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Coupling-constant analyticity and the renormalization group

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It is shown that for all definitions of coupling constants any analytic continuation in the coupling parameter of the Green's functions of quantum chromodynamics is limited to a horn-shaped domain with zero opening angle. This extends a result of 't Hooft's obtained for one specific choice of coupling parameter to all possible choices. For some choices of coupling parameter, it is also shown that the renormalization group itself leads to an analytic continuation in the coupling constant for the Green's functions regular in the above domain.

I. INTRODUCTION

The Green's functions of a quantum field theory are known to have a singularity at the origin in the coupling-constant plane which leads to the divergence of the perturbation series. It is important to investigate the analytic properties in the coupling constant in the neighborhood of the origin. This is essential for establishing summability methods[1] and it also has other applications.[2]

For a zero-mass field theory it has been known that analyticity in coupling is related to analyticity in momentum via the renormalization-group equations.[3] This relation can be made more precise by exploiting the freedom that one has in choosing the definition of the coupling parameter. 't Hooft introduced a definition of the coupling parameter such that the corresponding Gell-Mann–Low function has only two terms in its power-series expansion. Using this ingenious definition he showed that the renormalization-group equations imply that analytic continuation in the new coupling in the positive neighborhood of the origin is limited to a wedge bisected by the real axis and bounded above and below by circles which are tangent to the real axis at the origin.[4] This was done for quantum chromodynamics (QCD) but similar results could hold for massless $(\phi^4)_4$ theory and massless QED.

In this paper we shall extend 't Hooft's result to all definitions of the coupling. The main output is a theorem showing that if any analytic continuation in the coupling constant exists the domain of analyticity is limited to a wedge as above regardless of the coupling definition used. The main tool in proving this theorem is the explicit expression for the 't Hooft transformation obtained by McBryan and the author.[5]

While our main theorem indicates the severe restrictions placed by the renormalization group on analyticity in coupling it does not directly address the question of whether such analytic properties can be established. This is discussed in the last section where we show that for at least two definitions of the coupling parameter the renormalization-group equations do give us an analytic continuation into the wedge. We also show that this does not hold for general couplings without additional input beyond that used in this paper.

The results of this paper were established only for massless field theories or more specifically those theories which have exact renormalization-group equations with just one mass parameter. We also restrict ourselves to four dimensions. For massive field theories in two or three dimensions [$(\phi^4)_2$ and $(\phi^4)_3$ theories] much stronger analyticity results have been established by quite different methods.[1] For massive field theories in four dimensions the renormalization-group equations are more complicated and our method does not apply even though the results might still be true, see footnote 24 of Ref. 4.

II. LIMITS ON COUPLING-CONSTANT ANALYTICITY

The input of this paper consists of two general features of quantum field theories, namely, the renormalization-group equations and the standard momentum-plane analyticity. We shall present our results for the case of massless QCD (or pure Yang-Mills) field theories. At a later stage we will indicate the necessary modifications for our theorem to apply for massless $(\phi^4)_4$ theory or massless QED. In all cases we restrict ourselves to four dimensions.

We consider the Green's functions of the theory and limit our discussion only to two-point functions, $G(p^2;\alpha,\mu)$. Here $\alpha=g^2/4\pi$ and g is the renormalized coupling constant; μ is the renormalization point, and p is Euclidean, $p^2<0$. These functions satisfy a homogeneous renormalization-group equation:

$$\left[\lambda\frac{\partial}{\partial\lambda}-\beta(\alpha)\frac{\partial}{\partial\alpha}+2-2\gamma(\alpha)\right]G(\lambda p;\alpha\ ,\mu)=0\,.\ \text{(A)}$$

The Gell-Mann-Low function, $\beta(\alpha)$, has the property

$$\beta(\alpha)=a_2\alpha^2+a_3\alpha^3+O(\alpha^4), \quad a_2<0 \tag{1}$$

and the function γ behaves as

$$\gamma(\alpha)=b_1\alpha+O(\alpha^2). \tag{2}$$

The solution of the differential equation (A) is given by

$$G(\lambda^2p^2;\alpha,\mu) = \lambda^{-2}G(p^2;\bar{\alpha}(t,\alpha),\mu)\exp\left(2\int_\alpha^{\bar{\alpha}(t,\alpha)}\frac{\gamma(x)}{\beta(x)}dx\right), \tag{3}$$

where

$$\frac{d\bar{\alpha}}{dt}=\beta(\bar{\alpha}), \quad \bar{\alpha}(0,\alpha)=\alpha, \tag{4}$$

and $t\equiv\ln\lambda$.

