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Studies in the renormalization-prescription dependence of perturbative calculations

William Celmaster and Dennis Sivers
High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439
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Now that the quantitative testing of perturbative quantum chromodynamics (QCD) has become a major experimental and theoretical effort, it is important to understand the renormalization-prescription dependence of perturbative calculations. We stress the phenomenological importance of finding a definition of the QCD expansion parameter which reduces the magnitude of high-order corrections. We give explicit arguments suggesting that a choice of coupling based on momentum-space subtraction can be phenomenologically useful. Examples from QCD and QED are used to illustrate these arguments, and we also discuss possibilities for refining them.

I. INTRODUCTION

The process of proving or disproving the validity of quantum-chromodynamics (QCD) perturbation theory involves the use of certain experimental data to extract a fundamental parameter, the strong coupling, and the insertion of this parameter into calculations for other physical processes. However, a brief survey of the literature[1-6] makes it apparent that there are many possible definitions for the strong coupling. The procedure for defining the coupling experimentally is sometimes referred to as choosing a prescription. Establishing conventions or choosing prescriptions is necessary even for very simple systems. However, in quantum field theories the issue of prescriptions is tied to the problem of infinities. Consistent comparisons with experiment require the specification of infinities which occur in the calculation and the renormalization of physical quantities.

Most physicists are aware of the fact that the choice of renormalization procedure defines the coupling which occurs in the perturbation expansion. What we would like to stress here is that this is not just an empty formal exercise. The precise definition of an expansion parameter can have important phenomenological consequences. It can affect the magnitude of high-order corrections and, hence, the agreement between experiment and low-order calculations. Some papers which find uncomfortably large corrections in perturbative QCD (Refs. 7, 8) merely reflect an unfortunate choice of prescription.

The salient features of prescription dependence have *nothing* to do with the complexities of QCD. They are simply properties of perturbation theory—*any* perturbation theory. We will therefore begin our discussion by imagining a make-believe world whose dynamics is governed by an imaginary field theory, QID (quantum imaginary dynamics), which is exactly soluble.

Two high-precision experiments are conducted in the QID world. The first is a measurement of the "QID Josephson effect,"

$$J_I = 0.200\,000\,000(13)\,. \tag{1.1a}$$

The second experiment measures the anomalous magnetic moment of the QID electron,

$$a_{eI} = 0.333\,333\,33(12)\,. \tag{1.1b}$$

The *exact* solution of QID is given in terms of a parameter x (to be measured) and as predictions of the theory we find

$$J_I = x \tag{1.2a}$$

and

$$a_{eI} = \frac{x}{1-2x}\,. \tag{1.2b}$$

Then, if the parameter x is measured by J_I (so that $x = 0.2000\cdots$) we see that indeed $a_{eI} = 0.3333\cdots$. This is in splendid agreement with experiment.

Unfortunately for the inhabitants of this imaginary world, they have not succeeded in solving QID. The physicists in this world are forced to resort to "QID perturbation theory." The way they do this is with complicated diagrams. The answer is written in terms of a parameter y which appears naturally in the calculation of these diagrams. It happens that y and x are related by

$$x = y(1+10y)\,, \tag{1.3}$$

but the physicists are unaware of that. In fact, they attach absolutely no significance to x since this parameter arises *only* in the exact solution of QID—about which they know nothing.

By substituting Eq. (1.3) into Eqs. (1.2) and then making a Taylor expansion we easily see that

$$J_I = y + 10y^2\,, \tag{1.4a}$$

$$a_{eI} = y + 12y^2 + 44y^3 + \cdots\,. \tag{1.4b}$$

This expansion is extremely difficult for the physicists to obtain with QID perturbation theory, but

they manage, after a while, to compute the leading term in the above expansions.

The leading term of (1.4a) and the experimental value of J_I give a measurement of y—namely, $y = 0.20$. By substituting this in the equation $a_{eI} = y + \cdots$, the physicists arrive at a prediction, $a_{eI} = 0.2$, in bad disagreement with experiment. However, by perseverance they finally compute the next-to-leading terms of Eqs. (1.4a) and (1.4b). They solve (1.4a) to obtain

$$y + 10y^2 = 0.200\,00\cdots$$
$$\Rightarrow y = 0.10\cdots . \tag{1.5}$$

Substituting this into $a_{eI} = y + 12y^2 + \cdots$, they predict $a_{eI} = 0.22$. This still disagrees with experiment *despite* the fact that $y = 0.1$ seems like a fairly small expansion parameter. The problem is closely related to the fact that the correction terms in (1.4a) and (1.4b) have large coefficients.

