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How super-renormalizable interactions cure their infrared divergences

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Perturbative expansions in models which possess super-renormalizable interactions of massless fields are beset by severe infrared divergences. We show that the complete theory is well defined and has no such divergences; rather the exact amplitudes are nonanalytic functions of the coupling constant and cannot be expanded in its powers. Typically, logarithms of the coupling constant occur, as well as analytic pieces. The analytic portions cannot be found in perturbation theory; they are determined by matrix elements of composite operators. But the nonanalytic behavior is completely fixed in terms of the theory's other parameters. The present investigation should be relevant to a study of physical quantum chromodynamics at its finite-temperature phase transitions.

I. INTRODUCTION

Super-renormalizable interactions are governed by coupling constants with dimensions of positive powers of mass. When the fields are also massless, perturbative expansions in the super-renormalizable coupling lead to infrared divergences, even in off-mass-shell amplitudes, for the following reason. Computation of any Green's function to some (high) order yields a formula which must involve, for dimensional reasons, a (high) power of the coupling constant, divided by a (high) power of a momentum variable, characteristic of the process in question. Upon inserting this result into some further diagram and attempting a further momentum integration, infrared divergences will in general be encountered, as a consequence of the momenta in the denominator.

In this paper we examine how these infrared singularities are healed in several interesting models; how perturbation theory must be modified in order to avoid them. The result is that the dimensional coupling constant provides an infrared cutoff and is a source of nonanalyticity in various amplitudes. An expansion in the coupling is possible, but logarithms of the coupling constant occur, in addition to powers. The coefficients of the leading nonanalytic logarithms are calculable in conventional perturbation theory, but the coefficients of the analytic powers require nonperturbative information.

Super-renormalizable field theories arise in studies of symmetry-changing phase transitions at finite temperature.[1] When one examines a physical theory in four space-time dimensions and at a finite temperature close to the critical temperature, a three-dimensional field theory becomes effective in the description of dynamics near the transition. Moreover, at high temperatures, even away from a critical point, the infrared behavior of any theory is described by the same theory in one lesser dimension. If the four-dimensional theory is governed by renormalizable, dimensionless coupling constants, those of the effective three-dimensional theory are super-renormalizable and dimensionful. They are related to the physical ones by inverse powers of the temperature.[2] We shall disregard this physical context for super-renormalizable models, and shall study them in their own right, as interesting examples of quantum field-theoretical phenomena.

When perturbation theory is beset by infrared divergences of the type discussed here, a very simple mechanism may intervene to heal the infrared singularities: The field may acquire a mass in perturbation theory through a tadpole mechanism. The mass provides an infrared cutoff, and this is a trivial solution to the problem, which happens, for example, in theories with scalar fields. We are interested in the more interesting situation where a mass cannot be generated in perturbation theory because it is prohibited by a symmetry or by some other principle. Prime instances are gauge theories, where gauge invariance prevents the gauge fields from acquiring a mass. Also massless spinors will not generate a perturbative mass term. Even scalar fields at the phase transition can have no mass, since the critical temperature is defined as precisely that temperature at which the tadpole-induced, temperature-dependent contribution to the mass cancels the bare, zero-temperature mass. (There may of course be nonperturbative mass generation by dynamical symmetry breaking. We have nothing to say about this, and ignore it henceforth.)

In the interesting situation wherein a perturbatively produced mass term does not heal the infrared singularities, another mechanism operates, whereby coupling-constant-dependent logarithms replace those that are infrared divergent. This was noted many years ago[3] but was not publicized at the time, since no physical setting was envisioned; a situation which has changed with the advent of gauge theories and finite-temperature field theory. The emergence of coupling-constant logarithms is somewhat analogous to bound-state perturbation theory; $\alpha \ln\alpha$ contributions to the Lamb shift are familiar.[4] Also it is similar to chiral perturbation theory, where logarithms of the chirality-breaking parameter are encountered.[5,6]

In Sec. II we present an integral equation for a scalar amplitude, in a highly truncated four-dimensional field theory, with super-renormalizable cubic couplings.[3] Although not particularly realistic, the example provides a simple setting for the effects that we wish to discuss: emergence of coupling-constant logarithms with coefficients that are perturbatively calculable, and analytic terms which require nonperturbative information. The results are relevant to realistic models which are studied in the following sections. In Sec. III we analyze three-dimensional massless spinor electrodynamics (QED),[7] and in Sec. IV, three-dimensional Yang-Mills theory in interaction with massless fermions, quantum chromodynamics (QCD).[8] The latter, which should be relevant to the physical quantum-chromodynamical theory at the deconfining phase transition,[9] follows the behavior of the former; but differences are present which reflect the greater complexity in the infrared of the non-Abelian interactions.

II. A SIMPLE SCALAR MODEL

Consider the following integral equation (in Minkowski space)[3]:

$$\Gamma(p^2) = 1 \mp ig^2 \int \frac{d^4r}{(2\pi)^4} \frac{\Gamma(r^2)}{[(r-p)^2+i\epsilon](r^2+i\epsilon)^2} . \tag{2.1}$$

Equation (2.1) cannot be solved perturbatively, because the first iteration is infrared divergent. With an infrared cutoff in the double propagator, one gets

$$\begin{aligned}\Gamma(p^2) &= 1 \mp ig^2 \int \frac{d^4r}{(2\pi)^4} \frac{1}{[(r-p)^2+i\epsilon](r^2-\mu^2+i\epsilon)^2} + O(g^4) \\ &= 1 \mp \frac{g^2}{16\pi^2(-p^2-i\epsilon)} \ln\left(1-\frac{p^2+i\epsilon}{\mu^2}\right) + O(g^4) .\end{aligned} \tag{2.2}$$

Nevertheless, a solution exists for (2.1).

The integral equation is here presented as a mathematical model for the phenomenon that will concern us in the remainder of the paper. With the positive sign, it is a highly truncated Bethe-Salpeter equation for the Fourier transform of the amplitude

$$\begin{aligned} F(x) &= \left\langle 0 \middle| T\varphi(x)\varphi(0) \int d^4z : \tfrac{1}{2}\varphi^2(z): \middle| 0 \right\rangle_C , \\ \int d^4x\, e^{ipx} F(x) &= -\frac{\Gamma(p^2)}{(p^2+i\epsilon)^2} \end{aligned} \tag{2.3}$$

(C = connected part) in a massless, scalar, four-dimensional field theory with a $(g/3!)\phi^3$ interaction. The truncation consists of ignoring quartic couplings which are necessary to stabilize the cubic interaction, dropping tadpoles which give rise to a mass, and keeping only the lowest-order ladder in the Bethe-Salpeter kernel. Consequently, (2.1) is a very unrealistic approximation, but it is useful for us as an example of the infrared-curing mechanism, which we shall explore for realistic models in Secs. III and IV. Note that the cubic coupling is super-renormalizable; g carries dimension of mass. Graphically (2.1) is represented in Fig. 1.

Equation (2.1) may be easily solved, since it is of the Volterra type. Upon recognizing that $[(r-p)^2+i\epsilon]^{-1}$ is a Green's function for the d'Alembertian with respect to p, we may convert (2.1) to a differential equation by operating with $\Box_p$. Equivalently, we may rotate to Euclidean space, perform the angular integrations in (2.1), and find a one-variable equation:

$$\begin{aligned} f(x) &= 1 \mp \int_0^x dy\, f(y) \mp x \int_x^\infty \frac{dy}{y} f(y) , \\ \Gamma(-p^2) &= f\left(\frac{g^2}{16\pi^2 p^2}\right) . \end{aligned} \tag{2.4}$$

Perturbation theory now corresponds to solving (2.4) by a power series in x, a procedure which again yields logarithmic divergences. But the differential equation which follows from (2.4),

$$f''(x) \mp \frac{f(x)}{x} = 0 , \tag{2.5}$$

has well-behaved solutions. We discuss the two cases, corresponding to the two signs, separately.

A. Negative sign

For the negative sign, the solution to (2.5) involves modified Bessel functions and two constants:

$$f(x) = A 2\sqrt{x} K_1(2\sqrt{x}) + B 2\sqrt{x} I_1(2\sqrt{x}) . \tag{2.6}$$

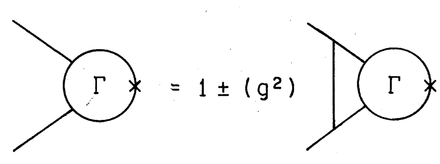

FIG. 1. Graphical representation of Eq. (2.1). The × depicts an insertion of $\int d^4z : \frac{1}{2}\phi^2(z):$.