Our second input is the analyticity in p^2 for fixed α and μ. In standard field theories this is well known for a two-point function. In QCD complications might arise because of confinement or gauge dependence. To avoid these we take G, as in Ref. 4, to be the Fourier transform of the time-ordered product of two quark bilinears $\bar{\psi}(0)\psi(0)$ and $\bar{\psi}(x)\psi(x)$ corresponding to a physical channel. Our input is then the following:

$G(p^2)$ is analytic in the p^2 plane cut along the positive real axis with multiparticle singularities extending to infinity as

$$p^2\to+\infty. \tag{B}$$

As we shall note below the presence of actual singularities, not just a branch cut, for arbitrarily large but positive values of p^2, is important for the proof of our theorem.

Finally, it is convenient to factor out a $(p^2)^{-1}$ from G and write

$$G(p^2;\alpha,\mu)\equiv\frac{D(p^2;\alpha,\mu)}{p^2}, \tag{5}$$

where we now have from Eq. (3)

$$D(\lambda^2p^2;\alpha,\mu)=D(p^2;\bar{\alpha}(t,\alpha),\mu)\exp\left(2\int_\alpha^{\bar{\alpha}}\frac{\gamma(x)}{\beta(x)}dx\right). \tag{6}$$

Our main theorem can now be stated as follows:

Theorem 1: Given (A) and (B) then if $D(p^2,\alpha,\mu)$, $\beta(\alpha)$, and $\gamma(\alpha)$ have an analytic continuation in α in the neighborhood of $\alpha=0$, $\mathrm{Re}\,\alpha>0$, the maximal domain of analyticity is a wedge $\mathcal{S}$,

$$\mathcal{S}=\left\{\alpha\,\middle|\; |\alpha|<r_0;\ \mathrm{Re}\,\alpha>0;\ |\mathrm{Im}\,\alpha^{-1}|<\left|\frac{\pi a_2}{2}\right|\right\}. \tag{7}$$

The theorem is true regardless of the renormalization scheme used in defining α.

Proof: The strategy of the proof is to assume analyticity in a domain larger than $\mathcal{S}$ and show that it leads to a contradiction with (A) and/or (B).

Let us assume $D(p^2;\alpha,\mu)$, $\beta(\alpha)$, $\gamma(\alpha)$ are all analytic in a sector S_ϵ with opening angle ϵ:

$$S_\epsilon\equiv\{\alpha\,|\; |\alpha|<r_\epsilon;\ \mathrm{Re}\,\alpha>0;\ |\arg\alpha|<\epsilon\}. \tag{8}$$

We proceed to show that this assumption leads to a contradiction. To achieve that we first make a change of variables,[4,5]

$$\alpha\to\alpha_R,$$

where we define α_R by

$$\alpha_R\equiv G_R(\alpha)=\alpha+O(\alpha^3) \tag{9}$$

and

$$\beta_R(\alpha_R)\equiv\beta(\alpha)\frac{dG_R}{d\alpha}. \tag{10}$$

As in Ref. 5. We choose G_R such that

$$\beta_R(\alpha_R)=a_2\alpha_R{}^2+a_3\alpha_R{}^3, \tag{11}$$

with a_2 and a_3 being the same coefficients as in Eq. (1).