It is that realization which eventually saves the day in this imaginary world. Some physicists realize that y may not be a sensible expansion parameter. By careful inspection of their calculations and "diagrams" they *guess* that they should reexpand all of their answers in terms of a new parameter, $x = y + 10y^2$. This new parameter is just the x of Eq. (1.3). A reexpansion of Eqs. (1.4a) and (1.4b) through next-to-leading order gives

$$J_I = x + O(x^3), \tag{1.6a}$$
$$a_{eI} = x + 2x^2 + O(x^3). \tag{1.6b}$$

[We can also obtain these expansions directly from the exact solution, Eqs. (1.2).] By fitting J_I to experiment, the physicists find $x = 0.20$ and predict $a_{eI} = 0.28$. Although this is not in perfect agreement with the experimental result of $0.333\cdots$, it is a large improvement over the prediction that was made with the expansion in terms of the y variable. Furthermore, the correction terms now have reasonably behaved coefficients and the reliability of the perturbation expansion appears less questionable than the results expressed in terms of y.

This dependence of coefficients and predictions upon the choice of expansion parameter is known as "prescription dependence." The scenario depicted above illustrates a number of important points which we list here.

(1) Unless we know the exact solution to a field theory, we have no way of being certain which parameter is likely to lead to the most reliable perturbation expansion.

(2) An inappropriate choice of parameter can lead to apparent disagreement between theory and experiment.

(3) An inappropriate choice of parameter can lead to large coefficients for correction terms. In fact, this may be the most obvious symptom of the bad choice.

(4) Even though a reexpansion may lead to a *larger* parameter ($y = 0.10$ becomes $x = 0.20$ in the example above), it can still yield a *more reliable expansion.*

(5) If the y expansion had been carried out to the third term $[O(y^3)]$, the agreement with experiment would have been better. However, by an appropriate choice of parameter (x) one can get equally good agreement without going beyond the second term. This is an obvious advantage since it is excruciatingly difficult to compute beyond the second term.

The real world, where QCD is assumed to govern the strong interactions, is rather similar to the imaginary world described above. The value of the QCD expansion parameter (α_s/π) is of order 0.1. Also, since we have no idea how to solve QCD, we do not know which expansion parameter is "best." Many recent calculations of next-to-leading coefficients[1,7] seem to give large numbers. This has sometimes been interpreted as a signal that the QCD asymptotic expansion is failing, but it may simply be a consequence of using a "bad" parameter. Unfortunately, the real world is unlike the imaginary world in that there does not yet exist any high-precision QCD measurement, so it is difficult to test improvements in perturbation theory.

Nevertheless, there are two independent pieces of information which make us believe that reexpansions can improve our predictions. First of all, experimental evidence in many processes supports the leading-order predictions of the scale-breaking parton model. If the series truly began to diverge at the second term, then there would be little reason to trust the leading predictions to even an order of magnitude. Secondly, and more compelling, is our success in understanding QED through many orders (the *fourth-order* correction to the electron's anomalous magnetic moment is almost complete[9]). The perturbation expansions appear well behaved and the agreement with experiments is spectacular. QED is, of course, less complicated than QCD and unlike QCD, there is a unique *physically motivated* choice for the QED coupling constant. Furthermore, $\alpha_{\rm QED} \ll \alpha_s$. However, what is relevant is the size of coefficients rather than the size of the expansion parameters. From a purely mathematical viewpoint the two theories have rather similar perturbative structures, so it

would be surprising if QCD perturbative theory were intrinsically much less well behaved than QED. If we can understand the *mathematical* reason for the success of $\alpha_{\rm QED}$ as an expansion parameter, then this may lead to a choice of α_s which might be expected to give a series with small coefficients.

The plan of this paper is as follows. In Sec. II we review the machinery for defining a prescription and discuss explicitly three distinct schemes: minimal subtraction[2] (MS), modified minimal subtraction[5] ($\overline{\rm MS}$), and momentum-space subtraction[3,4] (MOM). The two "minimal" schemes are conceptually simple and convenient. The goal behind momentum-space subtraction is to suppress the contribution of high-order diagrams at a given momentum. We compare these QCD prescriptions with the more familiar case of QED.

