may couple. Since at temperatures far above T_c the only dimensional parameter to be encountered is the temperature, classically all scalars are essentially massless. In the same way, within the static theory the contribution of electric fields to the action density is

$$G_{0i}{}^2\sim|D_iA_0|^2. \qquad (1.2)$$

Therefore, the A_0 field couples covariantly as a massless scalar to the vector potential A_i for the three-dimensional gauge theory.

What remains for distances $\xi\gg\beta$ is an effective three-dimensional theory of gauge bosons coupled only to scalar bosons, which are massless at the tree-graph level. Let us now consider quantum corrections to the effective theory. Any loop integral over a virtual momentum ($p_0,\bar{p}$) is of the form

$$T\sum_{n=-\infty}^{+\infty}\int d^3\bar{p}.$$

Ignoring possible ultraviolet divergences for the moment, the decoupling of finite-energy ($n\neq 0$) modes gives a three-dimensional gauge theory with a dimensional coupling constant αT.[12]

A three-dimensional gauge theory is superrenormalizable, with ultraviolet divergences arising only from mass renormalization. Because the original gauge theory is renormalizable at zero temperature, any such apparent divergences will yield a finite, temperature-dependent mass renormalization when the sum over all finite-energy modes is performed.[13] We shall find it convenient simply to ignore the finite-energy modes, but to retain the temperature as an ultraviolet cutoff.

To one-loop order, there are contributions to the self-masses for both scalar and electric (A_0) fields. With the temperature as a cutoff for the linear divergence, all scalars develop a mass $m_S{}^2\sim\alpha T^2$ from one-loop diagrams. Therefore, correlations between scalar fields or electric fields are screened over distances $\xi\gg\beta/\sqrt{\alpha}$. Explicit calculation shows that contributions to $m_S{}^2$ are always positive definite.[1,14] Intuitively, this reflects the fact that temperature always results in screening, not antiscreening, of scalar fields.

For the electric field this effect is the familiar Debye screening. However, there is an essential distinction between the behavior of hot electric and hot magnetic fields with low momentum, which can be directly understood from how gauge invariance works at finite temperature. The one-particle-irreducible self-energy for the gauge field, $\Pi_{\mu\nu}$, must satisfy the Ward identity

$$p^\mu p^\nu \Pi_{\mu\nu}(p_0,\bar{p})=0. \qquad (1.3)$$

In the infrared limit $p_0=0,\bar{p}^2\to 0$, $\Pi_{\mu\nu}$ becomes [$\Pi_{0i}(0,\bar{p})=0$]

$$\begin{aligned}\Pi_{00}(0,0)&=m_{\mathrm{el}}{}^2(\sim\alpha T^2),\\ \Pi_{ij}(0,\bar{p})&=(\delta_{ij}-\hat{p}_i\hat{p}_j)\Pi(\bar{p}^2).\end{aligned} \qquad (1.4)$$

At zero energy, gauge invariance places no restriction on the value of $\Pi_{00}(0,0)$, which we define as the electric mass squared, $m_{\mathrm{el}}{}^2$. A positive-definite $m_{\mathrm{el}}{}^2$ at one-loop order is common to both Abelian and non-Abelian theories. On the other hand, gauge invariance does restrict the form of the magnetic self-energy since $\Pi_{ij}(0,\bar{p}^2)$ must be transverse. In contrast to electric fields the infrared behavior of hot magnetic fields, which is determined by the dependence of $\Pi(\bar{p}^2)$ on $\bar{p}^2$ near zero momentum, depends on whether the group is Abelian or non-Abelian.

In summary, for distances $\xi\gg\beta/\sqrt{\alpha}$ electric and scalar fields decouple to leave a pure three-dimensional gauge theory. For an Abelian group, since the photon couples only to massive particles the infrared structure is trivial; to any finite order in perturbation theory, $\Pi(\bar{p}^2)$ always vanishes like $\bar{p}^2$ for $\bar{p}^2\to 0$.[15] For example, at one-loop order

$$\Pi(\bar{p}^2)\underset{\bar{p}^2\to 0}{\sim}\bar{p}^2\frac{\alpha T}{m_S}\sim\sqrt{\alpha}\,\bar{p}^2. \qquad (1.5)$$

The factor of $\bar{p}^2$ above must arise if Π_{ij} is to be transverse. It is easy to show that each higher order in a loop expansion contributes a higher power of $\alpha T/m_S\sim\sqrt{\alpha}$ to $\Pi(\bar{p}^2)$. Thus the perturbative expansion is infrared finite to all orders in a loop expansion. That is, although there is electric screening in hot QED at one-loop order, the long-range perturbative interaction between (purely) Abelian magnetic fields is always that of a free field.[16]

For a non-Abelian group, the effective theory for $\xi \gg \beta/\sqrt{\alpha}$ is a three-dimensional theory of unbroken Yang-Mills fields, a representative example being three-dimensional QCD (QCD_3). Since gluons couple to themselves, we expect the infrared behavior to be quite complex. One natural question to ask is whether there is screening (magnetic screening for hot QCD_4) at any order of perturbation theory. A necessary condition for magnetic screening to occur is that the gluon propagator be infrared soft: the renormalized inverse propagator must vanish less quickly than $\bar{p}^2$ about zero momentum. Equivalently, $\Pi(\bar{p}^2)/\bar{p}^2$ must be singular about $\bar{p}^2=0$.

Consider then a calculation of the gluon and ghost self-energies in perturbation theory. We can immediately understand why there are difficulties in the infrared region. A dimensionless quantity such as $\Pi(\bar{p}^2)/\bar{p}^2$, if it is free of divergences, can only depend on the dimensionless ratio $\alpha T/p$, so that each order in a loop expansion is expected to be proportional to a power of $\alpha T/p$. We refer to this property of the loop expansion as infrared sensitivity: the infrared behavior becomes more singular with each higher order in α. At one-loop order

$$\Pi(\bar{p}^2) \sim \bar{p}^2 \frac{\alpha T}{p} \sim \alpha T p \,, \tag{1.6}$$

and already $\Pi(\bar{p}^2)/\bar{p}^2$ is not regular about the origin. At two loops naive power counting indicates that $\Pi(\bar{p}^2)/\bar{p}^2 \sim (\alpha T)^2/\bar{p}^2$, suggesting the possibility of "magnetic mass" developing dynamically in QCD_3.[2,3,7]

In this paper we shall concentrate on the structure of QCD_3. We argue that the analytic structure of $\Pi(\bar{p}^2)/\bar{p}^2$ is much more complex than a simple pole in $\bar{p}^2$ at $\bar{p}^2=0$. Although a complete solution to the small-momentum behavior is not produced, the loop expansion exhibits features which are closely tied to large-distance questions such as possible magnetic screening. To motivate our discussion of these features, we remind the reader that the existence of a phenomenon like magnetic screening would be rather surprising. Electric screening can easily be understood since any charged particle carries an electric charge. To one-loop order, thermal fluctuations pull charged pairs out of the vacuum to screen external charges. However, there are no fundamental particles in any gauge theory which carry a magnetic charge; viz., $D_i\tilde{G}_{ij}=0$ is always an identity. Magnetic screening can presumably then only arise from *non*perturbative fluctuations which carry magnetic charge.

We are thus led to the question of nonperturbative vacuum structure in QCD_3. Of course, we cannot hope to calculate nonperturbative effects from perturbation theory. However, the presence of nonperturbative vacuum structure is augured by the infrared sensitivity of the loop expansion—in particular, by the presence of logarithmic infrared divergences in Euclidean Green's functions at a finite order in perturbation theory.

This aspect of QCD_3 is of considerable interest beyond its implications for hot four-dimensional theories. Forgetting finite-temperature physics for the moment, QCD_3 is an important (albeit unphysical) model to be analyzed. It is intermediate in complexity between QCD_4 and the nearly trivial two-dimensional model QCD_2. In four dimensions, the nonperturbative vacuum structure which must occur in order to produce confinement cannot be seen to any finite order in perturbation theory. In QCD_3, on the other hand, the vacuum intrudes at a finite order in the loop expansion. Still, unlike QCD_2, the problem of confinement for QCD_3 remains difficult and unsolved.