However, since I_1 grows exponentially at large x, it would produce a divergence in the integral equation; hence $B=0$. Also, $A=1$, since (2.4) fixes f at the origin:

$$f(0)=1 . \tag{2.7}$$

Thus the integral equation is solved by

$$\begin{aligned}\Gamma(-p^2)&=\left(\frac{g^2}{4\pi^2p^2}\right)^{1/2}K_1\left(\left(\frac{g^2}{4\pi^2p^2}\right)^{1/2}\right)\\&=1+\frac{g^2}{16\pi^2p^2}\ln\frac{g^2}{16\pi^2p^2}\\&\quad+\frac{g^2}{16\pi^2p^2}(2\gamma-1)+O(g^4)\,,\end{aligned} \tag{2.8}$$

where γ is Euler's constant.

Note that the infrared divergence has disappeared and the infrared cutoff in (2.2) has been replaced by the coupling constant

$$\mu^2\to g^2/16\pi^2 . \tag{2.9}$$

Indeed the coefficient of the nonanalytic piece, viz., of the logarithm in (2.8), is exactly the same as that of the infrared-divergent logarithm in (2.2). But the analytic, nonlogarithmic $O(g^2)$ contribution to (2.8) cannot be found in (2.2).

We may understand why perturbation theory cannot yield the nonlogarithmic terms from a further examination of the equations. Because the integral in (2.3) ranges to infinity, a power series, even when modified by logarithms, cannot be used for f. However, the differential equation (2.5) allows an expansion about $x=0$. The boundary condition at the origin, (2.7), gives rise to a logarithmic singularity $\sim x\ln x$; so we try

$$\begin{aligned}&f(x)=f_1(x)+f_2(x)\ln x\,,\\&f_1(0)=1,\quad f_2(0)=0\,.\end{aligned} \tag{2.10}$$

The two functions $f_{1,2}$ can be expanded in powers of x. When this expansion is substituted into (2.5), one finds that f_2 is completely determined, but the differential equation leaves $f_1'(0)$ arbitrary. Of the two boundary conditions on f, needed for a unique solution, one is given at the origin and can be incorporated in the perturbation series. The second, which follows from (2.4), is

$$\lim_{x\to\infty}f'(x)=0 . \tag{2.11}$$

This condition is not at $x=0$; it cannot be incorporated in the expansion of f about $x=0$. In other words, the perturbative solution leads to a family parametrized by $f_1'(0)$, but only for one value of this constant is (2.11) satisfied. This quantity is not locally determined at $x=0$, but requires knowledge of the complete solution for all x.

The following sum rule, which is easily derived from (2.4), (2.5), and (2.10), shows this:

$$\begin{aligned}f_1'(0)&=-1-\int_0^\infty\frac{dx}{x}[f(x)-\theta(1-x)]\\&=-1+\int_0^\infty dx\,\ln x f'(x)\,.\end{aligned} \tag{2.12}$$

In conclusion, let us rewrite the integral equation in a way that permits a direct perturbative solution, with one undetermined constant. We introduce an arbitrary scale in (2.4):

$$\begin{aligned}f(x)&=1-\int_0^x dy\,f(y)-x\int_x^\Lambda\frac{dy}{y}f(y)\\&\quad-x\int_\Lambda^\infty\frac{dy}{y}f(y)\,.\end{aligned} \tag{2.13a}$$

Guided by the "renormalization-group" analysis of the infrared behavior in the Bloch-Nordsieck model,[2] we want to send Λ into the ultraviolet region (small Λ). But this would produce ultraviolet divergences in the last two integrals of (2.13a). To overcome the problem, we rewrite (2.13a) as

$$\begin{aligned}f(x)&=1-\int_0^x dy\,f(y)-x\int_x^\Lambda\frac{dy}{y}[f(y)-1]\\&\quad+x\ln x-x\left[\ln\Lambda+\int_\Lambda^\infty\frac{dy}{y}f(y)\right].\end{aligned} \tag{2.13b}$$

Since Λ is arbitrary we may now set it to zero, and derive

$$\begin{aligned}[f(x)-1]&=x\ln x+xf_1'(0)-\int_0^x dy[f(y)-1]\\&\quad+x\int_0^x\frac{dy}{y}[f(y)-1]\,.\end{aligned} \tag{2.14}$$

With the exception of one constant, the equation can be solved iteratively by expanding $f-1$ in powers of x^n and $x^n\ln x$.

B. Positive sign

With the positive sign in (2.1), which corresponds to the field-theoretic model, the situation differs in that both of the differential equation's solutions, which are now ordinary Bessel functions,

$$f(x)=A2\sqrt{x}\,N_1(2\sqrt{x})+B2\sqrt{x}\,J_1(2\sqrt{x})\,, \tag{2.15}$$

oscillate at infinity and lead to a convergent integral. Thus while the boundary condition at the origin fixes A at $-\pi/2$, B remains undetermined, since $2\sqrt{x}\,J_1(2\sqrt{x})$ vanishes at the origin, and its derivative vanishes at infinite x. One must go outside the truncated Bethe-Salpeter framework to fix the other constant.

The sum rule for the unknown quantity

$$f_1'(0)=1+\int_0^\infty \frac{dx}{x}\,[f(x)-\theta(1-x)] \tag{2.16a}$$

may be recast in terms of the field-theoretical amplitude. We return to the original notation and rewrite (2.16a) as

$$f_1'(0)=1+\frac{1}{\pi^2}\int \frac{d^4p}{p^4}\,\Gamma(-p^2)\Big|_{\rm reg}\,. \tag{2.16b}$$

Here "reg" indicates that an ultraviolet regulation [the step function in (2.16a)] should be inserted. [There is no infrared divergence, since $\Gamma(0)=f(\infty)=0$.] In Minkowski space (2.16b) is reexpressed by (2.3) to give

$$\begin{aligned} f_1'(0)&=1-8\pi^2 i\int\frac{d^4p}{(2\pi)^4}\frac{\Gamma(p^2)}{(p^2+i\epsilon)^2}\Big|_{\rm reg}\\ &=1+8\pi^2 i\left\langle 0\left|T\varphi^2(0)\int d^4z\,{:}\tfrac{1}{2}\varphi^2(z){:}\right|0\right\rangle_C\Big|_{\rm reg}\,. \end{aligned} \tag{2.17}$$

Evidently, the nonperturbative information, needed for the matrix element of $\phi(x)\phi(0)$, is contained in the matrix element of $\phi^2(0)$.[10]

C. Summary

The lessons to be drawn from the above exercise are the following. Infrared divergences arising from super-renormalizable interactions can be cured, even when masses are not generated. It is necessary to allow for coupling-constant logarithms in the perturbative expansion. The coefficients of these logarithms are found in perturbation theory; however, there remain terms, not involving logarithms, that are not computable perturbatively. Rather they are determined by matrix elements of composite operators. All this also happens in more realistic theories, to which we now turn.

III. THREE-DIMENSIONAL SPINOR ELECTRODYNAMICS

We consider massless fermions ψ interacting with a massless Abelian gauge field A_μ in three space-time dimensions (QED),

$$\begin{aligned} \mathcal{L}&=i\bar\psi\gamma^\mu(\partial_\mu-ieA_\mu)\psi-\tfrac14 F^{\mu\nu}F_{\mu\nu}\,,\\ F_{\mu\nu}&=\partial_\mu A_\nu-\partial_\nu A_\mu\,. \end{aligned} \tag{3.1}$$

The square of the coupling constant e has dimensions of mass; the interaction is super-renormalizable. In three space-time dimensions the Dirac matrices can be chosen to be the 2×2 Pauli matrices

$$\gamma^0=\sigma^3,\quad \gamma^1=i\sigma^1,\quad \gamma^2=i\sigma^2, \tag{3.2}$$

and ψ is a two-component spinor.

There does not appear in $\mathcal{L}$ a fermion mass $-m\bar\psi\psi$, nor a photon mass, which in three dimensions can have a gauge-invariant form: $\frac14\mu\epsilon^{\alpha\beta\gamma}F_{\alpha\beta}A_\gamma$. We must ensure that such masses do not arise in perturbation theory. Note there is no chiral symmetry for the massless fermions, since no matrix anticommutes with all the Dirac (Pauli) matrices.