Section III discusses the choice of a prescription for physical calculations. We review the procedure of extracting the coupling from one experiment and using it in an asymptotic expansion for other processes. Arguments based on the analysis of high-order diagrams suggest that the procedures behind the technique of momentum-space subtraction might give a well-behaved perturbation series. We show how important it is to make a good choice for the coupling.

Sections IV and V contain all our examples from QED and QCD. We show that the difference between a good and a bad prescription should be considered when comparing experiments to theory. Not all experiments are equally sensitive to prescription. We compare explicitly expansions in the MS, $\overline{\rm MS}$, and MOM schemes for several processes. We discuss calculations for the decay of paraquarkonium, the total e^+e^- annihilation cross section, and structure functions. These results provide support to the point of view that momentum-space subtraction provides the most useful expansion. We also reexamine the calculation for the decay of parapositronium in QED.

In Sec. VI we consider the possibility of generalizing our arguments for how to choose a best prescription. Is it possible to turn the "art" of choosing a prescription into a "science"? Finally, in Sec. VII we give a summary of our results. Appendices are included on the asymptotic expansion of the Euler Γ function and on the prescription dependence of the QCD β function.

II. SIMPLE PRESCRIPTIONS FOR RENORMALIZATION

In order to calculate matrix elements for physical processes in a renormalizable field theory, it is necessary to assign values to a finite number of integrals which are formally divergent. We can illustrate this process of "renormalization" for the theory of QCD with n_f flavors of massless quarks. The Feynman rules for the perturbation expansion of the theory in covariant gauges can be found in the review articles in Ref. 1.

There are seven primitive divergences which must be specified in the theory. Diagrams for these processes through one loop are shown in Fig. 1 where we define the related counterterms. The seven counterterms are, of course, not all independent but are constrained by the Ward identities

$$\frac{Z_1^F}{Z_2}=\frac{Z_1}{Z_3}=\frac{\tilde{Z}_1}{\tilde{Z}_3}, \qquad \frac{Z_4}{Z_3}=\left(\frac{Z_1}{Z_3}\right)^2, \quad Z_3^{-1}\xi_B=\xi\,. \tag{2.1}$$

Apart from these constraints, the Z's are completely arbitrary. This arbitrariness naturally leads to different possible definitions for the renormalized coupling given by

$$g=Z_1^{-1}Z_3^{3/2}g_B\,, \tag{2.2}$$

where g_B is the "bare" coupling which appears in the Lagrangian. For example, two different prescriptions for the Z's related by

$$Z_1'=Z_1(1+ag^2+\cdots)\,, \qquad Z_3'=Z_3(1+bg^2+\cdots) \tag{2.3}$$

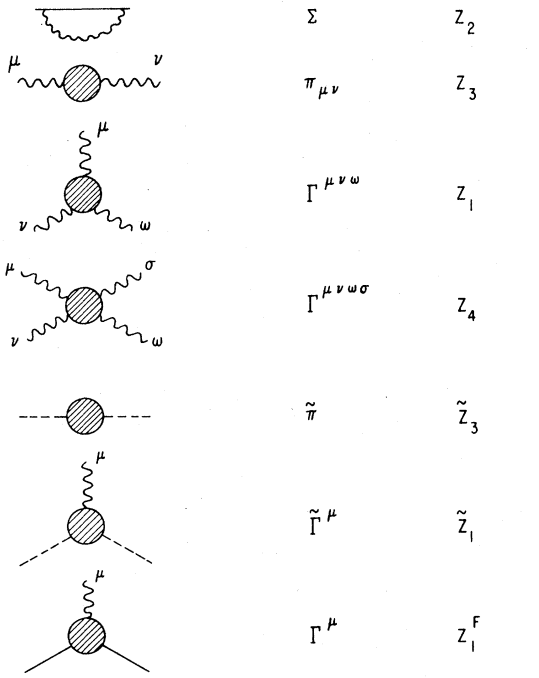

FIG. 1. The primitive divergences in covariant-gauge QCD and their counterterms.

will give different renormalized couplings related by

$$g = Z_1^{-1} Z_3^{3/2} (Z_3'^{-3/2} Z_1') g' ,$$
$$g = g'[1 + g'^2(a - 3/2b) + \cdots] . \tag{2.4}$$