Our analysis will proceed as follows. Consider an arbitrary local operator Ω with dimensions of $(\text{mass})^r$. If we were to calculate its vacuum expectation value $\langle\Omega\rangle$ in perturbation theory, the result would be of the form

$$\langle\Omega\rangle \sim \Lambda^r\left(1 + c_1\frac{\alpha}{\Lambda} + \cdots + c_r\frac{\alpha^r}{\Lambda^r}\ln\frac{\mu}{\Lambda} + \cdots\right), \tag{1.7}$$

where Λ is an ultraviolet cutoff independent of the fine-structure constant α (we have redefined αT as α) and the c_i's are constants. The important point is that, at rth order, $\langle\Omega\rangle$ develops a logarithmic infrared divergence, which we cut off with an infrared regulator μ. If we were to solve the theory exactly, the regulator μ would presumably be replaced by α times some function of the parameters of the gauge group [for example, $\mu = f(N)\alpha$ for an SU(N) gauge group]. The occurrence of a term like $\ln(\alpha/\Lambda)$ indicates we cannot consistently require that $\langle\Omega\rangle$ vanish in the physical vacuum.

This discussion can be systematized within an operator-product expansion (OPE). In the OPE, the operator $\Omega(\bar{p}^2)$ appears in the computation of some Euclidean Green's function, expanded about the limit of hard momentum, $\bar{p}^2 \gg \alpha^2$. The loop expansion of $\langle\Omega(\bar{p}^2)\rangle$ is then just that of $\langle\Omega\rangle$, except that the ultraviolet cutoff Λ is replaced by the hard momentum p. The appearance of logarithmic divergences in the Euclidean Green's function indicates that new operators are contributing to the OPE.

These conclusions hold for any operator, gauge invariant or not. Infrared divergences occur to lowest order in perturbation theory for quantities with the lowest dimensions of mass. In QCD_3, the operators with the lowest mass dimensionality are $\bar{A}^2$ ($\bar{A}$ is the gauge field) and $|\psi_{gh}|^2$ (ψ_{gh} is the ghost field). These, however, are gauge dependent and cannot enter the OPE for gauge-invariant Green's functions. Gauge-invariant operators such as $\langle G_{ij}^2\rangle$ [$\sim$(mass)^3] have higher dimensionality, requiring computation to higher order to be able to see the associated infrared logarithm. Consequently, in QCD_3 it is quite difficult to disentangle the role of gauge invariance.

In Sec. II, these questions will be addressed within the context of an Abelian model which can serve as a simplified version of QCD_3. The model is charged scalar electrodynamics in three dimensions (QED_3), where the renormalized scalar mass is fixed to be zero at each order in the loop expansion.[17] The principal virtue of massless scalar QED_3 is that gauge invariance is much easier to deal with since the photon self-energy is itself gauge invariant. Furthermore, if N is the number of scalars, the model is soluble as $N\to\infty$ within a $1/N$ expansion. Using the $1/N$ expansion, many questions which are otherwise insoluble can be answered. For example, it is used to show that the photon propagator is softened for infrared momenta $p\ll N\alpha$.

QCD_3 is discussed in Sec. III. The gluon and ghost self-energies are computed in perturbation theory, with convergence anticipated for hard momentum $p\gg\alpha$. We work both in covariant and three-dimensional Coulomb ($\partial_1A_1+\partial_2A_2=0$) gauges. We calculate the coefficient of the logarithmic infrared divergences which appear in the self-energies at two-loop order and discuss their gauge dependence. The connection to a gauge-dependent OPE is also established. In order to answer gauge-invariant questions, we examine Wilson loops suitable for both covariant and Coulomb gauges. Cancellation of infrared divergences corresponding to gauge-dependent operators is demonstrated in the Wilson loop through two nontrivial orders.

In Sec. IV, we conclude with a brief discussion of the open questions and possible implications of our results.

II. MASSLESS SCALAR QED_3

This section is devoted to an Abelian gauge theory which can, in many ways, serve as a prototype of QCD_3. The model is charged scalar electrodynamics in three dimensions (QED_3), with the renormalized scalar mass fixed to be zero.[18] By making this adjustment, scalar QED_3 becomes a theory with coupled massless fields, giving it an infrared structure similar to QCD_3. Indeed, purely on dimensional grounds, we see that since the loop-expansion parameter is of order α/p (p is the momentum scale of some correlation function), the loop expansion in massless scalar QED_3 is infrared sensitive. That is, perturbative calculations in α are valid at best for hard momenta $p\gg\alpha$.

Questions such as the nature of infrared softening and the role of the operator-product expansion in short-distance correlation functions can only be answered by going beyond finite orders in the loop expansion. There are two reasons why massless scalar QED_3 is ideal for understanding these questions of infrared physics. One is that since the model is Abelian, gauge invariance is much easier to deal with than in QCD_3. The second is that if there are N charged scalars, $1/N$ provides a new expansion parameter which allows the summation of infinite classes of Feynman graphs. The use of the $1/N$ expansion will be particularly useful as a means of developing intuition which would otherwise be lacking in QCD_3.

The invariant Lagrangian density for the model is[19]

$$\mathcal{L}=\tfrac{1}{4}F_{ij}^2+\sum_{a=1}^{N}|(\partial_i+ieA_i)\phi_a|^2\,, \tag{2.1}$$

where the ϕ_a are complex scalar fields and i,j run from one to three. We remind the reader that we shall only concern ourselves with the purely Euclidean theory, as directly relevant for hot four-dimensional theories beyond the Debye screening length. Calculations will be performed in a class of covariant gauges, in which the bare photon propagator is

$$D^0_{ij}(\bar{p})=(\delta_{ij}-\hat{p}_i\hat{p}_j)\frac{1}{\bar{p}^2}+\xi\,\frac{\hat{p}_i\hat{p}_j}{\bar{p}^2}\,. \tag{2.2}$$

The simplest gauge-invariant correlation function is the photon propagator, which we shall now examine both in the loop expansion and the $1/N$ expansion. The polarization tensor is defined as

$$\Pi_{ij}(\bar{p})=(\delta_{ij}-\hat{p}_i\hat{p}_j)\Pi_{ph}(\bar{p}^2)\,, \tag{2.3}$$

where $\Pi_{ph}(\bar{p}^2)$ is ultraviolet finite to all orders. At one-loop order, $\Pi_{ph}(\bar{p}^2)$ is also infrared finite and is given by

$$\Pi^{(1)}_{ph}(\bar{p}^2)=-\frac{N\alpha}{16}\,p\,, \tag{2.4}$$

where $\alpha=e^2$. To leading order in the large-N limit ($N\to\infty$ with αN fixed), the complete propagator is given by the iteration of the one-loop polarization tensor

$$D_{ij}^{(1)}(\bar{p})=\frac{\delta_{ij}-\hat{p}_i\hat{p}_j}{\bar{p}^2+(N\alpha/16)p}+\xi\frac{\hat{p}_i\hat{p}_j}{\bar{p}^2}. \qquad (2.5)$$

There are several comments to make about this result:

(i) It is an exact result to leading order in the $1/N$ expansion.

(ii) Only the transverse part of the propagator, which is independent of ξ, is renormalized by interactions.

(iii) Because of the sign of $\Pi_{\text{ph}}(\bar{p}^2)$, Eq. (2.4), there is no tachyonic pole in the photon propagator.

(iv) For small momenta, $p \ll N\alpha$, the dynamics softens the behavior of the bare propagator, but not enough to result in mass generation.