To find such a function $G_R(\alpha)$ one has to solve the differential equation

$$\frac{dG_R}{d\alpha}=\frac{1}{\beta(\alpha)}(a_2G_R{}^2+a_3G_R{}^3), \tag{12}$$

with (9) as a boundary condition for $\alpha\to0$. The solution is[5]

$$b\ln\left[\frac{1}{G_R(\alpha)}+b\right]-\frac{1}{G_R(\alpha)} = \int_0^\alpha dx\left[\frac{a_2}{\beta(x)}-\frac{1}{x^2}+\frac{b}{x}\right]-\frac{1}{\alpha}-b\ln\alpha, \tag{13}$$

with $b\equiv a_3/a_2$, $b>0$ in QCD with less than eight flavors.

The following properties of $G_R(\alpha)$ can easily be established:

(a) $G_R(\alpha)$ exists and has a unique inverse $dG_R/d\alpha>0$, in the interval $0\leq\alpha<\alpha_1$, where α_1 is defined by

$$b\ln b\equiv\int_0^{\alpha_1}dx\left[\frac{a_2}{\beta(x)}-\frac{1}{x^2}+\frac{b}{x}\right]-\frac{1}{\alpha_1}-b\ln\alpha_1. \tag{14}$$

Without loss of generality we choose r_ϵ such that $r_\epsilon<\alpha_1$.

(b) Given our analyticity assumption, $G_R(\alpha)$ is also analytic for $\alpha\in S_\epsilon$. To see this we write

$$F(\alpha)\equiv b\ln\left[\frac{1}{G_R(\alpha)}+b\right]-\frac{1}{G_R}(\alpha)$$

$$=\int_0^\alpha dx\left[\frac{a_2}{\beta(x)}-\frac{1}{x^2}+\frac{b}{x}\right]-\frac{1}{\alpha}-b\ln\alpha\,. \tag{15}$$

$F(\alpha)$ is analytic for $\alpha\in S_\epsilon$. This is so since $\beta(\alpha)$ is by assumption analytic for $\alpha\in S_\epsilon$. Furthermore, it is easy to check that the $\ln\alpha$ term in (15) does not lead to a singularity since it is canceled out by splitting the integral in (15) as $\int_0^\alpha=\int_0^{\alpha_0}+\int_{\alpha_0}^\alpha$ with α_0 real.

We have here assumed that $\beta(\alpha)$ has no zero for $\alpha\in S_\epsilon$. This clearly does not lead to a loss of generality, since if we have a finite number of zeros in S_ϵ we can always choose a new and smaller domain $S_{\epsilon'}$, $\epsilon'<\epsilon$, $r_{\epsilon'}<r_\epsilon$, such that β has no zeros for $\alpha\in S_{\epsilon'}$. The only case we cannot handle is an infinite set of zeros which accumulate at the origin and lie in $\mathcal{S}$. But this does not affect our theorem since we are asserting that the largest possible domain is $\mathcal{S}$ while the actual domain could very well be much smaller.

Finally in (15), $\ln[G_R^{-1}(\alpha)+b]$ has a branch point at $\alpha=\alpha_0$ where $G_R(\alpha_0)=-1/b$. Again we can choose $r_\epsilon<\alpha_0$ to locate this branch point outside S_ϵ. It therefore follows that $G_R(\alpha)$ is analytic for $\alpha\in S_\epsilon$.

(c) $dG_R/d\alpha\neq 0$, for $\alpha\in S_\epsilon$ and hence G_R is conformal for $\alpha\in S_\epsilon$ (i.e., we can choose ϵ and r_ϵ small enough but finite such that this is true). This follows easily from (12) and (9). Thus the mapping $\alpha_R\leftrightarrow\alpha$ is one to one for $\alpha\in S_\epsilon$ and $\alpha\equiv G_R^{-1}(\alpha_R)$ is an analytic function of α_R.