The form of the relationship (2.4) demonstrates that it is not possible to specify the value of the coupling without specifying the renormalization prescription in which it is defined. We are going to discuss explicitly three common renormalization prescriptions for QCD. These are minimal subtraction[2] (MS), modified minimal subtraction[5] ($\overline{\text{MS}}$), and momentum-space subtraction[3,4] (MOM). The starting point for each of these prescriptions will be the technique of dimensional regularization.[2,10] In dimensional regularization the infinities in the momentum integrals for the diagrams in Fig. 1 are displayed as poles in $\epsilon \equiv (N-4)$, where N is the dimension of space-time. Renormalized quantities are constructed by subtracting these poles through the introduction of counterterms with the appropriate singularities. A renormalization prescription is defined to be a complete specification of the primitive diagrams in Fig. 1 and the associated Z's. The counterterms in the three prescriptions we will be discussing each differ by finite amounts and, from (2.3) and (2.4), they will each lead to the definition of a different coupling. We hope to show that the usefulness of a perturbative expansion is sensitive to the precise definition of the expansion parameters. The motivation behind the different prescriptions is best explained through examples, so we will examine some of the primitive divergences through $O(g^2)$.

A. The gluon propagator and Z_3

The expression for the gluon propagator through $O(g^2)$ in QCD can be written (in $4+\epsilon$ dimensions)[3]

$$\pi_{ab}^{\mu\nu}(p^2) = -\frac{i\delta_{ab}}{p^2}[(g^{\mu\nu} - p^\mu p^\nu/p^2)h(p^2) + \xi p^\mu p^\nu/p^2]\mu^\epsilon , \tag{2.5}$$

where μ is an arbitrary mass parameter and the function $h(p^2)$ is given by

$$h(p^2) - 1 = \frac{g^2}{16\pi^2}\left\{\left(-\frac{13}{2} + \frac{3\xi}{2} + \frac{2}{3}n_f\right)\left[\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln\left(-\frac{p^2}{\mu^2}\right)\right] + \left(\frac{97}{12} + \frac{3\xi}{2} + \frac{3\xi^2}{4} - \frac{10}{9}n_f\right)\right\} - (Z_3\mu^{-\epsilon} - 1) , \tag{2.6}$$

where $\gamma_E = 0.5772\cdots$ is Euler's constant. The minimal-subtraction procedure defines the counterterm in (2.6) so that it removes only the $1/\epsilon$ pole,

$$Z_3^{\text{MS}}\mu^{-\epsilon} - 1 = \frac{(g^{\text{MS}})^2}{16\pi^2}(-13 + 3\xi + \tfrac{4}{3}n_f)\frac{1}{\epsilon} . \tag{2.7}$$

The procedure for the modified minimal-subtraction procedure (as introduced by Bardeen *et al.*[5]) subtracts the pole in the variable[3]

$$\frac{1}{\hat{\epsilon}} = \frac{1}{2}\left(\frac{2}{\epsilon} + \gamma_E - \ln(4\pi)\right) . \tag{2.8}$$

The construction of the modified minimal-subtraction prescription is based on the observation that the combination

$$\gamma_E - \ln(4\pi) = -1.954\cdots ,$$

which appears in several Green's functions in the minimal-subtraction scheme, is an "artifact" of dimensional regularization and that the perturbation expansion may be better behaved if it is removed. Both minimal procedures are defined in such a way as to make theoretical calculations in dimensional regularization convenient and straightforward.

For comparison, the counterterm for the momentum-space-subtraction prescription (MOM) is defined so that $h(p^2) - 1$ vanishes at $p^2 = -\mu^2$. From (2.6) we have

$$[h(p^2) - 1]^{\text{MOM}} = \frac{(g^{\text{MOM}})^2}{16\pi^2}\left(-\frac{13}{2} + \frac{3\xi}{2} + \frac{2}{3}n_f\right)\ln\left(-\frac{p^2}{\mu^2}\right) . \tag{2.9}$$

Hence, in contrast to the two minimal procedures, the expression for Z_3^{MOM} is more complicated. The general philosophy behind momentum-space subtraction is to specify desirable properties for the renormalized Green's functions at an appropriate momentum and then implicitly define the Z's to realize them. The properties themselves are chosen in order to lower the contribution of high-order corrections to physical calculations. The MOM prescription has as its goal a usable asymptotic expansion. The way in which the zero from the logarithm in (2.9) can suppress the contributions from high-order diagrams will be discussed in Sec. III.