To proceed further, beyond one loop in the polarization tensor and beyond leading order in the $1/N$ expansion, we shall next compute the scalar self-energy $\Pi_S(p^2)$ shown in Fig. 1. In order to keep the scalar massless, a subtraction will be performed at $p^2=0$, eliminating the second (tadpole) diagram. With the subtraction, the remaining diagram can be shown (unexpectedly) to be gauge invariant. Because of this, without loss of generality the self-energy can be written as

$$\Pi_S(\bar{p}^2)=\alpha\int\frac{d^3k}{(2\pi)^3}\frac{(2p+k)^i(2p+k)^j}{(\bar{p}+\bar{k})^2}\frac{(\delta_{ij}-\hat{k}_i\hat{k}_j)}{[k^2+(N\alpha/16)k]}$$

$$=\frac{\alpha p^2}{\pi^2}\int_0^\infty k\,dk\int_0^\pi\frac{\sin^3\theta\,d\theta}{(\bar{p}+\bar{k})^2[k+(N/16)\alpha]}. \qquad (2.6)$$

It is instructive to consider this expression in the limits of both large and small momentum. For large momentum $p \gg N\alpha$,

$$\Pi_S(\bar{p}^2)=\frac{\alpha}{4}p-\frac{1}{12\pi^2}\alpha(N\alpha)\ln\frac{p}{N\alpha}+O(N\alpha^2). \qquad (2.7)$$

The leading term results if the bare gauge propagator is used in Fig. 1 [$k^2+(N\alpha/16)k\rightarrow k^2$ above] and it corresponds to one-loop perturbation theory without the $1/N$ expansion. The presence of the $\ln(N\alpha)$ factor in the next term indicates that the perturbative expansion in α has broken down. An attempt to compute the scalar self-energy in the loop expansion will encounter a logarithmic infrared divergence at the two-loop level, which is not canceled by other diagrams. Without the $1/N$ resummation, or some other scheme for handling contributions from small momenta, only the coefficient of the infrared logarithm in Eq. (2.7) can be computed. The argument of the logarithm, or equivalently the constant term beyond the logarithm, receives contributions from all orders in the loop expansion. Physically, the meaning of this result is that at this order, even a short-distance correlation function becomes sensitive to long-distance effects.

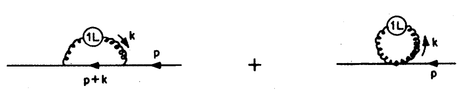

FIG. 1. The charged scalar self-energy in QED_3 to leading order in the $1/N$ expansion. The curly line represents the photon propagator of Eq. (2.5).

This sensitivity can be understood in the language of the operator-product expansion. Because the loop-expansion parameter is dimensional, the expansion will eventually lead to operators of higher dimension than the unit operator. The necessity of new operators in the OPE will be signaled by the occurrence of infrared divergences. At the two-loop level, the new operator appearing in $\Pi_S(p^2)$ is the vacuum expectation value of $\bar{A}^2$, $\langle\bar{A}^2\rangle$. Since the scalar self-energy has dimensions of (mass)^2 and $\langle\bar{A}^2\rangle$ has dimensions of mass, there must also be a single power of α accompanying the contribution of $\langle\bar{A}^2\rangle$. This factor of α describes the coupling of $\langle\bar{A}^2\rangle$ to the scalar fields and can be regarded as part of the Wilson coefficient function.

Of course, the operator $\bar{A}^2$ is gauge dependent and so it cannot appear in a gauge-invariant correlation function. Since the scalar propagator is itself gauge dependent the appearance of $\langle\bar{A}^2\rangle$ is possible. Loosely speaking, $\bar{A}^2$ couples to the charge carried by the scalar field.

In the small-momentum limit $p \ll N\alpha$, $\Pi_S(p^2)$ becomes

$$\Pi_S(\bar{p}^2)=\frac{64}{3\pi^2}\frac{\bar{p}^2}{N}\ln\left(\frac{N\alpha}{p}\right)+O\left(\frac{p^2}{N}\right). \qquad (2.8)$$

The important point to be made about this result is that $\Pi_S(p^2)$ vanishes like $p^2\ln(p)$. From the form of the propagator for hard momenta, we might have expected terms proportional to p or $\ln(p)$ in the infrared. The $1/N$ expansion shows that the behavior of the propagator in the ultraviolet is completely misleading in predicting the leading infrared behavior.

The factor of $\ln(N\alpha/p)$ in Eq. (2.8) is worthy of special mention. We naturally expect corrections in the $1/N$ expansion to be of order $1/N$. Equation (2.8), however, suggests that the effective expansion parameter is actually of order $(1/N)\ln[(N\alpha)/p]$. Even in the $1/N$ expansion, which sums an infinite class of diagrams, arbitrarily small momenta cannot be probed. The $1/N$ expansion will only converge for momenta $p \gg N\alpha e^{-N}$. The loop expansion was found to be infrared sensitive in a powerlike manner. We now see that the $1/N$ expansion is also expected to be infrared sensitive, if only by powers of logarithms.

All the above questions should finally be studied for gauge-invariant correlation functions. For that purpose, we examine the leading correction

$\Pi_{ij}^{(2)}(\bar{p}^2)$ to the polarization tensor in the $1/N$ expansion. It is possible to determine both the large- and small-momentum limits of $\Pi_{ij}^{(2)}$ by power counting and by paying careful attention to gauge invariance. The contributions are shown in Fig. 2 with the internal photon propagator given by Eq. (2.5). Two-loop perturbation theory corresponds to using the bare photon propagator [Eq. (2.2)] instead.

We first consider $\Pi_{ij}^{(2)}$ in the small-momentum limit. A typical diagram, say Fig. 2(a), gives a contribution of the form

$$\sim N\alpha \int \frac{d^3q}{(2\pi)^3} \frac{(2q+p)^i(2q+p)^j}{(\bar{p}+\bar{q})^2(\bar{q}^2)^2} \Pi_S(\bar{q}^2), \qquad (2.9)$$

where $\Pi_S(\bar{q}^2)$ is the scalar self-energy of Eq. (2.6). Because $\Pi_S(\bar{q}^2) \sim q$ for large q, this expression has an ultraviolet-divergent part proportional to δ_{ij}. However, this divergence must cancel against other diagrams, since there is no gauge-invariant counterterm to remove it. A reliable estimate of $\Pi_{ij}^{(2)}$ is extracted from the part of Eq. (2.9) proportional to $p_i p_j$. This piece behaves as

$$\sim \frac{N\alpha}{(2\pi)^3} p_i p_j \int \frac{d^3q\, \Pi_S(\bar{q}^2)}{(\bar{p}+\bar{q})^2(\bar{q}^2)^2}. \qquad (2.10)$$

Using the form of $\Pi_S(q^2)$ for small momentum [Eq. (2.8)], the behavior of the expression (2.10) for $p \ll N\alpha$ is found to be

$$\sim \frac{32}{3\pi^4} \alpha \hat{p}_i \hat{p}_j p \ln\left(\frac{N\alpha}{p}\right). \qquad (2.11)$$

The other graphs in Fig. 2 are found to behave similarly. Thus, for momenta $p \ll N\alpha$, the photon propagator through second order in $1/N$ is

$$D_{ij}^{(2)}(\bar{p}) = \frac{\delta_{ij} - \hat{p}_i \hat{p}_j}{\bar{p}^2 + (N\alpha/16)p[1 + O((1/N)\ln(N\alpha/p))]} + \xi \frac{\hat{p}_i \hat{p}_j}{\bar{p}^2}. \qquad (2.12)$$

We offer three comments about this result:

(1) Like the scalar self-energy, the form of the propagator in the infrared region is very different from what might be expected from hard momenta. Two-loop perturbation theory for $p \gg N\alpha$ appears to indicate the development of a photon mass $m^2 \sim N\alpha^2$. This is just wrong. The $1/N$ expansion shows that the would be mass actually behaves as $N\alpha p \ln(N\alpha/p)$.

(2) The logarithm in Eq. (2.12) again suggests that the $1/N$ expansion is infrared sensitive with an effective expansion parameter $(1/N)\ln(N\alpha/p)$. It might be possible to go beyond finite orders in $1/N$ by renormalization-group methods, but we shall not pursue this here.

(3) For a three-space-time-dimensional theory, the logarithmic potential corresponding to a $1/\bar{p}^2$ propagator is softened in the infrared region to become a $1/r$ potential, up to logarithmic corrections in the $1/N$ expansion. Because of the logarithms, the form of the potential is not trustworthy for distances $r \gg e^N/N\alpha$.