After changing variables from α to α_R we have

$$\gamma_R(\alpha_R)\equiv\gamma(\alpha)=\gamma(G_R^{-1}(\alpha_R)) \tag{16}$$

and

$$D_R(p^2;\alpha_R,\mu)\equiv D(p^2;\alpha,\mu)=D(p^2;G_R^{-1}(\alpha_R),\mu)\,, \tag{17}$$

where $\alpha=G_R^{-1}(\alpha_R)$ and the inverse exists and is unique and analytic for $\alpha\in S_\epsilon$. One can now write a new renormalization-group equation in terms of α_R and solve it to get

$$D_R(\lambda^2p^2;\alpha_R,\mu)$$

$$=D_R(p^2;\bar\alpha_R(t,\alpha_R),\mu)\exp\left(2\int_{\alpha_R}^{\bar\alpha_R}\frac{\gamma_R(x)}{\beta_R(x)}\,dx\right), \tag{18}$$

where $\beta_R(x)=a_2x^2+a_3x^3$ and

$$\frac{d\bar\alpha_R}{dt}=a_2\bar\alpha_R{}^2+a_3\bar\alpha_R{}^3\,,$$
$$\bar\alpha_R(0,\alpha_R)=\alpha_R\,. \tag{19}$$

From the conformal property of $G_R(\alpha)$ it follows that we can find a sector in the α_R plane, $S_{\epsilon'}$,

$$S_{\epsilon'}=\{\alpha_R\mid\ |\alpha_R|<r_{\epsilon'};\ \mathrm{Re}\alpha_R>0;\ |\arg\alpha_R|<\epsilon'\}; \tag{20}$$

with $\epsilon'>0$ and

$$\epsilon'\leqslant\epsilon,\quad r_{\epsilon'}\leqslant r_\epsilon\,,$$

such that $D_R(p^2;\alpha_R,\mu)$, $\gamma_R(\alpha_R)$, and $\beta_R(\alpha_R)$ are all analytic for $\alpha_R\in S_{\epsilon'}$. This all is a consequence of the assumption in the proof and the nature of the 't Hooft transformation. To show that this leads to a contradiction we write the solution of Eq. (19) as

$$b\ln\left(\frac{1}{\bar\alpha_R}+b\right)-\frac{1}{\bar\alpha_R}=a_2t+C(\alpha_R)\,,$$
$$C(\alpha_R)=b\ln\left(\frac{1}{\alpha_R}+b\right)-\frac{1}{\alpha_R}\,. \tag{21}$$

Taking p^2 fixed and $p^2<0$, we set

$$\bar\alpha_R(t,\alpha_R)\equiv z\,,\quad t_1\leqslant t\leqslant t_2\,, \tag{22}$$

where z is real and $z_1\leqslant z\leqslant z_2$. We choose t_1 and t_2 large enough so that both z_1, $z_2\in S_{\epsilon'}$. Then we rewrite Eq. (18) as

$$D_R(e^{2T(z)}p^2;\alpha_R,\mu)=D_R(p^2;z,\mu)\exp\left(2\int_{\alpha_R}^{r_{\epsilon'}}\frac{\gamma_R(x)}{\beta_R(x)}dx\right)$$
$$\times\exp\left(2\int_{r_{\epsilon'}}^{z}\frac{\gamma_R(x)}{\beta_R(x)}dx\ ,\right) \tag{23}$$

where

$$T(z)\equiv\frac{1}{a_2}\left[b\ln\left(\frac{1}{z}+b\right)-\frac{1}{z}\right]-\frac{C(\alpha_R)}{a_2}\,. \tag{24}$$

At this stage Eq. (23) holds for real z in the interval $z_1\leqslant z\leqslant z_2$ where by construction $z_1<r_{\epsilon'}$ and $z_2<r_{\epsilon'}$. Now we analytically continue both sides in z for fixed α_R and p^2. The first term on the right is analytic for $z\in S_{\epsilon'}$, the second term does not depend on z, and the third term is also analytic for $z\in S_{\epsilon'}$. This is true by hypothesis. However, for the left-hand side $T(z)$ is analytic for $z\in S_{\epsilon'}$ but the left-hand side by (B) is singular when $\exp[2T(z)]p^2$ becomes timelike, i.e., on the curves given by

$$\mathrm{Im}T(z)=\pm(2n+1)\frac{\pi}{2}\,,\quad n=0,1,2,\ldots\,. \tag{25}$$

For small enough z these curves, as can be seen from (24), are approximately circles tangent to the origin and will always have a segment that lies inside $S_{\epsilon'}$. For small $|z|$ we have