B. The triple-gluon vertex and Z_1

At the symmetric point $p^2 = q^2 = r^2$ the vertex function $\Gamma_{\mu\nu\omega}^{abc}(p,q,r)$ coupling three gluons is given through order g^3 by[3]

$$\Gamma^{abc}_{\mu\nu\omega}(p,q,r)=gf_{abc}\Big\{[g_{\mu\nu}(p-q)_\omega+g_{\nu\omega}(q-r)_\mu+g_{\omega\mu}(r-p)_\nu][G_0(p^2)+Z_1\mu^{-\epsilon}]$$
$$+(q-r)_\mu(r-p)_\nu(p-q)_\omega\frac{1}{p^2}G_1(p^2)+(r_\mu p_\nu q_\omega-r_\nu p_\omega q_\mu)\frac{1}{p^2}G_2(p^2)\Big\}\mu^\epsilon, \tag{2.10}$$

with

$$G_0(p^2)=\frac{g^2}{16\pi^2}\Big\{\tfrac{1}{8}(\xi^3-15\xi^2-9\xi-9+8n_f)+\frac{I}{24}(6\xi^2-27\xi+23-\tfrac{32}{3}n_f)$$
$$+\tfrac{1}{4}(17-9\xi-\tfrac{8}{3}n_f)\Big[\frac{2}{\epsilon}+\gamma_E-\ln(4\pi)+\ln\Big(-\frac{p^2}{\mu^2}\Big)\Big]\Big\}, \tag{2.11}$$

$$G_1(p^2)=\frac{g^2}{16\pi^2}[(-\tfrac{97}{72}+\tfrac{9}{4}\xi-\tfrac{5}{8}\xi^2+\tfrac{1}{6}\xi^3-\tfrac{4}{27}n_f)+I(\tfrac{67}{72}-\tfrac{1}{4}\xi+\tfrac{3}{8}\xi^2-\tfrac{1}{12}\xi^3+\tfrac{8}{27}n_f)], \tag{2.12}$$

$$G_2(p^2)=\frac{g^2}{16\pi^2}[(-\tfrac{77}{12}+\tfrac{15}{4}\xi+\tfrac{9}{4}\xi^2-\tfrac{1}{4}\xi^3+\tfrac{2}{9}n_f)+(-\tfrac{41}{12}+\tfrac{5}{4}\xi-\tfrac{1}{2}\xi^2+\tfrac{8}{9}n_f)I], \tag{2.13}$$

where

$$I=-2\int_0^1\frac{\ln x\,dx}{x^2-x+1}=2.3439\cdots. \tag{2.14}$$

In analogy to the case for Z_3, the MS prescription defines Z_1 to cancel only the $1/\epsilon$ poles in (2.10) and (2.11):

$$Z_1^{\rm MS}=\mu^\epsilon\Big[1-\frac{(g^{\rm MS})^2}{16\pi^2}(\tfrac{17}{2}-\tfrac{9}{2}\xi-\tfrac{4}{3}n_f)\frac{1}{\epsilon}\Big]. \tag{2.15}$$

The modified minimal-subtraction procedure gives the same form for Z_1 with the $1/\epsilon$ in (2.15) replaced by $1/\hat{\epsilon}$ as defined by (2.8). In defining the momentum-space-subtraction procedure we have some flexibility because of the tensor structure of (2.10). The idea behind momentum-space subtraction is to remove (as much as possible) the contribution of higher-order terms. One feasible way to achieve this aim is to define $Z_1^{\rm MOM}$ to cancel $G_0(p^2)$ at the scale $p^2=-\mu^2$,

$$Z_1^{\rm MOM}=\mu^\epsilon[1-G_0(-\mu^2)], \tag{2.16}$$

so that at the symmetric point $p^2=q^2=r^2$,

$$\Gamma^{abc}_{\mu\nu\omega}(p,q,r)=g^{\rm MOM}f_{abc}\Big\{[g_{\mu\nu}(p-q)_\omega+g_{\nu\omega}(q-r)_\mu+g_{\omega\mu}(r-p)_\nu]\Big[1+\frac{(g^{\rm MOM})^2}{16\pi^2}(\tfrac{17}{4}-\tfrac{9}{4}\xi-\tfrac{2}{3}n_f)\ln\Big(\frac{-p^2}{\mu^2}\Big)\Big]$$
$$+(q-r)_\mu(r-p)_\nu(p-q)_\omega\frac{1}{p^2}G_1(p^2)+(r_\mu p_\nu q_\omega-r_\nu p_\omega q_\mu)\frac{1}{p^2}G_2(p^2)\Big\}. \tag{2.17}$$

We will discuss possible modifications of the MOM scheme in Sec. VI, but all our examples will be based on the definition (2.16).