In three dimensions the situation is somewhat unexpected and requires discussion. In fact, neither mass will arise perturbatively, if it is absent from the Lagrangian, because both violate P and T invariance. Correspondingly, if one is inserted in the Lagrangian, the other will be induced by radiative corrections. To see this, let us first recall that in two spatial dimensions, parity corresponds to inverting one axis, say the x axis. (Inverting both would be a rotation.) One verifies that the theory (3.1) is invariant under the following parity transformation P:

$$\begin{aligned} \mathcal{P}\psi(t,\vec{x})\mathcal{P}^{-1}&=\sigma_1\psi(t,\vec{x}')\,,\\ \mathcal{P}A^0(t,\vec{x})\mathcal{P}^{-1}&=A^0(t,\vec{x}')\,,\\ \mathcal{P}A^1(t,\vec{x})\mathcal{P}^{-1}&=-A^1(t,\vec{x}')\,,\\ \mathcal{P}A^2(t,\vec{x})\mathcal{P}^{-1}&=A^2(t,\vec{x}')\,,\\ \vec{x}=(x,y),&\quad \vec{x}'=(-x,y)\,. \end{aligned} \tag{3.3}$$

Also time inversion T is a symmetry,

$$\begin{aligned} \mathcal{T}\psi(t,\vec{x})\mathcal{T}^{-1}&=\sigma_2\psi(-t,\vec{x})\,,\\ \mathcal{T}A^0(t,\vec{x})\mathcal{T}^{-1}&=A^0(-t,\vec{x})\,,\\ \mathcal{T}\vec{A}(t,\vec{x})\mathcal{T}^{-1}&=-\vec{A}(-t,\vec{x})\,. \end{aligned} \tag{3.4}$$

It is now easy to check that both the fermion and gauge-field mass terms are odd under P and T.[11]

To begin our study of the infrared structure, we compute first-order corrections to the fermion and gauge-field propagators:

$$\mathcal{D}_{\mu\nu}(p)=\int d^3x\,e^{ipx}\langle 0|TA_\mu(x)A_\nu(0)|0\rangle\,, \tag{3.5a}$$

$$\mathcal{S}(p)=\int d^3x\,e^{ipx}\langle 0|T\psi(x)\bar\psi(0)|0\rangle\,. \tag{3.5b}$$

The self-energies are defined by

$$\mathcal{D}^{-1}{}_{\mu\nu}=iP_{\mu\nu}[p^2-\Pi(p^2)]+\frac{i}{\alpha}p_\mu p_\nu\,, \tag{3.6a}$$

$$\begin{aligned} P_{\mu\nu}&=g_{\mu\nu}-p_\mu p_\nu/p^2\,,\\ \mathcal{S}^{-1}(p)&=\frac{1}{i}[\not p-\Sigma(p)]\,. \end{aligned} \tag{3.6b}$$

We shall always work in a class of covariant gauges, parametrized by the constant α, and we shall describe our results as gauge invariant when they are α independent. The lowest-order formulas for $\Pi_{\mu\nu}(p)=p_{\mu\nu}\Pi(p^2)$ and $\Sigma(p)$ are

$$\Pi_{\mu\nu}(p) = -ie^2 \int \frac{d^3k}{(2\pi)^3} \operatorname{tr}\gamma_\mu S(p+k)\gamma_\nu S(k) + O(e^4)\,, \tag{3.7a}$$

$$\Sigma(p) = -ie^2 \int \frac{d^3k}{(2\pi)^3} \gamma^\mu S(p+k)\gamma^\nu D_{\mu\nu}(k) + O(e^4)\,, \tag{3.7b}$$

where $D_{\mu\nu}$ and S are the free propagators,

$$D_{\mu\nu}(p) = \frac{-i}{p^2+i\epsilon} P_{\mu\nu} - \frac{i\alpha}{(p^2+i\epsilon)^2} p_\mu p_\nu\,, \tag{3.8a}$$

$$S(p) = \frac{i}{\not p}\,. \tag{3.8b}$$

The integrals are elementary; no infrared divergences are encountered. In spite of superficial ultraviolet divergences, they too are absent, by symmetric integration in the fermion case and by gauge invariance in the gauge-field case. (In this simple evaluation no regulators are needed, but if they are used, one must respect the masslessness of the fermion; otherwise a mass for the gauge field will be generated.)

The results are

$$\Pi(p^2) = \frac{e^2}{16} (-p^2 - i\epsilon)^{1/2} + O(e^4)\,, \tag{3.9a}$$

$$\Sigma(p) = -\frac{e^2\alpha}{16} \frac{\not p}{(-p^2 - i\epsilon)^{1/2}} + O(e^4)\,. \tag{3.9b}$$

The gauge-dependent fermion correction vanishes in the Landau gauge ($\alpha = 0$); the gauge-invariant vacuum polarization is positive for spacelike momenta, as it should be.

Next we attempt to calculate $O(e^4)$ terms. Of the several relevant graphs, the only ones which are potentially infrared divergent involve the insertion of one of the above self-energies into an internal propagator. The problematical graphs are depicted in Figs. 2 and 3, with wavy lines representing gauge-field propagators and solid lines representing fermion propagators. One finds the insertions into fermion lines [Figs. 2 and 3(i)] to be innocuous. The reason is that $\Sigma(p)$ for small p is $O(1)$; the two attached fermion propagators add a factor $1/p^2$, but the three-dimensional phase space can overcome this singularity. However, the fermion bubble in the gauge-field propagator [Fig. 3 (ii)] produces a divergence: $\Pi(p^2)$ is $O(p)$; the two attached photon propagators are $O(1/p^4)$, and the $O(1/p^3)$ singularity gives rise to a logarithmic divergence in the integral over loop momenta. We conclude that to $O(e^4)$, the gauge-field propagator remains finite, but that of the fermion acquires a logarithmic divergence, which we now show is cured by a nonanalytic $e^4 \ln e^2$ term.

We need the complete equations that determine

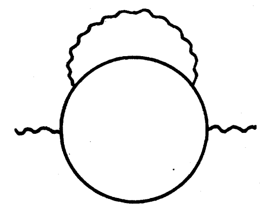

FIG. 2. Potentially infrared-divergent two-loop contribution to gauge-field self-energy.

$\mathcal{D}_{\mu\nu}$ and $\mathcal{S}$. The appropriate formalism is the one that expresses these objects as functionals of themselves. This procedure is well known: one writes the vacuum functional in terms of $\mathcal{D}_{\mu\nu}$ and $\mathcal{S}$, keeping only two-particle-irreducible graphs. The equations for the propagators follow by demanding that the variation of the functional with respect to the propagators vanishes.[12]

Of course there is no hope of solving the equations exactly; we shall be content merely to determine the $O(e^4)$ logarithms. To this end we need only keep truncated equations,

$$\mathcal{D}^{-1}{}_{\mu\nu}(p) = iP_{\mu\nu}p^2 - e^2 \int \frac{d^3k}{(2\pi)^3} \operatorname{tr}\gamma_\mu \mathcal{S}(p+k)\gamma_\nu \mathcal{S}(k) + \frac{i}{\alpha} p_\mu p_\nu + O(e^4)\big|_{\text{reg}}\,, \tag{3.10a}$$

$$\mathcal{S}^{-1}(p) = \frac{1}{i}\not p + e^2 \int \frac{d^3k}{(2\pi)^3} \gamma^\mu \mathcal{S}(p+k)\gamma^\nu \mathcal{D}_{\mu\nu}(k) + O(e^4)\big|_{\text{reg}}\,. \tag{3.10b}$$

Here $O(e^4)|_{\text{reg}}$ represents contributions that are regular to $O(e^4)$. The omitted terms do give rise to logarithms, but only in terms $O(e^6)$ and higher. [Observe a significant difference from the toy model of the previous section: There we find a single power of the logarithm; now because of nonlinearities, the $O(e^4)$ logarithm fuels higher logarithms in higher orders. Thus in $O(e^8)$ there is an $\ln^2 e^2$ term as well as $\ln e^2$. It is likely that all the leading logarithms can be explicitly calculated, but we have not done this here.[13]] To $O(e^4)$, the insertions into fermion lines are innocuous; hence, on the right-hand side of (3.10) we may replace $\mathcal{S}$ by S, its free-field part. Thus we arrive at the completely simplified equations,

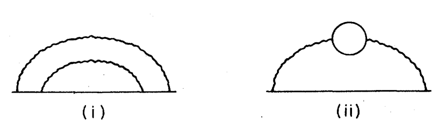

FIG. 3. Potentially infrared-divergent two-loop contributions to fermion self-energy.