$$T(z)\cong-\frac{1}{a_2z}\,.$$

For $n=0$ we get, from (25),

$$z \cong -\frac{1}{a_2\left(T_r \pm i\frac{\pi}{2}\right)}, \quad T_r \to \infty$$

or $\mathrm{Im}z \cong \pm(\pi a_2/2)\,(\mathrm{Re}z)^2$ as $|z| \to 0$. Thus we have a contradiction no matter how small we take ϵ and ϵ' to be. The only way to avoid the contradiction is to choose the boundary of our domain to be the circle for the $n=0$ case.

We conclude this section with a few relevant remarks.

(1) In generalizing this theorem to massless $(\phi^4)_4$ theory or massless QED one first has to consider the infrared limit $t \to -\infty$. The renormalization group then maps the low-mass end of the $p^2>0$ cut into circles that will lie inside S_ϵ. However, in this case the branch cut in the neighborhood of $p^2=0$ will have no other multiparticle singularities on it. Such cuts can be moved, and the validity of theorem 1 in massless $(\phi^4)_4$ theory or massless QED will depend on whether we can or cannot deform the cut so it lies outside S_ϵ. In QCD we cannot do that because there are actual singularities corresponding to multiparticle states that lie on the circles in the neighborhood of $z=0$, see Ref. 4.

(2) It is crucial for our theorem that we are dealing with four dimensions. In lower dimension, $(\phi^4)_2$ or $(\phi^4)_3$, $\beta(g)=g+O(g^2)$, and the existence of the g^1 term makes a difference. Furthermore, more powerful methods lead to stronger results in these cases.[1]

(3) As stressed above we have treated only massless field theories. These could include theories having particles with spontaneously generated mass. All we need are homogeneous renormalization-group equations with one mass parameter. Our method does not hold for massive field theories although the results could still be true.[4]

(4) Strong Borel summability is ruled out whenever (A) and (B) are true.

III. ANALYTIC CONTINUATION FOR 't HOOFT COUPLINGS

In Sec. II we showed that the renormalization group puts a restriction on any analytic continuation in the coupling constant regardless of the scheme used in defining the coupling. The question naturally arises as to whether one can actually perform such a continuation into the wedge $\mathcal{S}$ defined in Eq. (7), using the inputs (A) and (B). It is clear that using our broad definition of coupling-parameter transformations we can always choose an α' such that at least $\beta'(\alpha')$ has a singularity in $\mathcal{S}$. Thus the analytic continuation into $\mathcal{S}$ can only be done at best for some restricted choices of α unless one is given additional information.

To perform the continuation we start with any standard α and transform to a new coupling parameter α_S defined by

$$\alpha_S \equiv G_S(\alpha) = \alpha + 0(\alpha^2), \tag{26}$$

and we choose

$$\frac{dG_S}{d\alpha} \equiv \frac{a_2}{b_1}\,\frac{\gamma(\alpha)}{\beta(\alpha)}\,G_S(\alpha), \tag{27}$$

where a_2 and b_1 are given in Eqs. (1) and (2), and

$$\beta_S(\alpha_S) \equiv \beta(\alpha)\,\frac{dG_S}{d\alpha}. \tag{28}$$

This transformation exists and is given by

$$G_S(\alpha_S) = \alpha\,\exp\left\{\frac{a_2}{b_1}\int_0^\alpha dx\left[\frac{\gamma(x)}{\beta(x)} - \frac{b_1}{a_2 x}\right]\right\}. \tag{29}$$

It has a unique inverse in the interval $0<\alpha<\alpha^*$, where α^* is the first zero of $\gamma(\alpha)$ or $\beta(\alpha)$, whichever is smaller. G_S^{-1} is singular at $\alpha=\alpha^*$.