C. Fermion-propagator renormalization

We can write the correction to the fermion (quark) propagator through one loop in the form

$$\Sigma^{(2)}_{jk}(\not p)=\frac{i\not p}{p^2}\delta_{jk}\Sigma^{(2)}(p^2)\mu^\epsilon, \tag{2.18}$$

where

$$\Sigma^{(2)}(p^2)=\tfrac{4}{3}\xi\frac{g^2}{16\pi^2}\Big[\frac{2}{\epsilon}+\gamma_E-\ln(4\pi)+\ln\Big(\frac{-p^2}{\mu^2}\Big)-1\Big]$$
$$-(Z_2\mu^{-\epsilon}-1). \tag{2.19}$$

(In the Landau gauge, $\xi=0$, $\Sigma^{(2)}$ vanishes to this order.)

By now the procedures for removing divergences in the three regularization prescriptions should be familiar. In the minimal-subtraction prescription we choose

$$Z_2^{\rm MS}=\Big[1+\Big(\tfrac{8}{3}\xi\frac{(g^{\rm MS})^2}{16\pi^2}\frac{1}{\epsilon}\Big)\Big]\mu^\epsilon. \tag{2.20}$$

For the $\overline{\rm MS}$ prescription we change the factor $1/\epsilon$ in (2.20) to $1/\hat{\epsilon}$ as given by (2.8). The momentum-space subtraction procedure is defined so that

$$\Sigma^{(2)}(p^2)=\tfrac{4}{3}\xi\frac{(g^{\rm MOM})^2}{16\pi^2}\ln\Big(\frac{-p^2}{\mu^2}\Big). \tag{2.21}$$

D. Ghost-propagator renormalization

The ghost propagator in the covariant formulation of QCD can be written as

$$\Pi_{ab}^{\text{ghost}}(p^2) = \frac{i\delta_{ab}}{p^2}[I(p^2)]\mu^{\epsilon},$$

where

$$I(p^2) - 1 = \frac{g^2}{16\pi^2}\left\{(-\tfrac{9}{4} + \tfrac{3}{4}\xi)\left[\frac{2}{\epsilon} + \gamma_E - \ln 4\pi + \ln\left(\frac{-p^2}{\mu^2}\right)\right] + 3 - (\tilde{Z}_3\mu^{-\epsilon} - 1)\right\}. \quad (2.22)$$

The MS prescription defines the counterterm

$$\tilde{Z}_3^{\text{MS}} = \left(1 - \frac{(g^{\text{MS}})^2}{16\pi^2}(\tfrac{9}{2} - \tfrac{3}{2}\xi)\frac{1}{\epsilon}\right)\mu^{\epsilon} \quad (2.23)$$

to remove the ϵ pole while the $\overline{\text{MS}}$ prescription removes the pole in $\hat{\epsilon}$. In the MOM prescription,

$$I(p^2) - 1 = \frac{(g^{\text{MOM}})^2}{16\pi^2}(-\tfrac{9}{4} + \tfrac{3}{4}\xi)\ln\left(\frac{-p^2}{\mu^2}\right), \quad (2.24)$$

the corrections to the ghost propagator vanish at $-p^2 = \mu^2$.

E. Relating the different couplings

1. *Converting between MOM, MS, and $\overline{\text{MS}}$*

We have specified, in three different ways, the four independent Z's needed to completely define the theory. The other three counterterms can be defined by using the Ward identities in (2.1).

We could have chosen a different set of four independent Z's in order to specify the prescriptions. This would not affect the two minimal schemes since the pole terms satisfy the Ward identities. It would, however, change the MOM prescription since it is not possible to make all one-loop "corrections" vanish at $p^2 = -\mu^2$ while maintaining the Ward identities. We return to this problem in Sec. VI. From Eqs. (2.3) and (2.4) the renormalized couplings in each of these three schemes can be related to one another. We can write the expansions for $\alpha_s(\mu) = g^2(\mu)/4\pi$ through first order

$$\alpha_s = \alpha_s'\left\{1 + [8\pi^2(a - \tfrac{3}{2}b)]\left(\frac{\alpha_s'}{\pi}\right) + \cdots\right\}, \quad (2.25)$$

where a and b relate Z's as in (2.3) and the Z_i's are taken from the previous section. We may then construct a conversion matrix A_{ij} which relates the coupling constants by

$$\alpha_i(\mu) = \alpha_j(\mu)\left[1 + A_{ij}\left(\frac{\alpha_j(\mu)}{\pi}\right)\right]. \quad (2.26)$$

For instance, when there are four flavors and the Landau gauge ($\xi = 0$) is taken in the equations for the Z's,

$$\alpha_{\text{MOM}}(\mu) = \alpha_{\overline{\text{MS}}}(\mu)\left[1 + 7.29\frac{\alpha_{\text{MOM}}(\mu)}{\pi}\right]. \quad (2.27)$$

Other entries in this matrix are listed for three, four, and five flavors in Table I. We will return later to the question of gauge dependence. In what follows, we will choose $\xi = 0$ (Landau gauge) unless we specify otherwise.