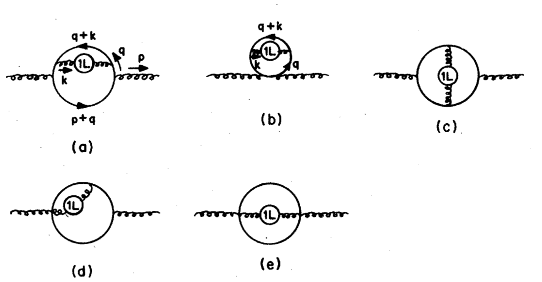

FIG. 2. Contributions to the vacuum polarization tensor in QED_3 to second order in the $1/N$ expansion. The conventions are those of Fig. 1.

To conclude this section, the large-momentum limit of $\Pi_{ij}^{(2)}(\bar{p})$ will be examined. The main observation, as for the scalar self-energy $\Pi_S(\bar{p}^2)$, will be the breakdown of the loop expansion and the appearance of a new operator in the OPE. Now, however, everything will be gauge invariant.

Consider the integration over q in Fig. 2(a). If $\Pi_S(\bar{q}^2)$ is approximated by its one-loop value $\alpha q/4$ [the first term in Eq. (2.7)], the q integral becomes logarithmically infrared divergent even at fixed external momentum p, indicating a breakdown in the loop expansion for the polarization tensor at the two-loop level. The divergence can be regulated by going beyond the loop expansion and using the full expansion for $\Pi_S(\bar{q}^2)$ [Eq. (2.6)] justified by the $1/N$ expansion. Because of the softening of $\Pi_S(\bar{q}^2)$ in the infrared [Eq. (2.8)], there is no longer an infrared divergence. In the large-p limit, the leading behavior of Fig. 2(a) can be read from Eq. (2.10). The integral over q is conveniently broken into a piece from $O \lesssim q \lesssim N\alpha$ and another piece from $N\alpha \lesssim q \lesssim p$, using the low- and high-momentum forms of $\Pi_S(\bar{q}^2)$. The dominant contribution can be shown to arise from the region $\alpha N \lesssim q \lesssim p$. Using this, Eq. (2.10) takes the large-momentum form

$$\frac{N\alpha}{2\pi^2} \frac{p_i p_j}{p^2} \int_{N\alpha}^{p} \frac{dq}{q^2} \frac{\alpha}{4} q \sim \frac{N\alpha^2}{8\pi^2} \hat{p}_i \hat{p}_j \ln \frac{p}{N\alpha}. \qquad (2.13)$$

Thus, for hard momentum $p \gg N\alpha$, we find

$$\Pi_{ij}^{(2)}(\bar{p}) \sim (\delta_{ij} - \hat{p}_i \hat{p}_j)\left[\frac{N\alpha^2}{8\pi^2} \ln(p/N\alpha) + O(N\alpha^2)\right]. \qquad (2.14)$$

The infrared divergence has been replaced by a logarithmic dependence on $N\alpha$. This sensitivity to large distances can be viewed as a contribution from the vacuum expectation value of a higher-dimension operator in the OPE. The only gauge-invariant operator which has a low enough dimension to arise at this level in the loop or $1/N$ expansions is $|\bar{\phi}|^2=\sum_{a=1}^{N}\phi_a^*\phi_a$. That this is indeed the relevant operator can be seen by short circuiting the propagator carrying the large momentum p in Fig. 2(a). The resulting graph describes $\langle|\bar{\phi}|^2\rangle$ to first nonleading order $[N\alpha\ln(N\alpha)]$ in the $1/N$ expansion. It then couples with strength α to the photon.

At higher orders in the loop or $1/N$ expansion, higher-dimension operators such as $|\bar{\phi}|^4$ or $F_{ij}{}^2$ will be encountered, all having nonzero vacuum expectation values. Gauge-dependent operators such as $\bar{A}^2$ should not appear. This operator, which appeared in the scalar propagator at two loops (order α^2), first makes a possible appearance in the polarization tensor at three loops (order α^3). Even though it can be seen in individual graphs, it must finally cancel for the full polarization tensor.

III. THREE-DIMENSIONAL QCD

We now turn our attention to three-dimensional Yang-Mills theories, generically referred to as QCD_3. The gauge group is taken to be SU(N) but our conclusions apply equally well to other groups relevant to grand unification. The invariant Lagrangian is

$$\mathcal{L}=\tfrac{1}{4}G_{ij}^aG_{ij}^a, \tag{3.1a}$$

where

$$G_{ij}^a=\partial_iA_j^a-\partial_jA_i^a+gf^{abc}A_i^bA_j^c. \tag{3.1b}$$

The quadratic Casimir eigenvalue of the adjoint representation, C_A, is defined by

$$f_{acd}f_{bcd}=C_A\delta_{ab}, \tag{3.2}$$

with $C_A=N$ for SU(N).

Although computational results will be presented in this section for general N, the kind of tractable $1/N$ expansion employed in Sec. II is not available here. The celebrated $1/N$ expansion for SU(N) gauge theories leads only to the dominance of planar diagrams.[20] Although QCD_2 is soluble in this approximation, the existence of transverse degrees of freedom in higher dimensions has so far prevented similar progress.[21]

The analysis of this section will therefore be restricted to finite orders in the loop expansion. Large-momentum Green's functions will be shown to contain infrared singularities associated with the breakdown of the loop expansion. The connection of this phenomenon to the operator-product expansion and the vacuum structure of the theory will be elucidated. The Green's functions of QCD_3 are all gauge dependent and, therefore, in order to ask gauge-invariant questions, Wilson loops will also be examined. It is instructive to perform the analysis both in covariant gauges and in a three-dimensional Coulomb gauge. This section will be divided accordingly.

A. Covariant gauge

In the class of gauges considered, the bare Euclidean gauge-boson propagator is given by

$$\delta^{ab}D_{ij}^{(0)}(p)=\delta^{ab}\left[(\delta_{ij}-\hat{p}_i\hat{p}_j)\frac{1}{\bar{p}^2}+\xi\frac{\hat{p}_i\hat{p}_j}{\bar{p}^2}\right]. \tag{3.3}$$

The self-energies for the Faddeev-Popov ghost and for the gauge boson will play an important role in subsequent discussions and we begin by computing them through one loop.

The one-loop ghost self-energy affords the simplest computation. There is one diagram, shown in Fig. 3, and it is both infrared and ultraviolet finite. Furthermore, it is found to be independent of the gauge-fixing parameter ξ in three dimensions. The result is

$$\Pi_{\rm gh}^{(1)}(\bar{p}^2)=\frac{N\alpha}{16}p, \tag{3.4}$$

where $\alpha\equiv g^2$. The gauge polarization tensor is defined as

$$\Pi_{ij}(\bar{p})=(\delta_{ij}-\hat{p}_i\hat{p}_j)\Pi_g(\bar{p}^2) \tag{3.5}$$

and the three contributing diagrams are shown in Fig. 4. The result is ultraviolet and infrared finite but gauge dependent, in the form of a quadratic polynomial in ξ. We find

$$\Pi_g^{(1)}(\bar{p}^2)=[(\xi+1)^2+10]\frac{N\alpha}{64}p. \tag{3.6}$$

The sign of $\Pi_{\rm gh}^{(1)}$ means that the ghost propagator

$$\delta^{ab}D_{\rm gh}(\bar{p}^2)\equiv\frac{\delta^{ab}}{\bar{p}^2-\Pi_{\rm gh}(\bar{p}^2)} \tag{3.7}$$

would develop a tachyonic pole if the one-loop result could be trusted for $p\sim O(N\alpha)$. It cannot, of course, since the expansion parameter is expected to be of order $N\alpha/p$. Similarly, the gauge

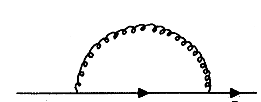

FIG. 3. The one-loop ghost self-energy is a covariant gauge. Curly lines represent gauge bosons, solid lines ghost quanta.

