PHYSICAL REVIEW D VOLUME 23, NUMBER 10 15 MAY 1981

# Fermionic expansion in quantum electrodynamics

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(Received 23 October 1980)

Within the framework of an Abelian U(1) gauge theory, with $N$ massive fermions, we develop the formalism of the $1/N$ expansion and compute explicitly the $\beta$ function of the on-shell renormalization scheme up to first nontrivial order. Borel summability is discussed.

## I. INTRODUCTION

In this paper the underlying field theory will be an Abelian gauge field theory with $N$ massive fermions. All of the fermions will be assumed to have the same mass $m$ and charge $e$. We will choose to renormalize our theory according to the traditional on-shell renormalization scheme: The coupling constant $\alpha$ will be the usual fine-structure constant and the renormalized mass will be defined as the position of the pole of the total propagator. (For a review of renormalization schemes in QED, see Ref. 1.) We will develop a perturbative approach according to increasing powers of $1/N$. One could call such an approach a "fermionic expansion" or "flavor expansion." Let us emphasize the fact that this expansion should not be confused with the "$1/N$ expansion" used in non-Abelian gauge theory where $N$ refers to the gauge group.[2,3] In Sec. II we will explain the basic features of the method, using the photon propagator as an example. Then, after introducing the usual Callan-Symanzik $\beta$ function, we will show in Sec. III how to develop it according to the powers of $1/N$. The first term of the expansion will be obtained in a straightforward manner, but the computation of the next term will involve an infinite number of Feynman diagrams. To tackle these diagrams, we will use the method introduced by deRafael and Rosner[4] that is reviewed in Sec. IV. As a by-product we will be able to exhibit analytically the contribution of diagrams of Fig. 1(a) to any finite order in $\alpha$, and will recover, of course, the results of Refs. 4 and 5.

In Sec. V we recall the notion of Borel summability and discuss the nature of the series whose formal sum is $\tilde{\beta}_2(K)$, with

$$\beta = \tilde{\beta}_1(K) + \tilde{\beta}_2(K)\frac{1}{N} + O\left(\frac{1}{N^2}\right) \quad \left(K = \frac{\alpha N}{\pi}\right).$$

Actually, we will find that $\tilde{\beta}_2$ is given formally by a non-Borel-summable series. In Sec. VI we will discuss the possible generalization of this approach to the case of a non-Abelian gauge group.

In this article we have deliberately avoided the use of functional integration to generate the $1/N$ expansion. This could, of course, have been done, but, for technical as well as "pedagogical" reasons, we have decided to use a more pedestrian (and diagrammatic) approach.

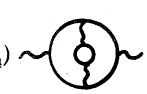
## II. THE FERMIONIC EXPANSION

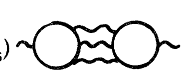
In order to explain the basic idea leading to the concept of fermionic expansion, we will use the photon propagator as an example. The first diagram shown below is of order $\alpha N$ because there are actually $N$ diagrams of this type with the same value. Let us define $K = \alpha N/\pi$ and keep $K$ as a constant. Then

$(i_1)$  is of order $\alpha N/\pi = K \times 1$ ,

$(i_2)$  is of order $(\alpha/\pi)^2 N = K^2 \times 1/N$ ,

$(i_3)$  is of order $(\alpha/\pi)^3 N = K^3 \times 1/N^2$ ,

$(i_4)$  is of order $(\alpha/\pi)^3 N^2 = K^3 \times 1/N$ ,

$(i_5)$  is of order $(\alpha/\pi)^4 N^2 = K^4 \times 1/N^2$ .

Notice that the diagrams $(i_3)$ and $(i_4)$ are of the same order in $\alpha$, but are not of the same order in $1/N$.

We will be mainly interested in the photon proper self-energy $i\Pi^{\mu\nu}$ since one can recover the complete propagator $-iD_{\mu\nu}$ using the relation $D = D^0 + D^0 \Pi D$ (where $-iD^0$ is the free propagator). The leading term in the fermionic expansion of $\Pi$ is obviously given by diagram $(i_1)$. It is easily seen that, of all the diagrams of a given order $\alpha^p$, the ones contributing to leading order in $1/N$ are those which maximize the number of fermionic loops, e.g., diagrams of Fig. 1(a). Notice that diagrams of Fig. 1(b) would also be of order $1/N$ but they are strictly equal to zero because of the Furry theorem. The fermionic expansion of $\Pi$

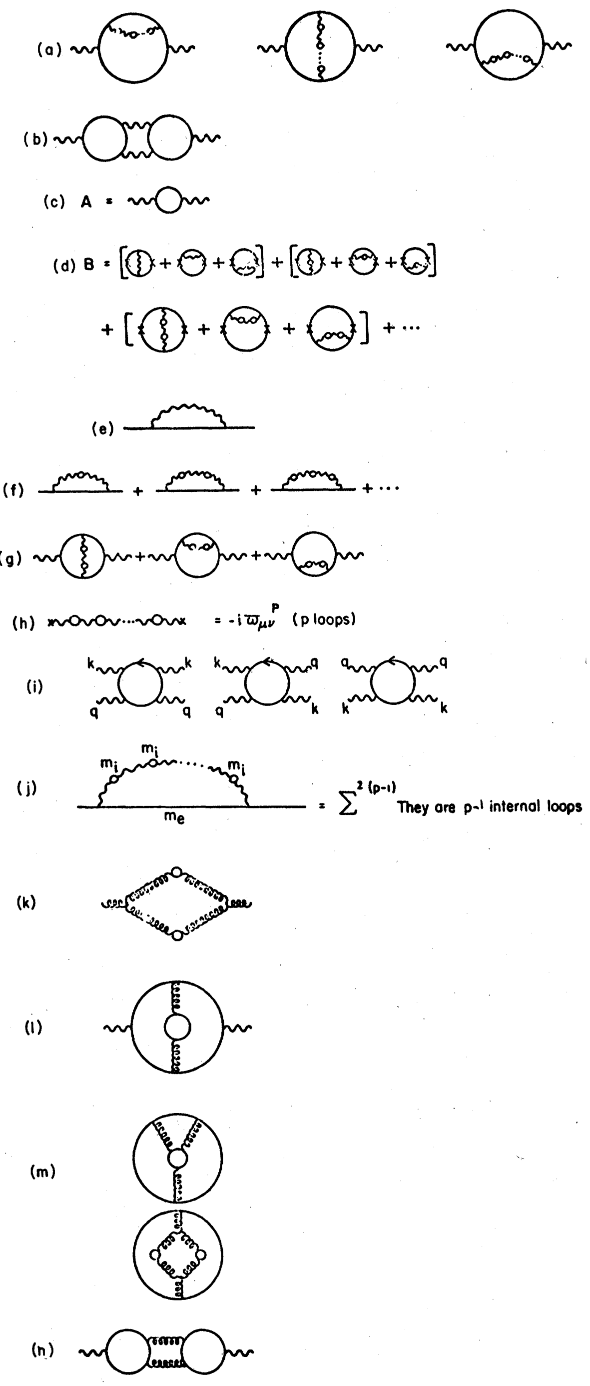



FIG. 1. Feynman diagrams referred to in the text.