2. *Scale dependence of $\alpha_s(\mu)$ and prescription dependence of the β function*

We can also relate coupling constants whose definitions differ only in that the Z's were subtracted at different mass scales. The one-loop result for this is

$$\alpha_i(\mu') = \alpha_i(\mu)\left[1 - \frac{33 - 2n_f}{6}\ln\left(\frac{\mu'}{\mu}\right)\left(\frac{\alpha_i(\mu)}{\pi}\right)\right]. \quad (2.28)$$

For instance, when there are four flavors,

$$\alpha_i(\mu') = \alpha_i(\mu)\left[1 - 4.17\ln\left(\frac{\mu'}{\mu}\right)\left(\frac{\alpha_i(\mu)}{\pi}\right)\right]. \quad (2.29)$$

This scale dependence of α_s can be derived from the definition of Z's in just the same way that the formulas in Table I are derived. However, Eq. (2.28) can also be derived by noting that the right-hand side consists of the first terms in an expansion of the solution of the well-known renormalization-group equation

$$\frac{d\alpha_s(\mu)}{d(\ln\mu^2)} = -4\pi\beta_0\alpha_s^2(\mu) - 16\pi^2\beta_1\alpha_s^3(\mu) + \cdots, \quad (2.30)$$

where

$$\beta_0 = (11 - \tfrac{2}{3}n_f)/16\pi^2$$

and

TABLE I. The conversion matrix A_{ij} which relates QCD couplings in different prescriptions, as defined in Eq. (2.26).

	Three flavors		
	MOM	MS	$\overline{\text{MS}}$
MOM	0	8.45	4.38
MS	−8.45	0	−4.07
$\overline{\text{MS}}$	−4.38	4.07	0
	Four flavors		
	MOM	MS	$\overline{\text{MS}}$
MOM	0	7.29	3.22
MS	−7.29	0	−4.07
$\overline{\text{MS}}$	−3.22	4.07	0
	Five flavors		
	MOM	MS	$\overline{\text{MS}}$
MOM	0	6.09	2.02
MS	−6.09	0	−4.07
$\overline{\text{MS}}$	−2.02	4.07	0

$$\beta_1 = (102 - \tfrac{38}{3} n_f)/(16\pi^2)^2 .$$

Equation (2.30) can be solved numerically as a function of μ' by solving the equation

$$\int_{\bar{g}(\mu)}^{\bar{g}(\mu')} \frac{dq}{-\beta_0 g^3 - \beta_1 g^5} = \ln(\mu'/\mu) \qquad (2.31)$$

and specifying (or measuring) the boundary value $g(\mu)$.

The resulting curve is plotted in Fig. 2. It is interesting to make the following observation: Through $O(\alpha_s^3)$, $\alpha_{\rm MS}(\mu')$, $\alpha_{\overline{\rm MS}}(\mu')$ and $\alpha_{\rm MOM}(\mu')$ all have the same scale dependence (β_0 and β_1 are prescription independent). It is likely, however, that they will differ in the α_s^4 terms (the prescription dependence of the β function is discussed in Appendix B). *If* one were to assume that $\beta^{\rm MOM}$ (the β function in the momentum-space-subtraction method) had small coefficients beyond $O(\alpha_s^3)$, then the same thing probably could *not* be said about $\beta^{\overline{\rm MS}}$. If indeed $\beta_i^{\rm MOM} \sim 0$, for $i > 1$, the solution to (2.31) gives the *exact* μ evolution of $g_{\rm MOM}(\mu)$. To find the evolution of $\alpha_{\overline{\rm MS}}(\mu')$ we then use (2.31) and Table I, resulting in the curve labeled B in Fig. 2. Of course, if $\beta_i^{\overline{\rm MS}} \sim 0$ for $i > 1$, then the evolution of $\alpha_{\overline{\rm MS}}(\mu)$ will be given by the "exact" curve of Fig. 2, although the entire curve might be shifted to the left or right depending on the boundary value $\bar{g}_{\overline{\rm MS}}(\mu)$. We would relate $\bar{g}_{\overline{\rm MS}}(\mu)$ to $\bar{g}_{\rm MOM}(\mu)$ at some fixed μ, but then allow $\bar{g}_{\overline{\rm MS}}(\mu')$ to evolve according to Eq. (2.30).