The advantage of this new variable can be seen when we calculate the new renormalization-group functions $\beta_S(\alpha_S)$ and $\gamma_S(\alpha_S)$. One gets

$$\gamma_S(\alpha_S) \equiv \gamma(G_S^{-1}(\alpha_S)) = \gamma(\alpha), \tag{30}$$

and from (28)

$$\beta_S(\alpha_S) = \frac{a_2}{b_1}\,\gamma(\alpha)G_S(\alpha) = \frac{a_2}{b_1}\,\alpha_S\gamma_S(\alpha_S). \tag{31}$$

Thus with this new variable we have a situation reminiscent of the case of the photon propagator in QED where β and γ differ only by a factor α. In this section we take D to represent the transverse gluon propagator in the Landau gauge, then our new renormalization-group equation is given by

$$\left[\lambda\frac{\partial}{\partial\lambda} - \frac{a_2}{b_1}\,\alpha_S\gamma_S(\alpha_S)\,\frac{\partial}{\partial\alpha_S} - 2\gamma(\alpha_S)\right]D_S(\lambda p;\alpha_S,\mu) = 0, \tag{32}$$

where

$$D_S(p;\alpha_S,\mu) \equiv D(p;\alpha,\mu), \quad 0 \le \alpha < \alpha^*. \tag{33}$$

With the exception of the factor a_2/b_1 and the factor 2, Eq. (32) is identical in form with the Callan-Symanzik equation for the photon propagator.[6]

The solution of Eq. (32) is now

$$D(\lambda^2p^2;\alpha_S,\mu) = D(p^2;\bar\alpha_S(t,\alpha_S),\mu)\left[\frac{\bar\alpha_S(t,\alpha_S)}{\alpha_S}\right]^{2b_1/a_2}, \tag{34}$$

where again

$$\frac{d\bar{\alpha}_S}{dt}=\frac{a_2}{b_1}\bar{\alpha}_S\gamma_S(\bar{\alpha}_S),\quad \bar{\alpha}_S(0,\alpha_S)=\alpha_S, \tag{35}$$

and μ is taken large enough so that $\alpha=G_S^{-1}(\alpha_S)<\alpha^*$.

Using a subtraction procedure where D is unity at $p^2=-\mu^2$, we simplify (34) further to

$$D(-e^{2t}\mu^2;\alpha_S,\mu)=\left[\frac{\bar{\alpha}_S(t,\alpha_S)}{\alpha_S}\right]^{2b_1/a_2}. \tag{36}$$

The running coupling constant in the α_S variable is then simply related to the gluon propagator in the Landau gauge,

$$\bar{\alpha}_S(t,\alpha_S)=\alpha_S[D(-e^{2t}\mu^2;\alpha_S,\mu)]^{a_2/2b_1}. \tag{37}$$

This again is analogous to the case of QED where indeed we have the special case $(a_2/2b_1)=1$.

We choose large real t_1 and t_2, $t_2>t_1>0$, and set

$$\bar{\alpha}_S(t,\alpha_S)\equiv z,\quad t_1\leq t\leq t_2. \tag{38}$$

From Eq. (36) we get

$$D(-e^{-2T_S(z)}\mu^2;\alpha_S,\mu)=\left(\frac{z}{\alpha_S}\right)^{2b_1/a_2},\quad z_1<z<z_2, \tag{39}$$

where

$$T_S(z)\equiv\int_{\alpha_S}^{z}\frac{dx}{\beta_S(x)}=\frac{b_1}{a_2}\int_{\alpha_S}^{z}\frac{dx}{x\gamma_S(x)}\,. \tag{40}$$

Up to this point all quantities in (39) and (40) are real. The analytic continuation in z can be easily performed in (39). The right-hand side is analytic except for a cut which we choose on the negative real axis.

Using (B) we see that, excluding the negative real axis then in the neighborhood of $z=0$, $T_S(Z)$ must have singularities along the curves

$$\operatorname{Im}T_S(z)=\pm(2n+1)\frac{\pi}{2},\quad n=0,1,2,\ldots\,. \tag{41}$$

For small enough z

$$T_S(z)\cong-\frac{1}{a_2z},$$

and (41) for $n=0$ leads us to the same boundary of analyticity as in the previous section.