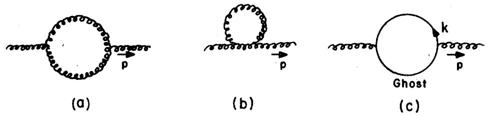

FIG. 4. The one-loop gauge self-energy in a covariant gauge.

propagator

$$\delta^{ab}D_{ij}(\bar{p}) \equiv \delta^{ab}\left[\frac{\delta_{ij}-\hat{p}_i\hat{p}_j}{\bar{p}^2-\Pi_g(\bar{p}^2)}+\xi\frac{\hat{p}_i\hat{p}_j}{\bar{p}^2}\right] \tag{3.8}$$

would exhibit a tachyonic pole if the one-loop result were used with abandon. Here, however, the gauge dependence of $\Pi_g^{(1)}$ further complicates the discussion. While $\Pi_g^{(1)}$ is positive for any real ξ, it can be made nonpositive or even zero with a suitable complex choice for ξ.

Before addressing the problem of gauge invariance, we examine some correlation functions at the next level of the loop expansion. Attention will necessarily be restricted to the short-distance regime $p \gg N\alpha$.

Consider first the gauge propagator. If $\Pi_g(\bar{p}^2)$ were again free of ultraviolet and infrared divergences at the two-loop level, then it would be of the form $(N\alpha)^2$. It is this guess for $\Pi_g^{(2)}(\bar{p}^2)$ which, when wrecklessly extrapolated to $p \lesssim N\alpha$, suggested the appearance of a magnetic mass in finite-temperature QCD. Our analysis of massless scalar QED_3, where a soluble $1/N$ expansion is available, has shown how wrong this is.

Even at large momentum, the form $(N\alpha)^2$ is incorrect for $\Pi_g^{(2)}(\bar{p}^2)$ since there are divergences in the class of diagrams shown in Fig. 5. In a gauge in which the one-loop insert [Eq. (3.6)] does not vanish, the integral over k will diverge logarithmically. Employing an infrared cutoff μ, the self-energy is of the form

$$\Pi_g^{(2)}(\bar{p}^2) \sim (N\alpha)^2 \ln(p/\mu)\,. \tag{3.9}$$

The logarithmic infrared sensitivity means that, at this level, the propagator is beginning to sample the large-scale structure of the vacuum. This sensitivity signals the appearance of a new operator in the OPE. The only allowed operators with low enough dimension to contribute at this level are $\bar{A}^2$ and $|\psi_{\text{gh}}|^2$, both having dimensions of mass and both being gauge dependent. By short circuiting the line carrying the large momentum in Fig. 5, we see that it is only $\langle\bar{A}^2\rangle$ which arises. It might be thought that $|\psi_{\text{gh}}|^2$ will also appear through the two-loop graph formed by inserting a one-loop ghost self-energy in Fig. 4(c). However, the integral is suppressed in the infrared at this order due to the gauge-ghost vertex structure.

Other Euclidean Green's functions will also develop infrared divergences at two loops. They always arise from one-loop insertions on internal lines and they are always associated with the appearance of $\bar{A}^2$ in the OPE. The coefficient of the logarithmic divergence can be computed at the two-loop level but the argument of the logarithm is sensitive to momenta on the order of $N\alpha$. Thus it is sensitive to all orders in the expansion and it can only be computed if some nonperturbative scheme is available.

All two-loop infrared divergences can be eliminated by choosing the gauge $(\xi+1)^2=-10$, in which $\Pi_g^{(1)}$ vanishes. They therefore have no physical significance and in any gauge they must cancel in the computation of a gauge-invariant correlation function. This expectation is further reinforced by the association of these divergences with the gauge-dependent operator $\bar{A}^2$ which cannot arise in the OPE of a gauge-invariant correlation function. In QCD_3, the gauge-invariant local operator of lowest dimension is $(G_{ij})^2 \sim (\text{mass})^3$. Thus, noncancelling infrared divergences should first appear in gauge-invariant correlation functions at the four-loop level. A convenient gauge-invariant object to study is the Wilson loop,

$$W=\left\langle P\exp\left(ig\oint_c dx_\mu A^\mu\right)\right\rangle. \tag{3.10}$$

An appropriate choice of the contour will allow, for example, the extraction of the static potential between two color charges in the $(2+1)$-dimensional theory. To analyze infrared divergences and their connection to the OPE, however, it is simpler to use a small contour such as the one shown in Fig. 6. If the contour has diameter $d \ll 1/N\alpha$, then only short distances are being probed and the OPE can be directly applied.

Dimensional analysis shows that if no infrared divergences are encountered, the loop expansion

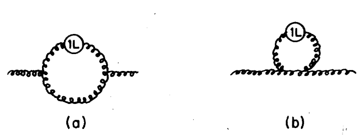

FIG. 5. Contributions to the two-loop gauge self-energy with infrared divergences.

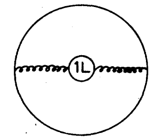

FIG. 6. A term of order α^2 in the perturbative expansion of a small Wilson loop.

must give the result

$$W=\sum_{n=0}^{\infty} C_n(N\alpha d)^n . \tag{3.11}$$

Since the local operator G_{ij}^2 first becomes infrared divergent at order α^3, and as two powers of α are required to make contact between the Wilson loop and this operator, we expect the expansion to break down at the $n=5$ level. The cancellation of infrared divergences in a general gauge has been checked through the $n=2$ level, where a typical contribution is shown in Fig. 6. The two-loop propagator divergence of Eq. (3.9) would feed into the Wilson loop at the $n=3$ level and it too must cancel.

B. Coulomb gauge

We now discuss the loop expansion for QCD_3 in a three-dimensional Coulomb gauge where

$$\vec{\nabla}\cdot\vec{A}=0 . \tag{3.12}$$

Throughout this section, we shall use the convention of denoting two-dimensional vectors by $\vec{A}$, so that a three-dimensional vector $\bar{A}=(\vec{A},A_3)$. We expect the results of computation in the Coulomb gauge to be most directly relevant not for QCD_3 considered as the limit of QCD_4 at high temperatures, but for QCD_3 considered as the Euclidean form of a Yang-Mills theory in two space and one time dimension.

In the Coulomb gauge there are quanta associated with Coulomb transverse and ghost fields. The bare Coulomb and ghost propagators are instantaneous in the 2 + 1 "time" coordinate x_3 behaving as $1/\vec{p}^2$ in momentum space. In contrast, the bare transverse gauge field propagates in both two-dimensional space $\vec{x}$ and time x_3, behaving as $1/\bar{p}^2$:

$$\delta^{ab}D_{ij}^{(0)}=\delta^{ab}(\delta_{ij}-\hat{p}_i\hat{p}_j)\frac{1}{\vec{p}^2+p_3^2} . \tag{3.13}$$

In this section, i and j run only between 1 and 2. We follow the same general method of attack as for covariant gauges, and so begin by discussing the self-energies to one-loop order.

The contributions of lowest order to the Coulomb self-energy $\Pi_c(p,p_3)$ $(p=|\vec{p}|)$ are sketched in Figs. 7(a) and 7(b).[22] Because the virtual Coulomb particle in Fig. 7(a) is instantaneous, its contribution to Π_c is independent of the $(2+1)$-dimensional energy p_3, and so it must be a number times $N\alpha p$. It is found to be $N\alpha p/\pi$. The contribution of Fig. 7(b) at $p_3=0$ is

$$N\alpha p\left(\frac{7}{32}-\frac{1}{\pi}\right),$$

so that

$$\Pi_c(p,0)=\tfrac{7}{32}N\alpha p . \tag{3.14}$$

Thus the sign of $\Pi_c(p,0)$ would be tachyonic if valid for $p\leq N\alpha$. The value of $\Pi_c(p,0)$ is of special interest as the sole contribution to the static potential at lowest order in perturbation theory. The general function $\Pi_c(p,p_3)$ is not of use to us here, and we only note that it vanishes when p does:

$$\Pi_c(0,p_3)=0 . \tag{3.15}$$

The transverse self-energy $\Pi_t(p,p_3)$, defined as

$$\Pi_{ij}(\vec{p},p_3)=(\delta_{ij}-\hat{p}_i\hat{p}_j)\Pi_t(p,p_3) , \tag{3.16}$$

is rather easier to calculate than might first be expected. Because graphs such as Figs. 7(c) and 7(d) involve emission and reabsorption of a virtual pair of quanta—both of which are instantaneous—their contribution is of the form

$$\sim\int\frac{(p+2k)^i(p+2k)^j d^2k}{\vec{k}^2(\vec{p}+\vec{k})^2}\int dk_3 . \tag{3.17}$$

Contributions like Eq. (3.17) are directly proportional to an ultraviolet cutoff as $\int dk_3$, and so vanish identically after regularization.