has the following structure:

$$\Pi = A + B\times\frac{1}{N} + C\times\frac{1}{N^2} + O\left(\frac{1}{N^3}\right).$$

The diagrams contributing to $A$ and $B$ are displayed in Figs. 1(c) and 1(d). The previous method could be applied as well to another Green's function and $S$-matrix elements. For example, studying the fermionic proper propagator, the leading term (of order 1) is given by Fig. 1(e) and the next leading term (of order $1/N$) is given by the infinite sum of diagrams in Fig. 1(f).

## III. THE CALLAN-SYMANZIK $\beta$ FUNCTION IN THE FERMIONIC EXPANSION

Let us recall some basic definitions and results. The renormalized photon proper self-energy tensor $i\Pi_R^{\mu\nu}(q)$ has the structure

$$i\Pi_R^{\mu\nu}(q) = -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi_R(q^2, m^2, \alpha)\,. \tag{3.1}$$

The general expression for the renormalized photon propagator is

$$\alpha D_R^{\mu\nu}(q) = -i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{\alpha d_R(q^2/m^2, \alpha)}{q^2} + \alpha(\zeta - 1)i\frac{q^\mu q^\nu}{q^2}\,, \tag{3.2}$$

where $\zeta$ is the gauge parameter. Moreover,

$$d_R = \frac{1}{1+\Pi_R} \text{ and } D^{\mu\nu} = D^{0\mu\nu} + D^{0\mu\rho}\Pi_{\rho\sigma}D^{\sigma\nu}\,. \tag{3.3}$$

If $Z_3$ denotes the renormalization constant of the photon field, we have the following relations between the bare and renormalized quantities:

$$A^\mu = \sqrt{Z_3}A_R^\mu\,, \tag{3.4}$$

$$D = Z_3 D_R\,, \tag{3.5}$$

$$(1+\Pi) = \frac{1}{Z_3}(1+\Pi_R)\,, \tag{3.6}$$

$$\Pi_R(q^2, m^2) = \Pi(\Lambda^2, q^2, m^2) - \Pi(\Lambda^2, 0, m^2)\,. \tag{3.7}$$

$Z_3 = 1 - \Pi(\Lambda^2, q^2=0, m^2, \alpha)$, where $\Lambda$ is a UV cutoff. Applying the operation $m d/dm$ to both sides of (3.6), one obtains the Callan-Symanzik equation[6-10]

$$\left[m\frac{\partial}{\partial m} + \beta(\alpha)\left(\alpha\frac{\partial}{\partial\alpha} - 1\right)\right][1+\Pi_R(q^2/m^2, \alpha)] = F(q^2/m^2, \alpha)\,, \tag{3.8}$$

where

$$\beta(\alpha) = \frac{m}{Z_3}\frac{dZ_3}{dm}\,. \tag{3.9}$$

By direct application of Weinberg's theorem on high-energy behavior,[10] it can be shown that $F(q^2/m^2, \alpha)$ vanishes at each order of perturbation theory when $-q^2/m^2 \to \infty$. Then, by denoting as $\Pi_R^\infty$ the asymptotic self-energy part of the photon propagator, i.e., deleting all terms which go to zero when $q^2 \to \infty$, one obtains

$$\left[m\frac{\partial}{\partial m}+\beta(\alpha)\left(\frac{\alpha\partial}{\partial\alpha}-1\right)\right][1+\Pi_R^\infty(q^2/m^2,\alpha)]=0\,. \quad (3.10)$$

In the case of a theory with $N$ fermions of the same mass $m$, one obviously has the same equation, but now $\beta$ and $\Pi_R^\infty$ will depend on $N$. According to our general philosophy, we will develop $\beta$ in powers of $1/N$, viz.,

$$\beta=\tilde{\beta}_1(K)+\tilde{\beta}_2(K)\times\frac{1}{N}+O\left(\frac{1}{N^2}\right). \quad (3.11)$$

Knowledge of $\tilde{\beta}_1$ is directly linked to the value of the diagram in Fig. 1(c). It is a well-known fact that, for $N=1$,

$$\Pi_R^\infty(q^2,m^2)=\frac{\alpha}{\pi}[a_1+b_1\ln(-q^2/m^2)]+O\left(\left(\frac{\alpha}{\pi}\right)^2\right), \quad (3.12)$$

with $a_1=\frac{5}{9}$ and $b_1=-\frac{1}{3}$. Then using (3.10) and the usual development for $N=1$, one finds

$$\beta(\alpha)=\sum\left(\frac{\alpha}{\pi}\right)^m\beta_m,\quad \beta_1=\tfrac{2}{3}. \quad (3.13)$$

Henceforth, for $N$ arbitrary one will obtain

$$\tilde{\beta}_1(K)=\frac{2}{3}\times\frac{\alpha}{\pi}\times N=\tfrac{2}{3}K\,. \quad (3.14)$$

Knowledge of $\tilde{\beta}_2$ is linked to the value of the diagrams in Fig. 1(d). In the $N=1$ case, let us denote as $\beta_n^{[n-1]}$ the contribution to $\beta$ of the diagrams of order $(\alpha/\pi)^n$ with $n-1$ fermionic loops. Of these $n-1$ loops, one is external and $n-2$ are internal. For example, $\beta_4^{[3]}$ is associated with the three diagrams of Fig. 1(g).

Then, if $N\neq 1$, one will obtain

$$\frac{1}{N}\tilde{\beta}_2=\sum_{n=2}^{\infty}\left(\frac{\alpha}{\pi}\right)^n\beta_n^{[n-1]}N^{n-1}=\frac{K^2}{N}\sum_{n=2}^{\infty}K^{n-2}\beta_n^{[n-1]}\,. \quad (3.15)$$

Thus,

$$\tilde{\beta}_2(K)=K^2C(K) \quad (3.16)$$

with

$$C(K)=\beta_2^{[1]}+\beta_3^{[2]}K+\beta_4^{[3]}K^2+\beta_5^{[4]}K^3+\cdots\,. \quad (3.17)$$

## IV. METHOD OF COMPUTATION

By using and generalizing the technique described in Ref. 4 for the case $n=3$, we will compute the quantities $\beta_n^{[n-1]}$ for arbitrary $n$. We will not repeat all the details of this method since they can be found elsewhere. However, for the sake of completeness, we will sketch the various steps of the procedure.