$$\mathfrak{D}^{-1}{}_{\mu\nu}(p)=ip^2P_{\mu\nu}-e^2\int\frac{d^3k}{(2\pi)^3}\,\mathrm{tr}\gamma_\mu S(p+k)\gamma_\nu S(k)+\frac{i}{\alpha}p_\mu p_\nu+O(e^4)\big|_{\mathrm{reg}},\tag{3.11a}$$

$$\mathcal{S}^{-1}(p)=\frac{1}{i}\not p+e^2\int\frac{d^3k}{(2\pi)^3}\gamma^\mu S(p+k)\gamma^\nu\mathfrak{D}_{\mu\nu}(k)+O(e^4)\big|_{\mathrm{reg}}.\tag{3.11b}$$

These are of course trivial to solve. From the lowest-order result (3.9a), we have an improved formula for the photon propagator:

$$\mathfrak{D}_{\mu\nu}(p)=-iP_{\mu\nu}\left[p^2+i\epsilon-\frac{e^2}{16}(-p^2-i\epsilon)^{1/2}\right]^{-1}-\frac{i\alpha p_\mu p_\nu}{(p^2+i\epsilon)^2}+O(e^4)\big|_{\mathrm{reg}}.\tag{3.12}$$

To evaluate $\mathcal{S}^{-1}(p)$, we merely need to insert (3.12) into (3.11b). Although the resulting integral is elementary, we prefer to proceed in a more deliberate fashion, in order to control the relevant momentum scales. It is useful to represent the improved photon propagator in a spectral form

$$\mathfrak{D}_{\mu\nu}(p)=-iP_{\mu\nu}\int_0^\infty d\mu\,\rho(\mu^2)\frac{1}{p^2-\mu^2+i\epsilon}-\frac{i\alpha p_\mu p_\nu}{(p^2+i\epsilon)^2}.\tag{3.13}$$

The spectral function for (3.12) is

$$\rho(\mu^2)=\frac{e^2}{8\pi}\frac{1}{\mu^2+(e^2/16)^2}.\tag{3.14}$$

Note that as a consequence of Lehmann's theorem, it is normalized to unity,

$$\int_0^\infty d\mu\,\rho(\mu^2)=1.\tag{3.15a}$$

Also its asymptotic form for $\mu\gg e^2$ is

$$\rho(\mu^2)\to\frac{e^2}{8\pi}\frac{1}{\mu^2}.\tag{3.15b}$$

Equation (3.13) shows that for the evaluation of (3.11b) we are instructed to compute a self-mass $\Sigma(p;\mu)$ with an infrared cutoff μ, and to integrate over the cutoff with the spectral function providing the relevant measure

$$\mathcal{S}^{-1}(p)=\frac{1}{i}\not p\left[1+\frac{e^2\alpha}{16(-p^2-i\epsilon)^{1/2}}-\sigma(p^2)\right]+O(e^4)\big|_{\mathrm{reg}},\tag{3.16a}$$

$$\not p\sigma(p^2)=\int_0^\infty d\mu\,\rho(\mu^2)\Sigma(p;\mu)=\Sigma(p;0)+\int_0^\infty d\mu\,\rho(\mu^2)[\Sigma(p;\mu)-\Sigma(p;0)],\tag{3.16b}$$

$$\Sigma(p;\mu)=-e^2\int\frac{d^3k}{(2\pi)^3}\gamma^\mu S(p+k)\gamma^\nu P_{\mu\nu}\frac{1}{k^2-\mu^2+i\epsilon}.\tag{3.16c}$$

Once $\Sigma(p;\mu)$ has been determined from (3.16c), the remaining task is to evaluate the integral in (3.16b).

An exact integration requires knowledge of $\rho(\mu^2)$ for all μ. Although our expression (3.14) provides a possible formula for $\rho(\mu^2)$, it is in fact unreliable for $\mu\lesssim e^2$, since in that region we may expect higher orders in e^2 to contribute. Fortunately, because we are interested only in nonanalytic $O(e^4)$ terms, involving logarithms of e^2, we may approximate (3.16b) in such a way that only reliable information about $\rho(\mu^2)$ is used. At the same time it will become obvious that a determination of the analytic $O(e^4)$ contribution needs the exact spectral function, i.e., the exact photon propagator. In other words, all orders of perturbation theory must be summed before the analytic term is known to $O(e^4)$.

Let us begin the evaluation of (3.16b) by leaving ρ unspecified, except for its form, which follows from dimensional analysis,

$$\rho(\mu^2)=\frac{e^2}{\mu^2}f\left(\frac{e^2}{\mu}\right).\tag{3.17}$$

Here f is a dimensionless function, which by virtue of (3.15a) satisfies

$$\int_0^\infty dx\,f(x)=1,\tag{3.18a}$$

while lowest-order perturbation theory [viz., Eq. (3.15b)] gives

$$f(0)=\frac{1}{8\pi}.\tag{3.18b}$$

Next we write $\Sigma(p;\mu)-\Sigma(p;0)$, which is explicitly determined by (3.16c), in terms of a dimensionless function $\Delta\Sigma$,

$$\Sigma(p;\mu)-\Sigma(p;0)=\frac{e^2\not p}{(-p^2-i\epsilon)^{1/2}}\Delta\Sigma\left(\frac{\mu}{(-p^2-i\epsilon)^{1/2}}\right).\tag{3.19}$$

With the new notation, (3.16b) becomes

$$\int_0^\infty d\mu\,\rho(\mu^2)[\Sigma(p;\mu)-\Sigma(p;0)]=\frac{e^2\not p}{(-p^2-i\epsilon)^{1/2}}\int_0^\infty dx\,f(x)\Delta\Sigma\left(\frac{e^2}{(-p^2-i\epsilon)^{1/2}x}\right).\tag{3.20a}$$

We wish to extract the $O(e^4)$ terms from (3.20a); however, a direct expansion of $\Delta\Sigma$ produces a logarithmic divergence, since $f(0)$ does not vanish. To proceed we rewrite the integral in (3.20a) as

$$e^2\int_0^\infty dx\, f(x)\,\Delta\Sigma\left(\frac{e^2}{(-p^2-i\epsilon)^{1/2}x}\right)=e^2 f(0)\int_0^1 dx\,\Delta\Sigma\left(\frac{e^2}{(-p^2-i\epsilon)^{1/2}x}\right)+e^2\int_0^1 dx\,[f(x)-f(0)]\Delta\Sigma\left(\frac{e^2}{(-p^2-i\epsilon)^{1/2}x}\right)+e^2\int_1^\infty dx\, f(x)\,\Delta\Sigma\left(\frac{e^2}{(-p^2-i\epsilon)^{1/2}x}\right) \tag{3.20b}$$

and observe that the last two terms do not give nonanalytic contributions to $O(e^4)$. In the first term on the right-hand side of (3.20b), we change variables, and integrate twice by parts, to find

$$\begin{aligned}e^2 f(0)\int_0^1 dx\,\Delta\Sigma\left(\frac{e^2}{(-p^2-i\epsilon)^{1/2}x}\right)&=\frac{e^4 f(0)}{(-p^2-i\epsilon)^{1/2}}\int_{e^2/(-p^2-i\epsilon)^{1/2}}^\infty \frac{dy}{y^2}\,\Delta\Sigma(y)\\&=\frac{e^4 f(0)}{(-p^2-i\epsilon)^{1/2}}\left\{-\Delta\Sigma'\left(\frac{e^2}{(-p^2-i\epsilon)^{1/2}}\right)\ln\frac{e^2}{(-p^2-i\epsilon)^{1/2}}\right.\\&\quad+\frac{(-p^2-i\epsilon)^{1/2}}{e^2}\,\Delta\Sigma\left(\frac{e^2}{(-p^2-i\epsilon)^{1/2}}\right)\\&\quad\left.-\int_{e^2/(-p^2-i\epsilon)^{1/2}}^\infty dy\,\ln y\,\Delta\Sigma''(y)\right\}.\end{aligned} \tag{3.20c}$$

Therefore, (3.16b) evaluates to

$$\begin{aligned}\not p\sigma(p^2)=\Sigma(p;0)+\frac{e^4\not p}{p^2+i\epsilon}\Big\{&f(0)\Delta\Sigma'(0)\ln\frac{e^2}{(-p^2-i\epsilon)^{1/2}}-\Delta\Sigma'(0)\int_0^\infty\frac{dx}{x}\,[f(x)-f(0)\theta(1-x)]\\&-f(0)\Delta\Sigma'(0)+f(0)\int_0^\infty dy\,\ln y\,\Delta\Sigma''(y)\Big\}+O(e^6).\end{aligned} \tag{3.20d}$$

Note that the coefficient of the coupling-constant-dependent logarithm involves the perturbatively determined quantities $f(0)$ and $\Delta\Sigma'(0)$. However, the analytic piece proportional to e^4 includes, in addition to perturbatively calculable numbers [the last two terms of (3.20d)], an integral over the spectral function $f(x)$ for all x, i.e., for all e^2. The expression for the fermion propagator which thus follows from $\Sigma(p;0)=0$, $\Delta\Sigma'(0)=1/6\pi$ is

$$\mathcal{S}^{-1}(p)=\frac{1}{i}\not p\left[1+\frac{e^2\alpha}{16(-p^2-i\epsilon)^{1/2}}-\frac{e^4}{48\pi^2(p^2+i\epsilon)}\ln\frac{e^2}{(-p^2-i\epsilon)^{1/2}}+O(e^4)\big|_{\rm reg}\right]. \tag{3.21}$$

The coefficient of the logarithm is gauge invariant (α independent).