We wish only to make the following point about the transverse self-energy. The contribution from Fig. 7(e) depends on both p and p_3, and is infrared finite for all p and p_3. For example, when $p=0$, it is a finite number times $N\alpha\,|p_3|$. Now consider the contribution of Fig. 7(f), which is due to the emission and reabsorption of a virtual Coulomb quantum by a transverse gluon. This graph behaves as

$$-\frac{N\alpha}{(2\pi)^3}\int\frac{(k_3+p_3)^2 d^2k\,dk_3}{(\vec{k}-\vec{p})^2(\vec{k}^2+k_3^2)} . \tag{3.18}$$

To deal with the logarithmic singularity in the integral over virtual two momenta k about $\vec{k}=\vec{p}$, we introduce an infrared regulator μ by drilling a hole of radius μ in momentum space about $\vec{k}=\vec{p}$. With this convention, Fig. 7(f) then contributes

$$\frac{N\alpha}{4\pi}\left(\frac{p_3^2}{p}+p\right)\ln\left(\frac{2p}{\mu}\right)-\frac{N\alpha}{2\pi}p , \tag{3.19}$$

which diverges as $\ln(\mu/p)$ for all values of p and p_3.

This infrared logarithm at one-loop order is at first rather surprising. However, it is just the appearance of the local operator A_3^2, which can easily be understood as follows. Since to this order the bare Coulomb field is instantaneous,

$$\langle A_3^2\rangle\sim\int dk_3\int\frac{d^2k}{\vec{k}^2}\sim\Lambda\ln\frac{\mu}{\Lambda} , \tag{3.20}$$

where Λ is an ultraviolet regulator. Hence, already at one-loop order, $\langle A_3{}^2\rangle$ develops an infrared logarithm. When we compute Fig. 7(f) as in Eq. (3.19), Λ is just replaced by p and p_3, but the $\ln\mu$ dependence persists.[23]

We do not pause to discuss the ghost self-energy, but proceed to sketch the origin of infrared logarithms at two-loop order. We concentrate on the Coulomb self-energy at $p_3=0$ because of its close relation to the static potential.

First of all, as in covariant gauges, the momentum structure of the ghost-gauge vertex ensures that the operator $|\psi_{gh}|^2$ does not appear to this order. Further, one-loop insertions for transverse gauge bosons as in Fig. 7(g) can be shown to be infrared finite in the integration over the loop momentum k. The only infrared divergence is inside the one-loop insert and that has already been identified with the appearance of $\langle A_3{}^2\rangle$ at one loop. Hence all that remains are one-loop insertions on the propagators of Fig. 7(a). An insert on the Coulomb leg of Fig. 7(a), as in Fig. 7(h), does not by power counting yield any infrared divergence [the fact that $\Pi_c(0,p_3)=0$, Eq. (3.15), must be used to show this]. The only infrared divergence occurs due to an insertion in the transverse gluon of Fig. 7(a) as in Fig. 7(i). Note that the infrared logarithm of Fig. 7(i) is directly associated with the appearance of $\vec{A}^2$, but that $A_3{}^2$ also enters indirectly through the contribution of Fig. 7(f) to the one-loop insert for the transverse gluon of Fig. 7(i). All these infrared divergences should, of course, cancel in a gauge-invariant correlation function.

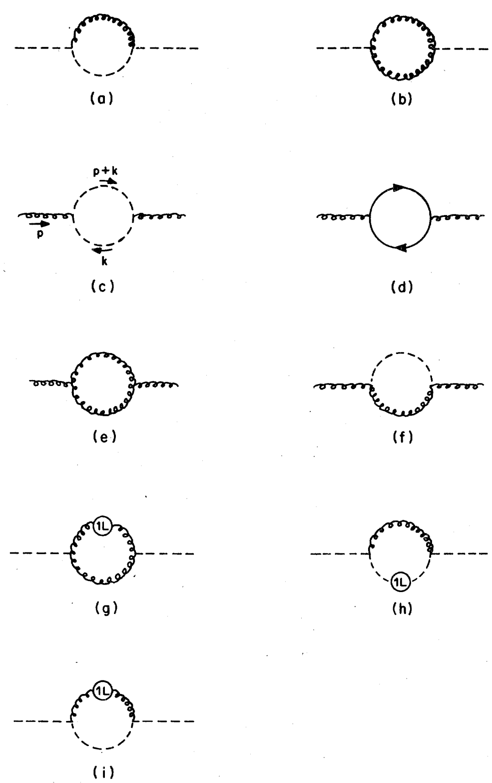

FIG. 7. Contributions to self-energies in Coulomb-gauge QCD_3. In this figure, dashed lines represent Coulomb quanta, curly lines transverse quanta, and solid lines ghost quanta. In Figs. 7(g)–7(i), 1L denotes one-loop inserts using the results of Figs. 7(a)–7(f).

We end this section with a short consideration of the Wilson loop in the Coulomb gauge.[24] It is most appropriate to consider long, thin loops, whose temporal duration in the x_3 direction τ, is much greater than the spatial extent R. For $\tau \gg R$, at lowest nontrivial order the dominant contribution to the Wilson loop is given by the exchange of a single Coulomb quanta across the loop. This term, which is of order $\alpha[\ln(R/\Lambda)]\tau$, is just the leading term for the static potential in a (2 + 1)-dimensional theory. In contrast, the exchange of a transverse quantum between the ends of the loop is of order αR and can be neglected.

Since a Wilson loop is directly related to the static potential for the (2 + 1)-dimensional theory only when $\tau \gg R$, we cannot directly carry over the remarks about the applicability of the OPE to Wilson loops as in the preceding section.

Nevertheless, it is direct to verify that infrared divergences in the static potential cancel to order α^2. The leading R-dependent term is given by the one-loop insertion for the Coulomb propagator, as in Fig. 8(a). Since the bare Coulomb field is instantaneous, to this order only $\Pi_c(p,p_3)$ at $p_3=0$ enters. This term itself is infrared divergent as

$$\int \frac{d^2p}{(\vec{p}^2)^2}\,\Pi_c(p,0) \sim \int \frac{d^2p}{|\vec{p}|^3}\,,$$

but the divergence cancels against R-independent terms as in Fig. 8(b). As mentioned at the end of Sec. III A, divergences should continue to cancel through four orders in α. The limit $\tau\to\infty$ can then be taken and the static potential can be extracted in the usual way.[24] At order α^5, large-distance quantum fluctuations may prevent the extraction of a static potential.

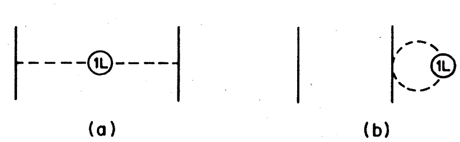

FIG. 8. Terms of order α^2 for a long, thin Wilson loop in the Coulomb gauge. The conventions are those of Fig. 7, except that the sides of the loop are drawn as solid vertical lines.

IV. CONCLUSIONS

We now summarize some of our main conclusions and avenues for further research.