*Step 1.* The Callan-Symanzik equation (3.10) leads to

$$\beta(\alpha)=\frac{m(\partial/\partial m)\alpha\tilde{\Pi}_R^\infty}{1-\alpha^2(\partial/\partial\alpha)\tilde{\Pi}_R^\infty}, \quad (4.1)$$

where $\tilde{\Pi}_R^\infty$ is defined by $\Pi_R^\infty=\alpha\tilde{\Pi}_R^\infty$.

*Step 2.* The numerator of (4.1) may be separated into two terms:

$$m\frac{\partial}{\partial m}\alpha\tilde{\Pi}_R^\infty=m_e\frac{\partial}{\partial m_e}\alpha\tilde{\Pi}_R^\infty(q^2,m_e^2,m_i^2,\alpha)\Big|_{m_i=m_e=m}+m_i\frac{\partial}{\partial m_i}\alpha\tilde{\Pi}_R^\infty(q^2,m_e^2,m_i^2,\alpha)\Big|_{m_i=m_e=m},$$

where the external and internal masses $(m_e,m_i)$ are defined in the following way. Assume that all internal photon self-energy parts in a vacuum polarization diagram are shrunk down to points. The mass in any remaining fermion loop will be called external; all other fermion masses in the internal diagram will be called internal. Then one can prove, using the Adler-Bardeen theorem,[11] that the $m_i\partial/\partial m_i$ part is exactly canceled by the denominator of (4.1). Hence,

$$\beta(\alpha)=m_e\frac{\partial}{\partial m_e}\alpha\tilde{\Pi}_R^\infty(q^2,m_e^2,m_i^2,\alpha)\Big|_{m_i=m_e=m}\,. \quad (4.2)$$

*Step 3.* If one calls $\Pi_{R(2p)}^{\gamma\nu[p-1]}$ the sum of the three diagrams of Fig. 1(a) with $p-2$ internal loops, one sees that this quantity can be obtained easily by inserting $\overline{\omega}_{\rho\tau}^{(p)}$, which is defined in Fig. 1(h), in the lowest-order light-by-light scattering amplitude corresponding to the forward scattering $q+k\to q+k$ [Fig. 1(i)]. This amplitude is described by the tensor $\Pi_{(0)}^{\mu\nu\rho\tau}(q,k,m^2)$ which, owing to gauge invariance, is a finite quantity and does not need renormalization. A direct consequence of (3.1) is that

$$g_{\gamma\nu}\Pi^{\gamma\nu}=-3q^2\Pi \quad (4.3)$$

and

$$g_{\alpha\beta}q^\mu q^\nu\frac{\partial}{\partial q_\alpha}\frac{\partial}{\partial q_\beta}\Pi_{\mu\nu}(\Lambda^2,q^2,m^2)=-6q^2\Pi(\Lambda^2,q^2,m^2,\alpha)\,. \quad (4.4)$$

Putting all things together, one obtains

$$\beta_p^{[p-1]}=m_e\frac{\partial}{\partial m_e}\text{asymp}\int\frac{d^4k}{(2\pi)^4}[-i\overline{\omega}_{\rho\sigma}^{[p]}(k^2,m_i^2)]g_{\mu\nu}\left[-\frac{1}{3q^2}\Pi^{\mu\nu\rho\sigma}(q,k,m_e^2,\Lambda^2)+\tfrac{1}{24}g_{\alpha\beta}\frac{\partial^2}{\partial q_\alpha\partial q_\beta}\Pi^{\mu\nu\rho\sigma}(q,k,m_e^2,\Lambda^2)\Big|_{q=0}\right]_{m_i=m_e=m}. \quad (4.5)$$

In this formula the symbol "asymp" refers to the asymptotic part. The quantity $\Pi^{\mu\nu\rho\tau}(q,k,m^2,\Lambda^2)$ differs from $\Pi^{\mu\nu\rho\tau}_{(0)}(q,k,m^2)$ by a mass renormalization counterterm associated with the diagram in Fig. 1(j). One can then move $m_e\partial/\partial m_e$ through $\bar{\omega}^{[p]}_{\rho\tau}(k^2,m_i^2)$ to obtain

$$\beta_p^{[p-1]}=\text{asymp}\int\frac{d^4k}{(2\pi)^4}[-i\bar{\omega}^{[p]}_{\rho\tau}(k^2,m^2)]g_{\mu\nu}m\frac{\partial}{\partial m}\left[-\frac{1}{3q^2}\Pi^{\mu\nu\rho\tau}(q,k,m^2,\Lambda^2)+\tfrac{1}{24}g_{\alpha\beta}\frac{\partial^2}{\partial q_\alpha\partial q_\beta}\Pi^{\mu\nu\rho\tau}(q,k,m^2,\Lambda^2)\bigg|_{q=0}\right]. \tag{4.6}$$

*Step 4.* One can then prove, owing to Weinberg's theorem,[10] that the contribution of the first term vanishes. Then,

$$\beta_p^{[p-1]}=\text{asymp}\int\frac{d^4k}{(2\pi)^4}[-i\bar{\omega}^{[p]}_{\rho\tau}(k^2,m^2)]g_{\mu\nu}m\frac{\partial}{\partial m}\left[\frac{1}{24}g_{\alpha\beta}\frac{\partial^2}{\partial q_\alpha\partial q_\beta}\Pi^{\mu\nu\rho\tau}(q,k,m^2,\Lambda^2)\bigg|_{q=0}\right]. \tag{4.7}$$

*Step 5.* The quantity $\bar{\omega}^{[p]}_{\rho\tau}$ is obtained in a straightforward way:

$$\bar{\omega}^{[0]}_{\rho\tau}=\frac{g_{\rho\tau}}{k^2+i\epsilon},$$

$$\bar{\omega}^{[1]}_{\rho\tau}=-\left(g_{\rho\tau}-\frac{k_\rho k_\tau}{k^2}\right)\frac{1}{k^2+i\epsilon}\Pi^{\infty}_{R(2)}, \tag{4.8}$$

$$\bar{\omega}^{[p-2]}_{\rho\tau}=(-1)^{p-2}\left(g_{\rho\tau}-\frac{k_\rho k_\tau}{k^2}\right)\frac{1}{k^2+i\epsilon}(\Pi^{\infty}_{R(2)})^{p-2},$$

with $\Pi^{\infty}_{R(2)}=(\alpha/\pi)[a_1+b_1\ln(-k^2/m^2)]$. One then obtains

$$\beta_p^{[p-1]}=\text{asymp}\int\frac{d^4k}{(2\pi)^4}(-1)^{p-2}\left(\frac{-i}{k^2+i\epsilon}\right)[a_1+b_1\ln(-k^2/m^2)]^{p-2}\frac{m\partial}{\partial m}\;\frac{g_{\rho\tau}g_{\mu\nu}g_{\alpha\beta}}{24}\;\frac{\partial^2\Pi^{\mu\nu\rho\tau}(q,k,m^2,\Lambda^2)}{\partial q_\alpha\partial q_\beta}\bigg|_{q=0}. \tag{4.9}$$