In addition, however, $T_S(z)$ could have singularities at those points z_0 which correspond to zeros of dD/dp^2 in the p^2 plane, i.e.,

$$\left.\frac{dD}{dp^2}\right|_{p^2=-e^{2T_S(z_0)}\mu^2}=0. \tag{42}$$

With the exception of such points it is clear that Eq. (39) provides us with an analytic continuation of $T_S(z)$ such that it is regular for $z\in\mathcal{S}$ with $\mathcal{S}$ defined as in Eq. (7).

From (40) we have

$$\gamma_S(z)=\left(\frac{b_1}{a_2}\right)\left(z\frac{dT_S}{dz}\right)^{-1}, \tag{43}$$

and hence $\gamma_S(z)$ is analytic in z for $z\in\mathcal{S}$ except at the zeros of dT_S/dz. The same is obviously true of $\beta_S(z)$. Finally, from Eq. (34) we have for $z_1<z<z_2$,

$$D_S(e^{2T_S(z)}p^2;\alpha_S,\mu)=D_S(p^2;z,\mu)\left(\frac{z}{\alpha_S}\right)^{2b_1/a_2}. \tag{34$'$}$$

This gives an analytic continuation of $D_S(p^2;z,\mu)$ for all $z\in\mathcal{S}$ (with the above exceptions).

In summary using the variable α_S does indeed give us an analytic continuation into $\mathcal{S}$ for $D_S(p^2;z,\mu)$, $\gamma_S(Z)$, $\beta_S(z)$.

The question is now can this be true for general choices of α? Clearly with the kind of general arguments we are using in this paper the answer is no. For we can always define an α_Q and a β_Q such that by construction $\beta_Q(z)$ has a singularity for $z\in\mathcal{S}$. The only fixed property of β is the invariance of the first two coefficients of the power-series expansion.

However, for specific choices of coupling one can get analyticity in some $\mathcal{S}'$, for example the 't Hooft coupling α_R, of Sec. II where $\beta_R(\alpha_R)\equiv a_2\alpha_R^2+a_3\alpha_R^3$. We consider the transformation $\alpha_S\to\alpha_R$ given by

$$\alpha_R\equiv H(\alpha_S)=\alpha_S+O(\alpha_S^2), \tag{44}$$

where

$$\frac{dH}{d\alpha_S}=\frac{1}{\beta_S(\alpha_S)}(a_2H^2+a_3H^3). \tag{45}$$

Following arguments exactly analogous to those of Sec. II, Eqs. (12)–(15), it is easy to see that since $\beta_S(z)$ is analytic for $z\in\mathcal{S}$, $H(z)$ is analytic there and also conformal such that $H^{-1}(z)$ is analytic in a smaller but similar wedge $\mathcal{S}'$. But we have the relation

$$\gamma_R(z)\equiv\gamma_S\circ(H^{-1}(z)). \tag{46}$$

Hence, $\gamma_R(z)$ is analytic in some similar wedge $\mathcal{S}''$, $\mathcal{S}''\subset\mathcal{S}$, since both H^{-1} and γ_S are analytic. Furthermore, $\beta_R(z)=a_2z^2+a_3z^3$ and therefore trivially analytic for $z\in\mathcal{S}''$. The renormalization group now guarantees us that $D_R(p^2;z,\mu)$ is analytic for $z\in\mathcal{S}''$.

It is clear that in general without additional input we cannot carry out the argument that took us from α_S to α_R to go from α_S to any α. The argument is crucially dependent on the simple form of $\beta_R(\alpha_R)$. One can consider the transformation $\alpha_S\to\alpha_Q$ where the relevant β_Q is chosen, for example, as $\beta_Q=\alpha_Q^4(\delta^2-\alpha_Q^2)^{1/2}+a_2\alpha_Q^2+a_3\alpha_Q^3$, with δ as small as one wishes. This leads to a $\beta_Q(z)$ which has a singularity for $z \in \mathcal{S}$ and gives us a counterexample to a generalization of the argument for α_R to all α, without new information not included in this paper.

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