First of all, the Abelian gauge model analyzed in Sec. II deserves much more study. The $1/N$ expansion, which has led to interesting and nontrivial behavior for both large and small distances, should be examined more thoroughly. A $\lambda\phi^4$ interaction can be included along with the gauge-coupling and renormalization-group methods studied. A comparison with the ϵ expansion about four dimensions should be especially interesting.

In QCD_3, the $1/N$ expansion leads only to planar-diagram dominance just as in four dimensions. Nevertheless, since spin-one particles have only one degree of freedom in three space-time dimentions, perhaps the planar-diagram approximation is more manageable.

In both QCD_3 and the Abelian model, an infrared breakdown of the loop expansion for Euclidean Green's functions has been discovered. In each case it has been connected with the appearance of new local operators in the operator-product expansion for the large-momentum Green's functions. The OPE should be studied further for these models. For example, the factorization of infrared and ultraviolet physics (the basic physical content of the OPE) should be established by examining higher orders in the loop expansion.

On a deeper level the infrared physics embodied in the vacuum expectation values of local gauge-invariant operators appearing in the OPE should be explored. Although these operators have appeared through the infrared sensitivity of the loop expansion, their computation and physical understanding surely take one beyond the loop expansion. In the Abelian model, vacuum matrix elements such as $\langle|\vec{\phi}|^2\rangle$ and $\langle F_{ij}^2\rangle$ can be studied in the $1/N$ expansion.

In QCD_3, the vacuum expectation $\langle G_{ij}^2\rangle$ is of most immediate interest. The question of its sign, for example, is important. A positive sign is anticipated by considering QCD_3 as the Euclidean form of a $(2+1)$-dimensional theory: the Euclidean term $G_{ij}^2=\frac{1}{2}(E^2+B^2)$ becomes $G_{ij}^2=\frac{1}{2}(-E^2+B^2)$ in Minkowski space. If $(2+1)$-dimensional QCD is to confine electric flux (in the fundamental representation), then it must be a magnetic condensate for which $B^2>E^2$, so $\langle G_{ij}^2\rangle>0$.[25]

We conclude with some remarks as to why there might be magnetic screening of hot non-Abelian fields if infrared softening is sufficiently strong. Infrared softening implies that the ability of the physical vacuum to sustain long-range magnetic fields is reduced by interactions. If this reduction takes place, the loop expansion suggests that it will set in over distances $\xi\gtrsim\beta/(N\alpha)$ for $SU(N)$.

We want to stress that no direct evidence for this infrared softening has emerged from the analysis of this paper. We can only study QCD_3 in the loop expansion, and thus are restricted to distances $\xi\ll\beta/(N\alpha)$. Thus we cannot compute anything about the infrared structure of QCD_3. In contrast, for the Abelian model we *were* able to analyze the theory for distances $\xi\gtrsim\beta/(N\alpha)$ using the $1/N$ expansion. In the Abelian model, we found a mild softening in that a $1/p^2$ propagator became $1/p$ in the infrared limit. No magnetic mass was found in this model of massless scalar QED_3.

Nevertheless, it is worth exploring the consequences of the assumption that in QCD_3 there *is* sufficient infrared softening to give a finite range [of order $\beta/(N\alpha)$] to the magnetic fields of hot QCD_4. Whether it is precisely of the form of a mass term or whether the analytic structure is much more involved, as is certainly likely, may not be that important. The gross qualitative features of the physics may not depend on the detailed form of the gauge-boson propagator about zero momentum.

Consider then the introduction of external magnetic charge into a hot thermal bath of Yang-Mills fields, as in the early universe. When the effects of infrared softening are sufficiently great there will only be two alternatives: screening or confinement. Screening will always occur if there are vacuum excitations with the quantum numbers of the external charge, confinement if not.

To decide between the two, we need a convenient way of introducing external sources for magnetic charge. Let us then extend the gauge group to be semisimple,[12] such as $G'\times U(1)$, at temperatures below restoration to a simple gauge symmetry. For example, consider the Georgi-Glashow SU(5) model at temperatures above ~250 GeV and below $\sim 10^{14}$ GeV. There are then massive Abelian magnetic monopoles M, with $m_M\sim 10^{16}$ GeV. The Abelian monopole M acts as a source for both Abelian [U(1)] and non-Abelian [$SU(2)\times SU(3)$] fields.

The question of magnetic screening vs confinement hinges upon whether there is a topological invariant associated with magnetic charge. However, it is easy to understand why non-Abelian magnetic charge has no topological significance. Non-Abelian magnetic charge Q_n is defined as the integral of the magnetic field B_i^a over spatial infinity as

$$Q_n\sim\int B_i^a dS_i\,.$$

Now, the magnetic charge Q_n must be linear in the magnetic field so that they both transform in the same manner under discrete symmetries such as P and T. But then the only gauge-invariant charge we can form is $\mathrm{tr}(Q_n)$, which always vanishes. In contrast, Abelian magnetic monopoles are topologically stable since, with an adjoint scalar field ϕ^a (with mass typically $\sim 10^{14}$ GeV in the Georgi-Glashow model) not zero at spatial infinity, a gauge-invariant charge can be formed as

$$Q \sim \int B_i^a \phi^a dS_i .$$

Since nothing topological prevents non-Abelian magnetic fields from being screened, it is possible for it to occur over distances $\sim \beta/(N\alpha)$. Magnetic screening can be imagined to result from a condensate of Wu-Yang monopoles, with a spatial density of order $(\beta/N\alpha)^3$. Such a monopole condensate would also generate a nonzero value for $\langle G_i{}^2\rangle$. This picture has been discussed elsewhere.[7]

But consider then what would happen if there were infrared softening for hot *Abelian* magnetic fields. In that case the only excitations which could screen the Abelian magnetic charges are virtual $M\bar{M}$ pairs. In any realistic grand unified model, enormous energies are needed to pull very massive $M\bar{M}$ pairs out of the vacuum, and Abelian magnetic monopoles would be effectively confined.

Demonstrating that the Abelian part of $G' \times \mathrm{U}(1)$ is sufficiently infrared soft is difficult. Naive power counting indicates an Abelian magnetic mass (Sec. I) could occur beginning at fifth-loop order $m_\gamma{}^2 \sim \alpha^4 T^2$. If true, the length ξ_c over which $M\bar{M}$ confinement occurs would be $\xi_c \sim \beta/\alpha^2$. Of course, at such high orders in a loop expansion a very careful analysis is required in order to be able to make definite statements.

Nevertheless, we would like to suggest the possibility that for any semisimple model such as the Weinberg-Salam model the infrared sensitivity of hot non-Abelian magnetic fields will eventually infect hot Abelian magnetic fields, in that the infrared behavior of hot Abelian fields is softened from that of free fields. The demonstration of this might provide a solution to the vexing problem of too many primordial magnetic monopoles in the early universe.[26,27]

ACKNOWLEDGMENTS

One of us (TA) would like to thank Martin Einhorn, Willy Fischler, and Michael Dine for several helpful conversations during the course of this work. R. D. P. would like to thank David Gross and Larry Yaffe for discussions at the beginning of the work, and A. Linde and Larry McLerran for sharing their understanding at the Conference on Quarks and Hadrons, Bielefield, Germany, 1980. As we were completing this investigation, we became aware of three works which overlap ours. In his lectures at Schladming, Austria, 1980, G. 't Hooft discusses several points of common interest, including perturbative infrared singularities in QCD_3 and the occurrence of nonzero vacuum expectation values for operators like $\bar{A}^2$ and $G_{ij}{}^2$ in QCD_3. R. Jackiw and S. Templeton [Phys. Rev. D 23, 2291 (1981)] have examined fermionic QED_3 and QCD_3 in the loop expansion. Their computational results for QCD_3 are in agreement with ours. We are grateful to Roman Jackiw for keeping us informed of their preliminary results and for a very stimulating discussion here at Yale. J. Schonfeld [Fermilab report, 1980 (unpublished)] has examined gauge theories in 2 + 1 dimensions, considering especially the effects of a gauge-invariant masslike term proportional to $A_i B_i$. This research was supported in part by the U. S. Department of Energy under contract No. EY-76-C-02-3075.