*Step 6.* One now separates the mass renormalization counterterm which is implicit in the definition of $\Pi^{\mu\nu\rho\tau}(q,k,m^2,\Lambda^2)|_{q=0}$. We shall write

$$\beta_p^{[p-1]}=\beta_p^{[p-1]}(\text{direct})+\beta_p^{[p-1]}(\text{mass counterterm}). \tag{4.10}$$

The direct contribution is equal to

$$\beta_p^{[p-1]}(\text{direct})=\int\frac{d^4k}{(2\pi)^4}(-1)^p\left(\frac{-i}{k^2+i\epsilon}\right)[a_1+b_1\ln(-k^2/m^2)]^{p-2}m\frac{\partial}{\partial m}\;\frac{1}{-k^2}\Xi\left(\frac{m^2}{-k^2}\right), \tag{4.11}$$

where $\Xi$ is defined as

$$\frac{\alpha^2}{-k^2}\Xi\left(\frac{m^2}{-k^2}\right)=\frac{g_{\mu\nu}g_{\rho\tau}g_{\alpha\beta}}{24}\;\frac{\partial^2\Pi^{\mu\nu\rho\tau}_{(0)}(q,k,m^2)}{\partial q_\alpha\partial q_\beta}. \tag{4.12}$$

The mass conterterm contribution is equal to

$$\left(\frac{\alpha}{\pi}\right)^{p-1}\beta_p^{[p-1]}(\text{mass conterterm})=-\tfrac{2}{3}m_e\frac{\partial}{\partial m_e}\frac{1}{m_e}\Sigma^{2(p-1)}_{(m_e,m_i)}\bigg|_{m_e=m_i=m}, \tag{4.13}$$

where $\Sigma^{2(p-1)}$ is the contribution of Fig. 1(j) to the $2(p-1)$ order of the unrenormalized self-mass of the fermions.

*Step 7.* Using Feynman parametrization, and performing a Wick rotation over the variable $k$, we obtain

$$\beta_p^{[p-1]}(\text{direct})=-\tfrac{1}{8}(-1)^p\int_0^\infty\frac{dz}{z}(a_1-b_1\ln z)^{p-2}z\frac{d}{dz}\Xi(z) \tag{4.14}$$

with $z=m^2/(-k^2)$, and

$$\beta_p^{[p-1]}(\text{mass counterterm})=-\tfrac{2}{3}(-1)^p\int_0^1 d\theta(\theta-2)\left[a_1+b_1\ln\frac{(1-\theta)^2}{\theta}\right]^{p-2}. \tag{4.15}$$

## V. ANALYTICAL AND NUMERICAL RESULTS

The function $\Xi(z)$ defined in (4.12) can be computed using either Feynman parametrization or the powerful Gegenbauer polynomial expansion technique.[4] The result as given in Ref. 4 is

$$\Xi(z)=-\tfrac{4}{3}\int_0^1 d\chi \sum_{n=1}^{4}\frac{\chi^{n-1}(1-\chi)^{n-1}}{[z+\chi(1-\chi)]^n}[B_n\chi^2(1-\chi)^2+E_n\chi(1-\chi)^3+D_n(1-\chi)^4]\,, \tag{5.1}$$

with

$$B_1=42,\quad E_1=-124,\quad D_1=24,$$
$$B_2=-170,\quad E_2=289,\quad D_2=-65,$$
$$B_3=190,\quad E_3=-274,\quad D_3=62,$$
$$B_4=-69,\quad E_4=90,\quad D_4=-21.$$

This parametric representation is not always convenient. Some pieces of this integral are singular but the final result turns out to be finite, owing to relations such as $E_4+B_4+D_4=0$ or $\sum_{i=1}D_i=0$. Actually, even a "good computer" runs into problems computing $\Xi(z)$ with sufficient precision to be able then to evaluate the integral (4.11). It appears that $\Xi(z)$ can be evaluated in a nice closed form provided we use the natural variable $\theta$, defined by

$$\left\{\begin{matrix}\theta=-\dfrac{1-(1+4z)^{1/2}}{1+(1+4z)^{1/2}}\\ 0<\theta<1\end{matrix}\right\}\Longleftrightarrow\left\{\begin{matrix}z=\dfrac{\theta}{(1-\theta)^2}\\ 0<z=m^2/(-q^2)<\infty\end{matrix}\right\}. \tag{5.2}$$

We obtain

$$\Xi(\theta)=-48\frac{\theta^2(1+\theta^2)\ln\theta}{(1+\theta)^5(1-\theta)}-4\frac{(1+10\theta^2+\theta^4)}{(1+\theta)^4}\,. \tag{5.3}$$

Note that

$$\lim_{\theta\to 0}\Xi(\theta)=-4,\quad \lim_{\theta\to 1}\Xi(\theta)=0\,.$$

Most of the algebraic substitutions and factorizations leading from (5.1) to (5.3) have used the symbolic system REDUCE[12] elaborated by Hearn. We will also need the derivative of $\Xi(\theta)$:

$$\frac{d\Xi(\theta)}{d\theta}=-\frac{96\theta(\theta^4-2\theta^3+4\theta^2-2\theta+1)}{(1+\theta)^6(1-\theta)^2}\ln\theta-\frac{48\theta(1+\theta^2)}{(1+\theta)^5(1-\theta)}-\frac{16(\theta-1)(\theta^2-4\theta+1)}{(1+\theta)^5}\,. \tag{5.4}$$