It is important to appreciate that only two properties of the photon spectral function are needed to derive (3.21): Lehmann's theorem (3.15a) or (3.18a), and the asymptotic form (3.15b) or (3.18b). The former is of course a general result; the latter is fixed by perturbation theory. In fact the number $1/8\pi$ controls the logarithmic infrared divergence, which is present in fourth order, before the healing is taken into account. [In Sec. IV, where we study the non-Abelian theory, a successful analysis depends critically on our ability to use only reliable information of the type (3.15) or (3.18) for the spectral function.]

Terms that have been dropped in arriving at (3.21) include the nonlogarithmic $O(e^4)$ contributions as well as all higher orders. The $O(e^4)$ terms arise from the next-order contributions to Eq. (3.11), i.e., from graphs depicted in Figs. 4 and 5, from the last two terms in (3.20d), and from the perturbatively noncalculable quantity which, from (3.17) and (3.20d), is seen to involve $[e^4/6\pi(p^2+i\epsilon)]c_4$, where

$$\begin{aligned}c_4&=\int_0^\infty\frac{dx}{x}[f(x)-f(0)\theta(1-x)]\\&=-\frac{1}{e^2}\int_0^\infty d\mu\,\ln\mu\,\frac{d}{d\mu}[\mu^2\rho(\mu^2)].\end{aligned} \tag{3.22}$$

The unknown constant c_4 satisfies a sum rule completely analogous to (2.12). Indeed one may relate it to a matrix element of an operator. We write

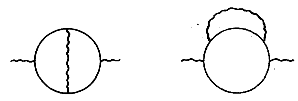

FIG. 4. Infrared-finite two-loop contributions to gauge-field self-energy.

$$c_4 = \frac{1}{e^2}\int_0^\infty d\mu\,\mu\,\rho(\mu^2)\big|_{\rm reg}\,. \tag{3.23a}$$

Here, as in (2.16b), reg indicates that an ultraviolet regulation is to be inserted. [The regulation corresponds to the step function in (3.22).] The spectral function is given by the absorptive part of the photon propagator; carrying out the appropriate integrations gives equivalently to (3.23a)

$$c_4 = \frac{1}{4e^2}\int \frac{d^3x}{(x^2 - i\epsilon)^{1/2}}\langle 0|F^{\mu\nu}(x)F_{\mu\nu}(0)|0\rangle\big|_{\rm reg}\,. \tag{3.23b}$$

In contrast to our toy model, in which the corresponding formula (2.17) involves a product of two fields at the same point, here occurs an operator product extended over x, with the extent somewhat damped by the $(x^2)^{-1/2}$ factor. The reason for the difference can be traced to the different dimensionalities in the two problems, and to the circumstance that here we need to extract an effect that arises from second-order perturbation theory, rather than first.

Although the logarithmic effect that we have exposed is gauge invariant (α independent), it occurs in a gauge-variant quantity—the fermion propagator. If one examines a gauge-invariant amplitude, then to leading order the infrared divergences are absent. Consider, for example, $\langle 0|T\bar\psi(x)\Gamma\psi(x)\bar\psi(y)\Gamma\psi(y)|0\rangle$, where Γ is any 2×2 matrix. Naively one would expect infrared divergences at the three-loop $O(e^4)$ level. There are two dangerous graphs, depicted in Fig. 6. For a finite evaluation we may extract the $O(e^4)$ terms from the same graphs constructed with the improved photon propagator (3.12); see Fig. 7, where the double wavy line depicts the improved propagator $\mathfrak{D}_{\mu\nu}$. But an explicit calculation shows that $e^4 \ln e^2$ is absent from the sum, even though each individual graph contains it.

Since the nonperturbative portion of the analytic $O(e^4)$ piece occurs in the same combination with the lowest-order logarithmic term [compare

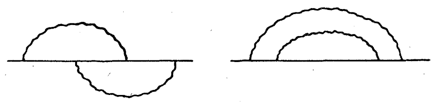

FIG. 5. Infrared-finite two-loop contributions to fermion self-energy.

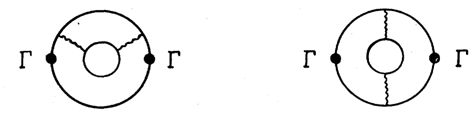

FIG. 6. Potentially infrared-divergent three-loop contributions to gauge-invariant fermion bilinears.

(3.20d)], it also cancels in the $O(e^4)$ contribution to the gauge-invariant amplitude.

One may understand the cancellation of the infrared divergence on the basis of guage invariance. The summed graphs of Fig. 7 can also be represented by

$$e^2\int\frac{d^3k}{(2\pi)^3}\mathfrak{D}_{\mu\nu}(k)T^{\mu\nu}(k,p)\,,$$

where $T^{\mu\nu}(k,p)$ is a forward Compton amplitude for the scattering of photons on the "particles" $\bar\psi\Gamma\psi$. Since $T^{\mu\nu}$ is gauge invariant, viz., transverse to k^μ, the integral is simply

$$-ie^2\int\frac{d^3k}{(2\pi)^3}\left[k^2 + i\epsilon - \frac{e^2}{16}(-k^2 - i\epsilon)^{1/2}\right]^{-1}T^\mu{}_\mu(k,p)\,.$$

The $O(e^4)$ contribution,

$$-\frac{ie^4}{16}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{(-k^2 - i\epsilon)^{3/2}}\,T^\mu{}_\mu(k,p)\,,$$

does not diverge, since by virtue of the transversality condition, $T^\mu{}_\mu$ vanishes at zero photon momentum. A higher-order calculation, which is seen to involve at least four loops, must be done to exhibit the nonanalytic and nonperturbative contributions to gauge-invariant amplitudes. We expect that nonanalytic, presumably logarithmic, dependence on e^2 does set in, since we know of no reason for $T^\mu{}_\mu(k,p)$ to vanish at $k=0$ faster than all powers of k.

In summary, three-dimensional massless QED cures its perturbative, infrared divergences by giving rise to coupling-constant logarithms. The effect is gauge invariant and first occurs in $O(e^4)$, but only to higher order in gauge-invariant amplitudes. Subdominant terms include nonlogarithmic $O(e^4)$ contributions, as well as double, triple, etc., logarithms in $O(e^8)$, $O(e^{12})$, etc. The coefficient of the leading logarithm is computable in perturbation theory, but the normalization of the logarithm, which affects the nonlogarithmic power term, is not. This nonper-

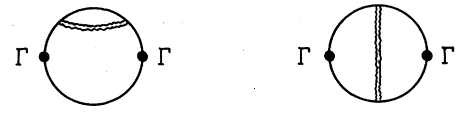

FIG. 7. Infrared-finite graphs from which the healed three-loop contributions to gauge-invariant fermion bilinears can be extracted.

turbative number is related to a matrix element of a gauge-invariant composite operator. In the further subdominant terms, new nonperturbative information is needed for the analytic terms, but the logarithms are then known. Our results should be valid in the region where $e^2/(-p^2)^{1/2} \ll 1$ and $|\ln[e^2/(-p^2]^{1/2}| \gg 1$, so $e^2/(-p^2)^{1/2}\ln[e^2/(-p^2)^{1/2}]$ also is small.