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[1]D. A. Kirzhnits and A. D. Linde, Phys. Lett. 42B, 471 (1972); L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974); S. Weinberg, *ibid.* 9, 3357 (1974); D. A. Kirzhnits and A. D. Linde, Ann. Phys. (N.Y.) 101, 195 (1976).

[2]A. D. Linde, Rep. Prog. Phys. 42, 389 (1979). This reference also contains some remarks about the infrared problem of finite-temperature Yang-Mills theories. They have been emphasized in a recent paper by the same author [Phys. Lett. 96B, 289 (1980)]. These remarks provided much of the stimulus for the present investigation which, we believe, has revealed even more infrared structure than previously suspected.

[3]A. M. Polyakov, Phys. Lett. 72B, 477 (1978).

[4]L. Susskind, Phys. Rev. D 20, 2610 (1979).

[5]T. Banks and E. Rabinovici, Nucl. Phys. B160, 349 (1979).

[6]L. McLerran and B. Svetitsky, Phys. Lett. 98B, 195 (1981); J. Kuti, J. Polónyi, and K. Szlachányi, *ibid.* 98B, 199 (1981).

[7]D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. 53, 43 (1981).

[8]For a short introduction to Ref. 7 and other aspects of finite-temperature QCD, see R. D. Pisarski, Yale report, 1980 (unpublished).

[9]The deconfining phase transition in quarkless QCD is precisely defined by the ability of the vacuum of

$SU(3)/Z_3$ gluons to support electric Z_3 flux from external sources in the fundamental representation. At zero temperature, confinement results from the infinite self-energy of electric Z_3 flux, while above the critical temperature Z_3 flux has finite self-energy in a thermal bath of gluons. In a realistic theory, Z_3 flux is screened by finite mass quarks for all temperatures. The physical phase transition in QCD is associated with the restoration of chiral symmetry for quarks with zero bare mass. By the arguments of Sec. I, since fermions decouple along with finite-energy modes, the critical temperature for the chiral phase transition in QCD, T_c^{ch}, must be on the order of a mass scale for the zero-temperature theory. Taking a typical mass of zero-temperature QCD as the renormalization mass scale, we estimate $T_c^{ch} \sim 200-400$ MeV.

[10]For a further discussion of this point see C. Bernard, Phys. Rev. D 9, 3312 (1974) and in particular Ref. 7.

[11]T. Appelquist and J. Carazzone, Phys. Rev. D 11, 2856 (1975).

[12]We shall only consider simple gauge groups. The extension of our results to semisimple groups G of the form $G = G_1 \times G_2 \times \cdots$ is straightforward if the $G_1, G_2, \ldots$ are all Abelian or all non-Abelian. The physically interesting case of $G = G' \times U(1)$, with G' not Abelian, is briefly discussed at the end of Sec. IV.

[13]The temperature-dependent mass is uniquely defined by requiring it to vanish at zero temperature. This also ensures that the mass is free of ultraviolet divergences.

[14]Positivity for all contributions to $m_S{}^2$ is only true when contributions from all finite-energy modes are included.

[15]E. S. Fradkin, Proc. Lebedev Phys. Inst. 29, 7 (1967).

[16]Strictly speaking, the photon is not free in the infrared limit since Eq. (1.5) represents a finite wave-function renormalization, as a power series in $\sqrt{\alpha}$. However, this is rather innocuous since $\Pi(\bar{p}^2)/\bar{p}^2$ is always regular about $\bar{p}^2 = 0$.

[17]The requirement that the renormalized scalar mass vanish is not unphysical, occurring in finite-temperature scalar QED exactly at the critical temperature (if the phase transition is second order, assuming of course that the scalar field develops a nonzero vacuum expectation value at zero temperature). Indeed, consider the phase transition for the evaporation of the Higgs effect for an arbitrary scalar field ϕ coupled to an arbitrary gauge group. From the discussion of Sec. I, at the critical point the infrared behavior is that of a three-dimensional gauge theory with (massless) scalars ϕ. The critical exponents for this effective theory at $T = T_c$ can be calculated by an ϵ expansion about $4-\epsilon$ dimensions, setting then $\epsilon = 1$. This has been carried out by P. Ginsparg [Nucl. Phys. B (to be published)]. Ginsparg interprets the presence of an infrared-stable fixed point in the ϵ expansion as indicative of a second-order phase transition, and the absence of a stable fixed point as evidence of a weakly first-order phase transition. However, Kirzhnits and Linde (Ref. 1, 1976) and Linde [A. D. Linde, Lebedev Phys. Inst. report, 1980 (unpublished)] have argued that mean-field theory in the scalar field ϕ indicates that the Weinberg-Salam and grand-unified phase transitions are strongly first order.

[18]From the comments of Ref. 17, massless scalar QED_3 should model the critical behavior of a (type-I) superconductor. Our results to leading order confirm the expected result that the phase transition is second order as $N \to \infty$ [B. I. Halperin, T. C. Lubensky, and S. K. Ma, Phys. Rev. Lett. 32, 292 (1974)]. In this context, we are struck by the presence of logarithmic corrections in the $1/N$ expansion at the critical point.

[19]We neglect any induced interactions as $(|\vec{\phi}|^2)^2$, which should enter at $\sim O(\alpha^2)$. A careful justification of this neglect would require a detailed renormalization-group analysis. However, in this work our interest in massless scalar QED_3 is simply as a model of QCD_3, and so we ignore such subtleties.

[20]G. 't Hooft, Nucl. Phys. B72, 461 (1974).

[21]For a recent effort to creep above two dimensions by studying QCD in $2+\epsilon$ dimensions, see M. Dine, C. Litwin, and L. McLerran, Phys. Rev. D 23, 451 (1981).

[22]In this section we use dimensional regularization and thus neglect all tadpole graphs.

[23]We should remark that a graph like Fig. 7(e) occurs as well for a two-dimensional Yang-Mills theory (QCD_2) in the Landau gauge. In both Coulomb-gauge QCD_3 and Landau-gauge QCD_2, Fig. 7(e) is infrared finite. On the other hand, Fig. 7(f) is peculiar to Coulomb-gauge QCD_3, with no counterpart in Landau-gauge QCD_2.

[24]For a perturbative analysis of the Wilson loop in Coulomb-gauge QCD_4, see T. Appelquist, M. Dine, and I. J. Muzinich, Phys. Lett. 69B, 231 (1977); Phys. Rev. D 17, 2074 (1978).

[25]Presumably, if the condensate in 2+1 QCD were electric rather than magnetic in which $B^2 < E^2$ and $\langle G_{ij}{}^2 \rangle > 0$, then magnetic rather than electric Z_3 flux would be confined.

[26]Ya. B. Zeldovich and M. Yu. Khlopov, Phys. Lett. 79B, 239 (1978); J. P. Preskill, Phys. Rev. Lett. 43, 1365 (1979); A. H. Guth and S.-H. H. Tye, *ibid.* 44, 631 (1980); M. B. Einhorn, D. L. Stein, and D. Toussaint, Phys. Rev. D 21, 3295 (1980).

[27]A. D. Linde [Phys. Lett. 96B, 293 (1980)] has argued that a non-Abelian magnetic mass confines rather than screens $M\bar{M}$ pairs. Although we disagree with this conclusion, his numbers are interesting: for the adiabatic invariant r, r = (monopole/photon) number, with $\xi_c \sim \beta/\alpha$ he finds at present $r \sim 10^{-10^{70}}$ [presented at the Conference on Quarks and Hadrons, Bielefield, Germany, 1980 (unpublished)]. With $\xi_c \sim \beta/\alpha^2$ as from an Abelian magnetic mass, preliminary estimates give $r \sim \exp(-10^5)$ (R. D. P., unpublished). This is to be compared with Preskill's bound (Ref. 26) that by the time of helium synthesis r should be $\lesssim 10^{-19}$. The point is that if $M\bar{M}$ confinement occurs even at a high order in the loop expansion, M and $\bar{M}$'s become correlated over large distances, and the effects on the evolution of primordial magnetic monopoles in the early universe are important.