Note that

$$\frac{d\Xi(0)}{d\theta}=16\,,$$

$$\frac{d\Xi(\theta)}{d\theta}\underset{\theta\to 1}{\sim}\frac{1}{x}[3-3]+[\tfrac{3}{2}-\tfrac{3}{2}]+x[\tfrac{7}{4}-\tfrac{3}{4}-1]+x^2[\tfrac{15}{8}-\tfrac{3}{8}-2]\sim-\frac{x^2}{2}$$

with $x=1-\theta$. Let us recall the basic formulas (with $a_1=\frac{5}{9}$, $b_1=-\frac{1}{3}$):

$$\beta=\tilde\beta_1(K)+\tilde\beta_2(K)\frac{1}{N}+O\left(\frac{1}{N^2}\right),\quad \tilde\beta_1(K)=\tfrac{2}{3}K,\quad \tilde\beta_2(K)=K^2C(K)\,,$$
$$C(K)=\beta_2^{[1]}+\beta_3^{[2]}K+\cdots+\beta_p^{[p-1]}K^{p-2}+\cdots,\quad \beta_p^{[p-1]}=\beta_p^{[p-1]}\,(\text{counterterm})+\beta_p^{[p-1]}\,(\text{direct})\,, \tag{5.5}$$

$$\beta_p^{[p-1]}\,(\text{direct})=-\tfrac{1}{8}(-1)^p\int_0^1\left[a_1+b_1\ln\frac{(1-\theta)^2}{\theta}\right]^{p-2}\frac{d}{d\theta}\Xi(\theta)d\theta\,, \tag{5.6}$$

$$\beta_p^{[p-1]}\,(\text{counterterm})=-\tfrac{2}{3}(-1)^p\int_0^1(\theta-2)\left[a_1+b_1\ln\frac{(1-\theta)^2}{\theta}\right]^{p-2}d\theta\,. \tag{5.7}$$

Despite the awkwardness of (5.3) and (5.4), the functions $\Xi(\theta)$ as well as $d\Xi(\theta)/d\theta$ are bounded in the interval [0,1]; their plots are drawn in Figs. 2(a) and 2(b). Numerical results for $\beta_p^{[p-1]}$ from $p=2$ to $p=20$ are given in Table I. As a special case we recover the results of Refs. 4 and 5:

$$\beta_3^{[2]}\,(\text{direct})=\tfrac{5}{18}-\tfrac{1}{3}=-0.0556\,,$$
$$\beta_3^{[2]}\,(\text{counterterm})=-\tfrac{5}{9}-\tfrac{1}{6}=-0.72\,,$$
$$\beta_4^{[3]}\,(\text{direct})=-\tfrac{19}{162}=-0.1167\,,$$
$$\beta_4^{[3]}\,(\text{counterterm})=\frac{89}{162}+\frac{2\pi^2}{27}=1.27\,.$$

This constitutes a check for the validity of formulas (5.3) and (5.4).

Let us define $C_p^D=\beta_{p+2}^{[p+2-1]}(\text{direct})$ and $C_p^{CT}=\beta_{p+2}^{[p+2-1]}$ (counterterm). Then, $C(K)=C^D(K)+C^{CT}(K)$ with

$$C^D(K)=\sum_{p=0}^{\infty}C_p^D K^p\,, \tag{5.8}$$

$$C^{CT}(K)=\sum_{p=0}^{\infty}C_p^{CT}K^p\,. \tag{5.9}$$

The perturbative polynomials are defined by

$$C_N^D(K)=\sum_{p=0}^{N}C_p^D K^p\,,\quad C_N^{CT}(K)=\sum_{p=0}^{N}C_p^{CT}K^p\,.$$

They are plotted in Fig. 3. It can be seen from the results of Table I and Fig. 3 that (5.9) is a divergent series whose coefficients grow very

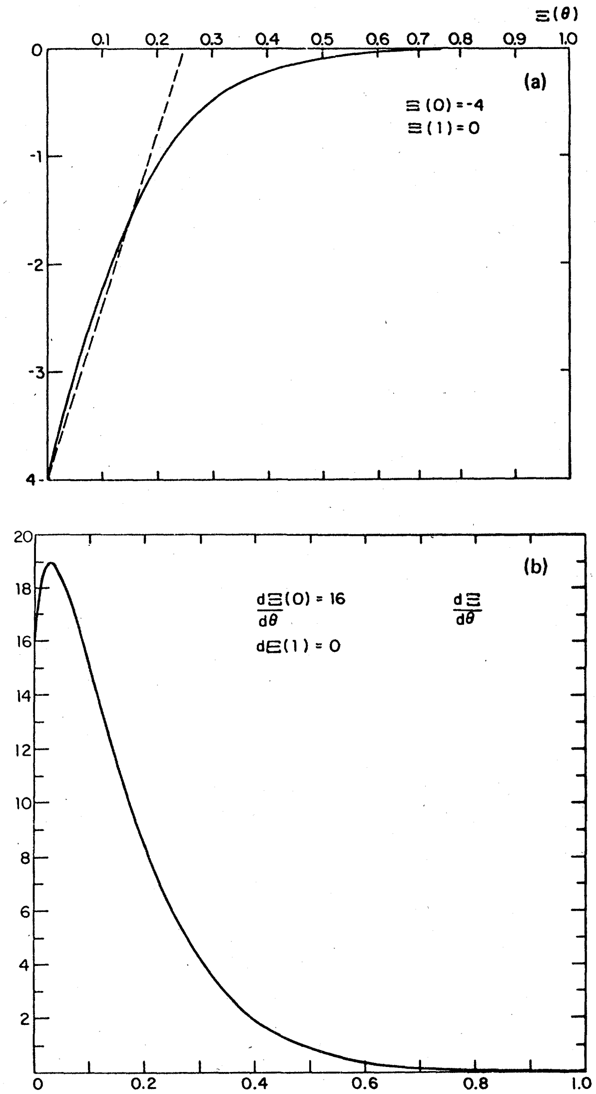



FIG. 2. Plots of (a) $\Xi(\theta)$ and (b) $d\Xi(\theta)/d\theta$.

rapidly with $p$. This will be confirmed later.

We will show in the following that the series (5.8) and (5.9) are not Borel summable. Actually, we will study in particular the mass counterterm contribution (5.9). The results will obviously be the same for the direct contribution (5.8) since the only difference comes from the replacement of $(\theta-2)$ by $d\Xi/d\theta$ which is also a well-behaved bounded function in the interval $[0,1]$.