IV. THREE-DIMENSIONAL YANG-MILLS THEORY

We now consider a non-Abelian gauge theory in three-dimensional space-time, governed by the following QCD Lagrangian:

$$\begin{aligned}\mathcal{L} &= i\bar{\psi}\gamma^\mu(\partial_\mu + gA^a_\mu T^a)\psi - \tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu},\\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^\mu_a + gf^{abc}A^b_\mu A^c_\nu.\end{aligned} \tag{4.1}$$

The representation matrices T^a span the Lie algebra of the gauge group under which the Fermi fields transform. The structure constants f^{abc}, defined by

$$[T^a, T^b] = f^{abc}T^c, \tag{4.2a}$$

are normalized conventionally: the fundamental (quark) representation matrices T^a_f satisfy

$$\mathrm{tr}\, T^a_f T^b_f = -\tfrac{1}{2}\delta^{ab} \tag{4.2b}$$

while for other representations the analogous trace involves the quadratic Casimir eigenvalue $Q(T)$ in that representation:

$$\begin{aligned}&\sum_a T^a T^a = -Q(T),\\ &\mathrm{tr} T^a T^b = -C(T)\delta^{ab},\\ &C(T) = Q(T)\frac{\text{dimension of representation}}{\text{dimension of group}}.\end{aligned} \tag{4.2c}$$

The gauge group we use is SU(N); then (4.2c) in the adjoint representation implies

$$f^{abc}f^{a'bc} = N\delta^{aa'}. \tag{4.2d}$$

The gauge-fixing and gauge-compensating terms to be added to the Lagrangian are

$$-\frac{1}{2\alpha}(\partial_\mu A^\mu_a)^2 + \bar{u}_a \Box u_a + \bar{u}_a\partial^\mu(gf^{abc}A^b_\mu u_c), \tag{4.3}$$

where u is the Faddeev-Popov ghost field.

No mass terms appear in the Lagrangian. Neither a fermion mass, $-m\bar{\psi}\psi$, nor a gauge-invariant gauge-field mass,

$$\frac{\mu}{4}\epsilon^{\alpha\beta\gamma}\left(F^a_{\alpha\beta}A^a_\gamma - \frac{g}{3}f^{abc}A^a_\alpha A^b_\beta A^c_\gamma\right),$$

will be generated in perturbation theory owing to P and T invariance.[14]

We shall study the various two-point functions

$$\delta^{ab}\mathfrak{D}_{\mu\nu}(p) = \int d^3x\, e^{ipx}\langle 0|TA^a_\mu(x)A^b_\nu(0)|0\rangle, \tag{4.4a}$$

$$\delta^{ab}\mathcal{G}(p) = \int d^3x\, e^{ipx}\langle 0|Tu_a(x)\bar{u}_b(0)|0\rangle, \tag{4.4b}$$

$$\mathcal{S}(p) = \int d^3x\, e^{ipx}\langle 0|T\psi(x)\bar{\psi}(0)|0\rangle. \tag{4.4c}$$

The self-energies are defined by

$$\mathfrak{D}^{-1}{}_{\mu\nu}(p) = iP_{\mu\nu}[p^2 - \Pi(p^2)] + \frac{i}{\alpha}p_\mu p_\nu, \tag{4.5a}$$

$$\mathcal{G}^{-1}(p) = i[p^2 - M(p^2)], \tag{4.5b}$$

$$\mathcal{S}^{-1}(p) = \frac{1}{i}[\not{p} - \Sigma(p)]. \tag{4.5c}$$

The lowest-order formulas are

$$\begin{aligned}\Pi^{\mu\nu}(p) = P^{\mu\nu}\Pi(p^2) = &-ig^2N\int\frac{d^3k}{(2\pi)^3}(p+k)^\mu k^\nu G(k)G(p+k) + g^2N\int\frac{d^3k}{(2\pi)^3}[D^{\mu\nu}(k) - g^{\mu\nu}D^\alpha_\alpha(k)]\\ &+\frac{ig^2N}{2}\int\frac{d^3k}{(2\pi)^3}V^{\mu\alpha\alpha'}(p,k)V^{\nu\beta\beta'}(p,k)D_{\alpha\beta}(k)D_{\alpha'\beta'}(p+k)\\ &-ig^2C_F\int\frac{d^3k}{(2\pi)^3}\mathrm{tr}\,\gamma^\mu S(p+k)\gamma^\nu S(k) + O(g^4),\end{aligned} \tag{4.6a}$$

$$M(p^2) = -ig^2N\int\frac{d^3k}{(2\pi)^3}p^\mu k^\nu D_{\mu\nu}(p+k)G(k) + O(g^4), \tag{4.6b}$$

$$\Sigma(p) = -ig^2Q_F\int\frac{d^3k}{(2\pi)^3}\gamma^\mu S(p+k)\gamma^\nu D_{\mu\nu}(k) + O(g^4). \tag{4.6c}$$

Here $D_{\mu\nu}$ and S are the free gauge-field and fermion propagators Eqs. (3.8), G is the free ghost propagator

$$G(k)=\frac{-i}{k^2+i\epsilon}\,, \tag{4.7}$$

and the vertex in (4.6a) is given by

$$V^{\mu\alpha\beta}(p,k)=(p+2k)^\mu g^{\alpha\beta}-(k+2p)^\alpha g^{\beta\mu}+(p-k)^\beta g^{\mu\alpha}. \tag{4.8}$$

C_F is the constant of (4.2c) and Q_F is the quadratic Casimir eigenvalue, both for the fermions' representation. To the gauge-field self-energy (4.6a), the first contribution comes from the ghost loop, the second, from the gauge-field contact term, the third, from the gauge-field bubble, and the fourth, from the fermion loop.

A lengthy but straightforward calculation yields

$$\Pi(p^2)=\left[-\frac{5N}{32}-\frac{N}{64}(1+\alpha)^2+\frac{C_F}{16}\right]g^2(-p^2-i\epsilon)^{1/2}, \tag{4.9a}$$

$$M(p^2)=-\frac{Ng^2}{16}(-p^2-i\epsilon)^{1/2}, \tag{4.9b}$$

$$\Sigma(p)=-\frac{N}{16}Q_F\alpha g^2\frac{\not p}{(-p^2-i\epsilon)^{1/2}}. \tag{4.9c}$$

The fermion result (4.9c) is completely analogous to that of QED; it offers no surprises.

To the numerical constant in the gauge-field polarization (4.9a), $-N/32$ comes from the ghost loop, $-(N/64)[8+(1+\alpha)^2]$ from the gauge-field loop, and $C_F/16$ from the fermion loop. The gauge-field and ghost loops contribute with a sign opposite to that of the fermions, a circumstance familiar from the four-dimensional calculation. Consequently, the vacuum polarization is negative for spacelike momenta, when C_F is not too large. The gauge dependence (α dependence) is to be expected, since we are calculating a gauge-variant quantity.

The ghost self-mass (4.9b) also is negative for spacelike momenta; moreover it has the surprising feature that it is unexpectedly gauge invariant (α independent).

Observe that the negative self-masses produce a pole in the gauge and ghost propagators at spacelike momenta. Admittedly, the location of the pole is at $g^2/(-p^2)^{1/2}\sim O(1)$, where we no longer rely on our approximate calculation. Nevertheless, it is puzzling to encounter this further infrared singularity, which is the three-dimensional residue of asymptotic freedom, and presumably signals the infrared instability of the theory.

In the next order, infrared divergences are encountered. Of the many fourth-order graphs, the potentially dangerous ones involve inserting the above self-energies into propagators. They are depicted in Figs. 8, 9, and 10, where wiggly lines are gauge fields; dashed ones, ghosts; solid ones, fermions. Just as in QED, insertions in fermion lines are infrared finite [Figs. 8 and 10(v)]. Also for the ghost insertions, the momentum in the vertex decreases the infrared singularity, so they too are innocuous [Figs. 8(ix) and 9(v)]. To heal the infrared singularity arising from the gauge-field insertions [Figs. 8(i)–8(viii), 9(i)–9(iv), and 10(i)–(iv)], we consider the complete equations for the propagators and truncate them so that only the $O(g^4)$ logarithms survive. The analogs of (3.11) now are

$$\mathfrak{D}^{-1}{}_{\mu\nu}(p)=iP_{\mu\nu}p^2-g^2N\int\frac{d^3k}{(2\pi)^3}(p+k)^\mu k^\nu G(k)G(p+k)-ig^2N\int\frac{d^3k}{(2\pi)^3}[\mathfrak{D}^{\mu\nu}(k)-g^{\mu\nu}\mathfrak{D}^\alpha_\alpha(k)]$$
$$+\frac{g^2N}{2}\int\frac{d^3k}{(2\pi)^3}V^{\mu\alpha\alpha'}(p,k)V^{\nu\beta\beta'}(p,k)\mathfrak{D}_{\alpha\beta}(k)\mathfrak{D}_{\alpha'\beta'}(p+k)$$
$$-g^2C_F\int\frac{d^3k}{(2\pi)^3}\operatorname{tr}\gamma^\mu S(p+k)\gamma^\nu S(k)+O(g^4)|_{\text{reg}}, \tag{4.10a}$$

$$\mathcal{G}^{-1}(p)=ip^2-g^2N\int\frac{d^3k}{(2\pi)^3}p^\mu k^\nu\mathfrak{D}_{\mu\nu}(p+k)G(k)+O(g^4)|_{\text{reg}}\,, \tag{4.10b}$$

$$\mathcal{S}^{-1}(p)=\frac{1}{i}\not p+g^2Q_F\int\frac{d^3k}{(2\pi)^3}\gamma^\mu S(p+k)\gamma^\nu\mathfrak{D}_{\mu\nu}(k)+O(g^4)|_{\text{reg}}. \tag{4.10c}$$