Let us notice first that the divergent nature of the series (5.8) and (5.9) shows that the family constituted by the contributions to the $\beta$ function from all possible Feynman diagrams does not constitute a summable family. If we were to suppose the contrary, every subfamily would be summable, in particular the one constituted from diagrams of

TABLE I. Numerical values for $\beta^{[p-1]}$ [of order $(\alpha N/\pi)^p N$].

| $p$ | $\beta_p^{[p-1]}$ (counterterm) | $\beta_p^{[p-1]}$ (direct) |
|---|---|---|
| 2 | 1 | −0.5 |
| 3 | −0.72 | −0.055 |
| 4 | 1.27 | −0.115 |
| 5 | −2.53 | −0.060 |
| 6 | 6.93 | −0.111 |
| 7 | −21.70 | −0.109 |
| 8 | 78.61 | −0.197 |
| 9 | −310.12 | −0.249 |
| 10 | 1309.12 | −0.453 |
| 11 | $0.57\times10^4$ | −0.623 |
| 12 | $0.26\times10^5$ | −1.15 |
| 13 | $-0.1\times10^6$ | −1.59 |
| 14 | $0.58\times10^6$ | −3.13 |
| 15 | $-0.27\times10^7$ | −3.9 |
| 16 | $0.13\times10^8$ | −9.1 |
| 17 | $-0.65\times10^8$ | −8.5 |
| 18 | $0.31\times10^9$ | −3.0 |
| 19 | $-0.15\times10^{10}$ | −3.9 |
| 20 | $0.76\times10^{10}$ | $-0.14\times10^3$ |

order $\alpha^p N^{p-1}=(\alpha N)^p/N$ for $p$ ranging from zero to infinity. In such a case, (5.8) and (5.9) would be convergent. Let us recall that a family $\{\alpha_p\}_{p\in I}$ is said to be summable if $\exists M$ such that, for all $j$ finite, $j\subset I, \sum_{p\in j}|a_p|<M$. However, it is very often possible to define what is the sum of a divergent series, especially when it is an alternating series.[13,14] There are many possible nonequivalent definitions, all of which have to be "regular," i.e., when applied to a convergent series, they have to lead to the correct sum. One of the best known definitions is that of Borel[14]: Suppose $\sum_{p=0}^\infty a_p$ is a divergent series. Multiplying and dividing by $p!$, we get $\sum_{p=0}^\infty p!\,a_p/p!$, which is exactly the same. Then expressing $p!$ as an Euler integral, we get

$$\sum_{p=0}^{\infty}\int_0^\infty dt\, e^{-t}t^p\frac{a_p}{p!}\,,$$

which is again the same. Finally, permuting the symbols $\sum$ and $\int$, we get

$$B=\int_0^\infty dt\, e^{-t}\sum_{p=0}^{\infty}t^p\frac{a_p}{p!}\,.$$

When $B$ exists, it is called the Borel sum of the series $\sum_{p=0}^\infty a_p$.

Let us study the case of the series $C^{\rm CT}(K)$,

$$C^{\rm CT}(K)=\sum_{p=0}^{\infty}C_p^{\rm CT}K^p \qquad (5.10)$$

with

$$C_p^{\rm CT}=-\tfrac{2}{3}(-1)^p\int_0^1 d\theta(\theta-2)[f(\theta)]^p \qquad (5.11)$$

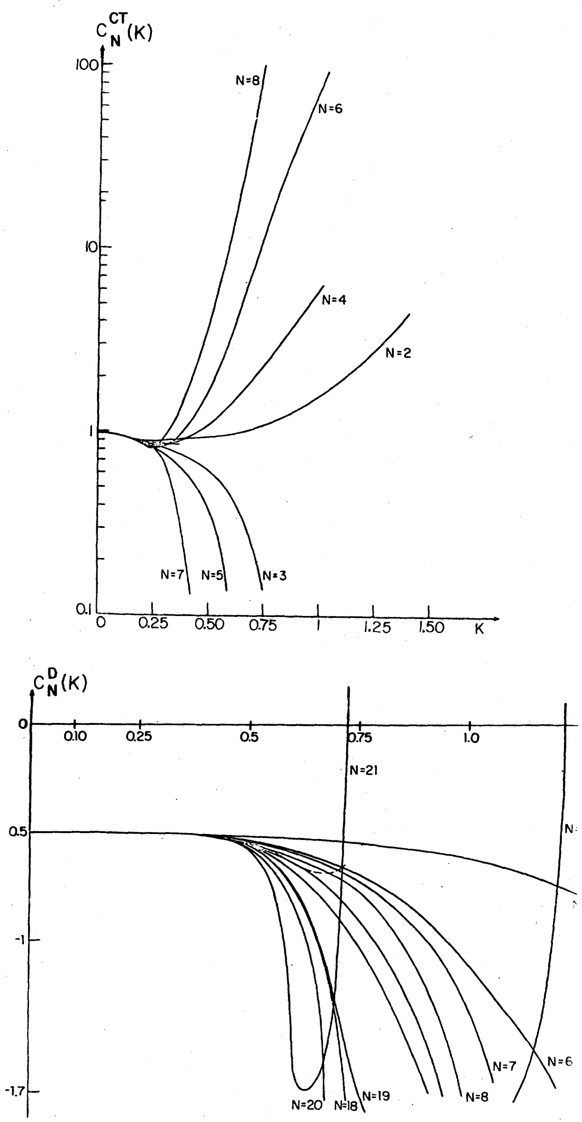



FIG. 3. Plots of perturbative polynomials.

and

$$f(\theta)=a_1+b_1\ln\frac{(1-\theta)^2}{\theta}\,. \tag{5.12}$$

It is clear that, for large $p$, this integral receives contributions primarily from the region near $\theta=0$ and $\theta=1$, where $|f(\theta)|>1$.

The sum of the contributions from these regions (obtained using the saddle-point method) is

$$C_p^{CT}=p!\,[(-1)^p(\tfrac{2}{3})^{p+1}e^{5/6}+4(\tfrac{1}{3})^{p+1}e^{-5/3}][1+O(1/p)]\,. \tag{5.13}$$

[The numerical values given in Table I for $\beta_p$ are calculated via numerical integration using the exact formulas (5.6) and (5.7). Actually, above $p$ $=10$, the numerical integration begins to fail; the approximation (5.13) would give a better value.] The Borel transform of the series (5.9) is

$$B^{CT}(K)=\sum_{p=0}^{\infty}\frac{C_p^{CT}}{p!}K^p\,. \tag{5.14}$$

The Borel sum of series (5.9), if it exists, would be

$$C_B^{CT}(K)=\int_0^{\infty}dt\,e^{-t}B^{CT}(Kt)\,. \tag{5.15}$$

We will show now that the series (5.9) is not Borel summable. One way to see it is to use the asymptotic expression (5.13). The first term of (5.13) which is asymptotically leading) is Borel summable, but the second is not:

$$\sum_{p=0}^{\infty}\frac{1}{p!}[p!(-1)^p(\tfrac{2}{3})^{p+1}e^{5/6}]K^p=\tfrac{2}{3}e^{5/6}\frac{1}{1+\frac{2}{3}K}\,, \tag{5.16a}$$

$$\sum_{p=0}^{\infty}\frac{1}{p!}[p!\,4(\tfrac{1}{3})^{p+1}e^{-5/3}]K^p=\tfrac{4}{3}e^{-5/3}\frac{1}{1-K/3}\,. \tag{5.16b}$$

The first term (5.16a) will lead to a finite integral (5.15) for all positive $K$, but the second will introduce a singularity for $K=3$.

Another way to prove this result is to proceed to the exact evaluation of $B^{CT}(K)$. Indeed, it is not hard to express $B^{CT}(K)$ in terms of the Euler $\Gamma$ function:

$$\begin{aligned}B^{CT}(K)&=\frac{2e^{-Ka_1}}{2-Kb_1}\,\frac{\Gamma(1+Kb_1)\Gamma(1-2Kb_1)}{\Gamma(1-Kb_1)}\\&=\frac{2e^{-5/9K}}{2+K/3}\,\frac{\Gamma(1-K/3)\Gamma(1+2K/3)}{\Gamma(1+K/3)}\,.\end{aligned} \tag{5.17}$$

It is clear that $\Gamma(1+Kb_1)$ is singular for $K=3,6,9,\ldots,(3p),\ldots$. As a consequence, the Laplace transform of $B^{CT}(K)$ leading to the Borel sum (5.15) does not exist.

If we permute $\int$ and $\sum$ in (5.10) and carry out the geometric sum, we get

$$C^D(K)=-\tfrac{1}{8}\int_0^1 d\theta\frac{1}{[1+Kf(\theta)]}\frac{d}{d\theta}\Xi(\theta)\,, \tag{5.18}$$

$$C^{CT}(K)=-\tfrac{2}{3}\int_0^1 d\theta\frac{1}{[1+Kf(\theta)]}(\theta-2)\,. \tag{5.19}$$

As one would expect, these integrals have no meaning (for $K>0$) because of the existence of a zero of the integrand between 0 and 1,

$$1+Kf(\theta)=0\rightarrow\begin{cases}\theta^2z+\theta(-2z-1)+z=0\,,\\ z=\exp\left(\dfrac{1+Ka_1}{Kb_1}\right)>0\,.\end{cases}$$

The discriminant of this second-degree equation is always positive; it is equal to $\Delta=4z+1$. Moreover, the product of the two roots is equal to 1, so there is always a root—which we will call $\theta_0(K)$—between 0 and 1.

In the neighborhood of $\theta_0$,

$$\ln\theta = \ln\left[1+\frac{\theta-\theta_0}{\theta_0}\right]+\ln\theta_0 \sim \frac{\theta-\theta_0}{\theta_0}+\ln\theta_0\,,$$

$$\ln(1-\theta)=\ln\left[1+\frac{(1-\theta)-(1-\theta_0)}{1-\theta_0}\right]+\ln(1-\theta_0)$$

$$\sim\frac{\theta_0-\theta}{1-\theta_0}+\ln(1-\theta_0)\,.$$

Hence,

$$1+Kf(\theta)\underset{\theta\to\theta_0}{\sim} Kb_1\frac{\theta_0+1}{\theta_0(1-\theta_0)}(\theta_0-\theta)\,.$$

We have a singularity of the type "$1/x$." The non-Borel summability of the previous series signals an ambiguity in QED. Another example of this phenomenon has been given in Ref. 15.

In order to give a precise meaning to the quantity $\tilde{\beta}_2$ appearing in the fermionic expansion, one has to handle the singularity of Eqs. (5.18) and (5.19). For instance, one can do this by subtracting a quantity that vanishes in perturbation theory, e.g.,

$$C^{D}_{\text{new}}(K)=-\tfrac{1}{8}\int_0^1 d\theta\frac{1}{1+Kf(\theta)}\left[\frac{d\Xi(\theta)}{d\theta}-\frac{d\Xi(\theta_0)}{d\theta}\right], \tag{5.20}$$

$$C^{\text{CT}}_{\text{new}}(K)=-\tfrac{2}{3}\int_0^1 d\theta\frac{1}{1+Kf(\theta)}[\theta-\theta_0]\,. \tag{5.21}$$

Another possibility would be to handle the first-order singularities in the Laplace transform (5.15) by giving a contour prescription (for example, a principal-value prescription).

Let us now return to the problem of obtaining analytic results from the expressions (5.6) and (5.7). It is clear that the Borel transform $B^{\text{CT}}(K)$ defined in (5.15) can be used as a generator function for quantities

$$\beta^{[p+2-1]}_{p+2}(\text{CT})=\left.\frac{\partial^p B^{\text{CT}}(K)}{\partial K^p}\right|_{K=0}. \tag{5.22}$$

Since the derivatives of $\ln\Gamma(1+x)$ are easier to write than those of $\Gamma(1+x)$, we find it convenient to write

$$B^{\text{CT}}(K)=\exp[G^{\text{CT}}(K)] \tag{5.23}$$

with

$$G^{\text{CT}}(K)=\ln B^{\text{CT}}(K)\,. \tag{5.24}$$

Then

$$B^{\text{CT}(1)}=B^{\text{CT}}G^{\text{CT}(1)}, \tag{5.25}$$

$$B^{\text{CT}(2)}=B^{\text{CT}}[(G^{\text{CT}(1)})^2+G^{\text{CT}(2)}], \tag{5.26}$$

$$B^{\text{CT}(3)}=B^{\text{CT}}[(G^{\text{CT}(1)})^3+3G^{\text{CT}(1)}G^{\text{CT}(2)}+G^{\text{CT}(3)}],\ldots\,. \tag{5.27}$$

It is then clear that $B^{\text{CT}(n)}(K)=(\partial^n/\partial K^n)B^{\text{CT}}(K)$ can be easily computed uniquely in terms of $B^{\text{CT}}(K)$ and the derivatives $G^{\text{CT}(m)}(K)=(\partial^m/\partial K^m)G^{\text{CT}}(K)$. The coefficients appearing in (5.25)–(5.27) are simply products of multinomial coefficients which are tabulated in all books of mathematical functions. To conclude we have only to compute the derivatives $G^{\text{CT}(m)}(K)=(\partial^m/\partial K^m)\ln B^{\text{CT}}(K)\,|_{K=0}$. This calculation, starting from (5.17), is straightforward, using the Euler $\psi$ function defined as

$$\psi(z)=\frac{d}{dz}\ln\Gamma(z)\,. \tag{5.28}$$

We find

$$\frac{\partial^p G^{\text{CT}}(K)}{\partial K^p}=\frac{b_1{}^p(p-1)!}{(2-Kb_1)^p}+b_1{}^p[\psi^{(p-1)}(1+Kb_1)+(-2)^p\psi^{(p-1)}(1-2Kb_1)-(-1)^p\psi^{(p-1)}(1-Kb_1)],\quad p\geq 2\,, \tag{5.29}$$

$$\frac{\partial G^{\text{CT}}(K)}{\partial K}=-a_1+\frac{b_1}{2-Kb_1}+b_1[\psi(1+Kb_1)-2\psi(1-2Kb_1)+\psi(1-Kb_1)]\,. \tag{5.30}$$