The gauge-field propagator $\mathfrak{D}_{\mu\nu}$ is determined by (4.10a), and then is used in evaluating $\mathcal{G}^{-1}$ and $\mathcal{S}^{-1}$. However, even in a truncated approximation, Eq. (4.10a) remains nonlinear and impossible to solve exactly. But for our purposes it is sufficient to replace $\mathfrak{D}_{\mu\nu}$ on the right-hand side by the improved formula arising from the lowest-order approximation to Π, Eq. (4.9a),

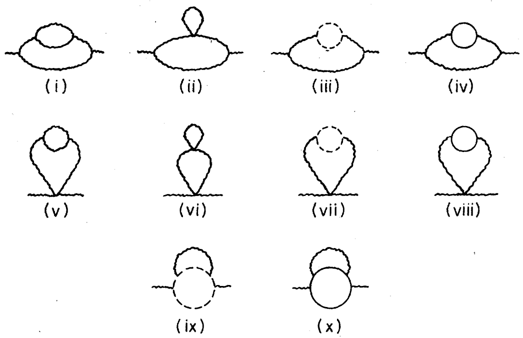

FIG. 8. Potentially infrared-divergent two-loop contributions to gauge-field self-energy.

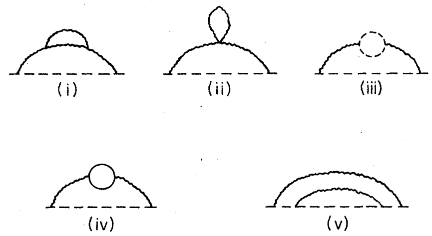

FIG. 9. Potentially infrared-divergent two-loop contributions to ghost self-energy.

$$\mathfrak{D}_{\mu\nu}(p) = -iP_{\mu\nu}[p^2 + i\epsilon - g^2 c(-p^2 - i\epsilon)^{1/2}]^{-1} - i\alpha \frac{p_\mu p_\nu}{(p^2+i\epsilon)^2},$$
$$c = -\frac{5N}{32} - \frac{N}{64}(1+\alpha)^2 + \frac{C_F}{16}. \tag{4.11}$$

We would like to use (4.11) to evaluate (4.10), but if we accept (4.11) for all momenta, we encounter the problem of spacelike singularities. Fortunately, our careful discussion of the Abelian case indicates how to proceed.

We replace the unphysical formula (4.11) by a spectral representation

$$\mathfrak{D}_{\mu\nu}(p) = -iP_{\mu\nu}\int_0^\infty d\mu \rho(\mu^2)\frac{1}{p^2 - \mu^2 + i\epsilon} - i\alpha\frac{p_\mu p_\nu}{(p^2+i\epsilon)^2}. \tag{4.12}$$

For the spectral function we assume Lehmann's theorem

$$\int_0^\infty d\mu\, \rho(\mu^2) = 1\,. \tag{4.13a}$$

Also we take the asymptote for large μ^2,

$$\rho(\mu^2) \to \frac{2g^2 c}{\pi\mu^2}, \tag{4.13b}$$

where the coefficient is set by lowest-order perturbation theory, so that the perturbative logarithmic divergence is reproduced if this formula is used all the way to $\mu = 0$. Just as in the electromagnetic case, Eqs. (4.13) are sufficient to determine the $O(g^4)$ nonanalytic pieces. After an extraordinarily lengthy calculation we find

$$\mathfrak{D}^{-1}{}_{\mu\nu}(p) = iP_{\mu\nu}\left[p^2 - g^2 c(-p^2 - i\epsilon)^{1/2} - \frac{g^4 cN}{3\pi^2}(2+\alpha)\ln\frac{g^2}{(-p^2-i\epsilon)^{1/2}}\right] - i\alpha\frac{p_\mu p_\nu}{(p^2+i\epsilon)^2} + O(g^4)|_{\text{reg}}, \tag{4.14a}$$

$$\mathcal{G}^{-1}(p) = i\left[p^2 + \frac{g^2 N}{16}(-p^2 - i\epsilon)^{1/2} + \frac{g^4 N}{3\pi^2}c\ln\frac{g^2}{(-p^2-i\epsilon)^{1/2}}\right] + O(g^4)|_{\text{reg}}, \tag{4.14b}$$

$$\mathcal{S}^{-1}(p) = \frac{1}{i}\not{p}\left[1 + \frac{g^2\alpha Q_F}{16(-p^2-i\epsilon)^{1/2}} - \frac{g^4 Q_F c}{3\pi^2(p^2+i\epsilon)}\ln\frac{g^2}{(-p^2-i\epsilon)^{1/2}}\right] + O(g^4)|_{\text{reg}}. \tag{4.14c}$$

Everything is gauge dependent. In the gauge-field propagator, the further α dependence of the logarithm's coefficient (beyond that contained in c) arises when one of the two gauge-field propagators in the gauge-field loop contributes its transverse part, while the other gives its α-dependent, longitudinal part. The regular $O(g^4)$ contributions are not perturbatively computable.

One may also consider gauge-invariant, group-singlet, fermion bilinears. The dangerous graphs

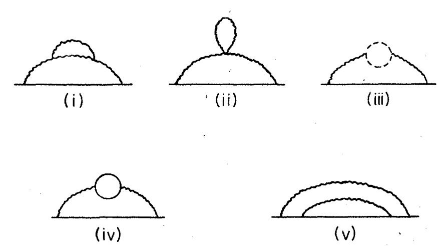

FIG. 10. Potentially infrared-divergent two-loop contributions to fermion self-energy.

are the same as in QED, Fig. 6, and are healed as in Fig. 7. As a consequence the three-loop $O(g^4)$ infrared divergences as well as the nonperturbative analytic pieces again cancel,[15] for the same reason, and one must proceed to a higher order to encounter the coupling-constant nonanalyticity.

In summary, to the order here investigated, the perturbative non-Abelian theory behaves similarly to the Abelian one, except for a greater gauge dependence and a characteristic reversal of signs.

V. CONCLUSION

It is amply clear that perturbative infrared divergences arising from super-renormalizable couplings in massless theories are a defect of the perturbation theory and are absent in the exact solution. The reason why the spurious singularities occur is that amplitudes are not analytic in the coupling constant; logarithmic singularities are present, which lead to logarithmic divergences when a power series is forced. Upon taking account of this phenomenon by expanding in powers of coupling-constant logarithms, one gets a completely finite perturbative expansion which, however, does not determine the analytic terms. They are fixed by nonperturbative values of matrix elements of various operators, and then the logarithmic coefficients can be calculated. The leading logarithms do not need the nonperturbative information; they are just the perturbative infrared divergences, with the coupling constant providing an infrared cutoff.

All the above is well illustrated in our toy model based on a four-dimensional scalar theory with cubic self-couplings. Physical theories in which this phenomenon is observed include three-dimensional Abelian (QED) and non-Abelian (QCD) gauge models, with massless fields. Our calculations produce no surprises in QED, while QCD acquires (presumably spurious) poles at spacelike momenta.

Much further investigation can be carried out. It should be possible to sum all the leading logarithms.[13] An effective way of extracting the nonanalytic behavior from gauge-invariant amplitudes should be developed. Since high orders of perturbation theory are involved, direct calculation becomes prohibitively lengthy. Finally a better understanding of the analytic terms, with nonperturbative coefficients, is needed.