If $K=0$, we get

$$G^{\text{CT}(1)}[0]=-a_1+\frac{b_1}{2}\,, \tag{5.31}$$

$$G^{\text{CT}(p)}[0]=\frac{b_1{}^p(p-1)!}{2^p}+b_1{}^p[\psi^{(p-1)}(1)+(-2)^p\psi^{(p-1)}(1)-(-1)^p\psi^{(p-1)}(1)]\,. \tag{5.32}$$

It is convenient to rewrite this result using the Riemann $\xi$ function which is related to $\psi^{(p)}(1)$ by

$$\psi^{(p)}(1)=(-1)^{p+1}p!\,\xi(p+1)\,. \tag{5.33}$$

Then, finally

$$G^{\text{CT}(p)}[0]=\frac{b_1{}^p(p-1)!}{2^p}\{1+\xi(p)2^p[(-1)^p+2^p-1]\}\,. \tag{5.34}$$

One could also write this result in terms of the polylogarithm function $\text{Li}_p(x)$ since $\xi(p)=\text{Li}_p(1)$.

In order to check the previous formulas, let us compute $\beta_2^1(\text{CT})$, $\beta_3^2(\text{CT})$, $\beta_4^3(\text{CT})$, and $\beta_5^4(\text{CT})$. Using $\beta^{\text{CT}}(0)=1$, we find

$$\beta_2^1(\text{CT})=1\,,$$

$$\beta_3^2(\text{CT})=-\tfrac{5}{9}-\tfrac{1}{6}\,,$$

$$\beta_4^3(\text{CT})=\frac{89}{162}+\frac{2\pi^2}{27}\,,$$

$$\beta_5^4(\text{CT})=[-\tfrac{1}{108}-\tfrac{4}{9}\xi(3)]+3[-\tfrac{13}{18}]\left[\frac{1}{36}+\frac{2\pi^2}{27}\right]+(-\tfrac{13}{18})^3,\ldots\,.$$

The expression (5.33), used in conjunction with (5.22) and (5.23), allows us to compute $\beta_p^{[p-1]}$ (counterterm) to any order. We have not attempted to find an analogous result for $\beta_p^{[p-1]}$ (direct), but it is clear that one could be worked out.

The previous calculations have been performed in the Feynman gauge. However, the result is gauge invariant for (at least) two reasons: since $\beta$ is gauge invariant, its $1/N$ expansion must be gauge invariant order by order. Moreover, the three diagrams of Fig. 1(a) constitute a gauge-invariant set of diagrams.

## VI. GENERALIZATION

The fermionic expansion in QED displays interesting features but its phenomenological application is not for today since, by now, only three kinds of fermions (electron, $\mu, \tau$) have been discovered; $N=3$ is not a big number. However, at tremendous energies, far above all the fermionic thresholds, such an approach would lead to new results. (The fact that all the fermions have been assumed to have the same mass would not play any role at such tremendous energies.)

It is clear that the most promising consequences of this approach have to be found in an extension of this procedure to the case of a non-Abelian gauge group. Until now, the topological expansion on the $1/N$ expansion [where $N$ refers to SU($N$)] has produced "qualitative" results, but very few quantitative results. One can hope that, using an expansion with respect to the number of flavors according to the previous ideas, one will be able to find explicit results such as those displayed in this article.

Finally, let us notice that the singularities appearing in the Borel transform $B^{CT}(K)$, Eq. (5.16), which are reminiscent of the presence of a "Landau ghost" ($b_1 < 0$), may be expected to be absent in the quantum-chromodynamics (QCD) case. However, one has to be aware of the fact that, in the pure QCD case, things are much more complicated since the $\beta$ function cannot be obtained from only the knowledge of the vacuum polarization. Even if one is only interested in the (gluonic) vacuum polarization, a diagram such as Fig. 1(k) is of order $\alpha_c^3 n^2 = (\alpha_c n)^3/n$, where $n$ is the number of flavors.

If one wants to apply the previous ideas to the study of the photon vacuum polarization with QCD corrections [which is of interest in order to compute $\tau$ ($e^+e^- \to$ hadrons)], things look better since, for example, the diagram in Fig. 1(l) is of order $(\alpha n)(\alpha_c n)^2/n$, and the diagrams in Fig. 1(m) are eliminated by a factor $1/n$: The first one is proportional to $(\alpha n)(\alpha_c n)^3/n^2$, the other one to $(\alpha n)(\alpha_c n)^4/n^2$. But the diagram in Fig. 1(n), which was equal to zero in the QED case because of Furry's theorem, is no longer equal to zero here since $\underline{8} \times \underline{8} = \underline{1} + \underline{8} + \underline{\bar{8}} + \underline{10} + \underline{\overline{10}} + \underline{27}$ has a singlet component. Thus, the generalization is not obvious but, at least in the case of the QCD corrections to photon vacuum polarization, does not seem impossible.


## ACKNOWLEDGMENTS

I would like to thank E. deRafael and J. Calmet (of the Centre de Physique Théorique in Marseille where part of this work was done) for helpful discussions and comments. This work was supported in part by the U. S. Department of Energy under Contract No. DE-AS02-76ER03227 with Harvard University.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[1] R. Coquereaux, Ann. Phys. (N.Y.) 125, 401 (1980).

[2] G. 't Hooft, Nucl. Phys. B75, 461 (1974); B72, 461 (1974).

[3] E. Witten, Nucl. Phys. B149, 285 (1979).

[4] E. deRafael and J. L. Rosner, Ann. Phys. (N.Y.) 82, 369 (1974).

[5] J. Calmet and E. deRafael, Phys. Lett. 56B, 181 (1975).

[6] C. Callan, Jr., Phys. Rev. D 2, 1541 (1970).

[7] K. Symanzik, Commun. Math. Phys. 18, 227 (1970).

[8] K. Symanzik, Commun. Math. Phys. 23, 49 (1971).

[9] S. L. Adler, Phys. Rev. D 5, 3021 (1972).

[10] S. Weinberg, Phys. Rev. 118, 838 (1960).

[11] S. L. Adler and W. A. Bardeen, Phys. Rev. D 4, 3045 (1971).

[12] A. C. Hearn, REDUCE 2, University of Utah, Salt Lake City, Utah (unpublished).

[13] G. H. Hardy, *Divergent Series* (Oxford University, London, 1949).

[14] E. Borel, *Lecons sur les series divergentes* (Paris, Gauthiers, Villars, ED 2, 1928); (CD 2).

[15] B. Lautrup, Phys. Lett. 69B, 109 (1977).