The investigation of the gauge models has been confined to perturbation theory, which is limited by the requirement $[e^2/(-p^2)^{1/2}]|\ln[e^2/(-p^2)^{1/2}]| \sim 1$. But one may inquire about the nonperturbative structure in the infrared region $p^2 \to 0$. Presumably the theories confine; in the classical approximation they do so with a logarithmic potential. However, in lowest-order perturbation theory for QED we find that the $p=0$ singularity in the gauge-field propagator is weakened: the inverse propagator goes as $e^2(-p^2)^{1/2}$ for small p, and screening sets in. How all the higher orders, which on dimensional grounds add a term proportional to e^4 times a function of e^2/p, modify this picture remains an open question. Especially intriguing is the difference observed between the Abelian and non-Abelian theories: the latter develop spacelike singularities. We do not know whether they are an annoyance of improper extrapolation and therefore unreliable, or if they are providing the first signal for instability in the perturbative theory.

We hope our considerations are applicable to physical four-dimensional QCD at finite temperatures in the vicinity of its phase transitions.

Note Added in Proof. The leading coupling-constant logarithms have now been summed to all orders, in the Abelian gauge theory [S. Templeton, MIT report (unpublished)]. Results for the fermion propagator and vacuum polarization tensor are as follows.

The exact functional integrals for these two amplitudes (in Euclidean space), when evaluated in an approximation which correctly summarizes the leading coupling-constant logarithms, lead to a representation in terms of ordinary integrals (viz. position-dependent fields become constant fields in this study of the infrared region):

$$\mathcal{S}(p) = \int d^3A\, e^{-A^2/2}(\not{p} + i\lambda \not{A})^{-1} \Big/ \int d^3A\, e^{-A^2/2},$$

$$\Pi^{\mu\nu}(p) = e^2 \int d^3A\, e^{-A^2/2} \int \frac{d^3k}{(2\pi)^3} \mathrm{tr}\gamma^\mu(\not{p} + \not{k} + i\lambda\not{A})^{-1}\gamma^\nu(\not{k} + i\lambda\not{A})^{-1} \Big/ \int d^3A\, e^{-A^2/2},$$

$$\lambda^2 = \frac{e^4}{48\pi^2}\ln p/e^2 > 0.$$

A formal expansion of the above in powers of λ gives the $e^4 \ln e^2$ series encountered in perturbation theory. (In the vacuum polarization tensor all terms beyond the first integrate to zero, since the leading coupling-constant logarithms are absent.) Evaluation of the integrals yields

$$\mathcal{S}(p)=\frac{1}{\not p}\left[1+\left(\frac{\pi}{2}\right)^{1/2}\frac{\lambda}{p}e^{p^2/2\lambda^2}\operatorname{erfc}\left(\frac{p}{\sqrt{2}\lambda}\right)-\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{\lambda}{p}+\frac{p}{\lambda}\right)e^{-p^2/2\lambda^2}\right],$$

$$\Pi^{\mu\nu}(p)=e^2P^{\mu\nu}\left[\frac{p}{16}+\frac{2\lambda}{(2\pi)^{3/2}}-\frac{p}{8\pi}\int_0^{\pi/2}d\theta\operatorname{erfc}\left(\frac{p\sin\theta}{2^{3/2}\lambda}\right)\right].$$

The results possess unexpected features. For the fermion propagator the first two terms in the brackets, when expanded for small λ in an asymptotic series, reproduce the Borel-summable perturbative series involving even powers of λ. [The first two contributions to that series agree with (3.21).] However, the last term possesses an essential singularity at $\lambda^2=0$ and is not seen in perturbation theory. For the vacuum polarization tensor, only the first term in the brackets reproduces the perturbative result (3.9a). The remainder is entirely nonperturbative; an asymptotic expansion for small λ yields a series in odd powers of $\lambda=(e^2/\sqrt{48}\pi)\ln^{1/2}p/e^2$, which do not occur in perturbation theory. Of course the reduction of the exact functional integrals to the ordinary integrals is justified by perturbation theory. Hence the significance of the nonperturbative contributions remains unclear.

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[1] D. Kirzhnits and A. Linde, Phys. Lett. 42B, 471 (1972); L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974); S. Weinberg, *ibid.* 9, 3357 (1974).

[2] For reviews, see S. Weinberg, in *Understanding the Fundamental Constituents of Matter*, edited by A. Zichichi (Plenum, New York, 1978); A. Linde, Rep. Prog. Phys. 42, 389 (1979).

[3] R. Jackiw, Ph.D. thesis, Cornell University, 1966 (unpublished).

[4] H. Bethe and E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Plenum, New York, 1977).

[5] H. Pagels, Phys. Rep. 16C, 219 (1975).

[6] Nonanalyticity in the coupling constant, which leads to infinities in perturbation theory, has been noticed in a variety of contexts, see, e.g., L. Cooper, Phys. Rev. 100, 362 (1955); K. Symanzik, Lett. Nuovo Cimento 8, 771 (1973); H. Fried, Nucl. Phys. B169, 329 (1980).

[7] Three-dimensional QED has also been examined by J. Cornwall, Phys. Rev. D 22, 1452 (1980). However, his formulation differs from ours; see Ref. 11.

[8] Three-dimensional QCD has also been examined by G. 't Hooft, Schladming lectures, 1980 (unpublished); by O. Kalashnikov and V. Klimov, Lebedev Report No. 129 (unpublished); as well as by T. Appelquist and R. Pisarski, Yale University report (unpublished).

[9] That such a transition takes place was first suggested by the work of J. Collins and M. Perry, Phys. Rev. Lett. 34, 135 (1975). In lattice-gauge theories, the issue has been examined by A. Polyakov, Phys. Lett. 72B, 477 (1978) and L. Susskind, Phys. Rev. D 20, 2610 (1979). For a review, see D. Gross, R. Pisarski, and L. Yaffe, Rev. Mod. Phys. 53, 43 (1981).

[10] That some parameters are not calculable in perturbation theory, but are related to matrix elements of operators was stressed by 't Hooft, Ref. 8 and private communication.

[11] It is possible to have P- and T-conserving fermion masses in three dimensions, if one doubles the number of fermion fields. In that case we use 4×4 Dirac matrices,

$$\gamma^0=\begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix},\quad \gamma^1=\begin{pmatrix}i\sigma^1 & 0\\ 0 & -i\sigma^1\end{pmatrix},\quad \gamma^2=\begin{pmatrix}i\sigma^2 & 0\\ 0 & -i\sigma^2\end{pmatrix}$$

and call the upper doublet of the 4-spinor ψ_1 and the lower, ψ_2. Parity and time reversal now interchange the doublets

$$\mathcal{P}\psi_1(t,\vec{x})\mathcal{P}^{-1}=\sigma_1\psi_2(t,\vec{x}'),$$
$$\mathcal{P}\psi_2(t,\vec{x})\mathcal{P}^{-1}=\sigma_1\psi_1(t,\vec{x}'),$$
$$\vec{x}=(x,y),\quad \vec{x}'=(-x,y),$$
$$\mathcal{T}\psi_1(t,\vec{x})\mathcal{T}^{-1}=\sigma_2\psi_2(-t,\vec{x}),$$
$$\mathcal{T}\psi_2(t,\vec{x})\mathcal{T}^{-1}=\sigma_2\psi_1(-t,\vec{x}).$$

Under these transformations the fermion mass $-m\bar{\psi}\psi=-m_1\psi_1^\dagger\sigma_3\psi_1+m\psi_2^\dagger\sigma_3\psi_2$ is invariant, as is the rest of the fermion Lagrangian. However, if no mass is inserted in the Lagrangian, none will be generated perturbatively since now chiral symmetry can be defined for the four-component realization, and it prohibits a mass term. The physical difference between the two-component theory and the four-component theory is that, in the former, the particle and the antiparticle each possesses one spin state, while in the latter, each possesses two. This is the formulation employed by J. Cornwall in Ref. 7, and we thank him as well as R. Giles for discussions.

[12] J. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D 10, 2428 (1974).

[13]This task is being pursued by one of us (S.T.); see *Note added in proof*.

[14]The gauge-field combination occurring in the mass term is familiar from the four-dimensional topological current, and the contribution to the three-dimensional action involves the four-dimensional topological charge. Consequently, if present, the mass term would be invariant only against "small" (deformable to the identity) three-dimensional gauge transformations, and not against "large" ones (not deformable to the identity). This intrusion of four-dimensional topological considerations is interesting, but has not been further explored by us. For a review of the topological properties of four-dimensional theories, see R. Jackiw, Rev. Mod. Phys. 52, 661 (1980). The gauge-invariant mass term has also been discovered and analyzed by J. Schonfeld, Fermilab report (unpublished).

[15]The cancellation of leading infrared divergences in color-singlet amplitudes is reminiscent of a similar phenomenon in the four-dimensional theory; see J. Cornwall and G. Tiktopoulos, Phys. Rev. D 13, 3370 (1976).