The difference between curves A and B on Fig. 2 is an indication of the possible importance of higher-order terms in the β function. In fact, the well-known curve $g(\mu')$ (known by many as the "running coupling constant") which solves (2.31) is valid in the low-μ range *only* for a prescription where the high-order terms of the β function are small. In Appendix B the prescription dependence of $\beta(g)$ is derived. If the β function is defined by

$$\frac{dg(\mu)}{d\ln(\mu)} = -\beta_0 g^3(\mu) - \beta_1 g^5(\mu) - \beta_2 g^7(\mu) + \cdots \qquad (2.32)$$

and g' is related to g through the prescription conversion

$$g' = g(1 + ag^2 + bg^4 + \cdots), \qquad (2.33)$$

then

$$\beta_0' = \beta_0, \quad \beta_1' = \beta_1$$

and

$$\beta_2' = \beta_2 - 2a\beta_1 + 2b\beta_0 - 3a^2\beta_0 ,$$

where β' is the β function appropriate to the "prime" prescription. A calculation of the three-loop β function (in various prescriptions) would be very interesting since it would show which prescription gives the smallest β_2.

It is worth noting, while on the subject of the scale dependence of $\alpha_s(\mu)$, that the solution to (2.30) can be approximated by the formula

$$4\pi\alpha_s(\mu') = \frac{1}{\beta_0 \ln(\mu'^2/\Lambda^2)} - \frac{\beta_1}{\beta_0^3} \frac{\ln\ln(\mu'^2/\Lambda^2)}{\ln^2(\mu'^2/\Lambda^2)} + O\left(\frac{1}{\ln^3(\mu'^2/\Lambda^2)}\right), \qquad (2.34)$$

where Λ is a parameter to be measured and is related to the boundary value $g(\mu)$ [in Eq. (2.31)] by simply letting $\mu' = \mu$ in (2.34), and solving for Λ. In Fig. 2 we show the comparison of the exact solution to (2.31) and the above solution in terms of Λ. The curves are quite similar. However, from the point of view of controlling coefficients in perturbation expansions, we believe results should be expressed in terms of $\alpha_s(\mu')$ rather than $1/\ln(\mu'^2/\Lambda^2)$. The reason is simply that Λ has only an indirect physical significance through (2.34). The scale Λ does provide a convenient way of parametrizing the coupling constant (provided that a prescription is specified) and is often quoted in QCD-related calculations and experimental results. From (2.34) we can deduce that

$$\Lambda_i/\Lambda_j = \exp(A_{ij}/8\pi^2\beta_0), \qquad (2.35)$$

where A_{ij} is the matrix of Eq. (2.26). In fact, if Λ is defined through all orders so that $d\Lambda/d\mu = 0$ [that is true in Eq. (2.34)] then (2.35) is exactly true through all orders of α_s—only the one-loop value for A_{ij} is needed in order to determine the ratio Λ_i/Λ_j.[3] Values for this ratio are given in

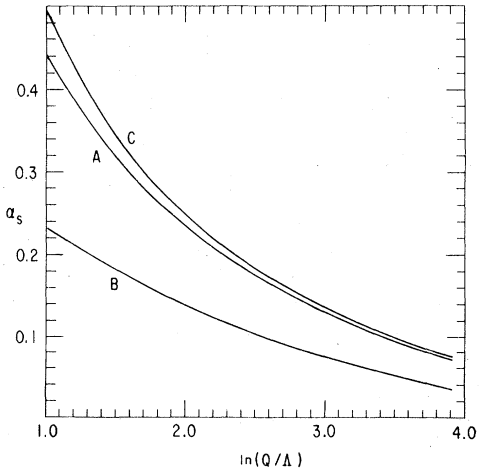

FIG. 2. The QCD running coupling. Curve A is obtained by integrating numerically Eq. (2.31). Curve B is obtained from curve A by a change in prescription as in Eq. (2.27). Curve C is obtained using the approximation (2.34) to the exact curve.

TABLE II. The ratios $\Lambda_i/\Lambda_j = B_{ij}$ for different prescriptions.

	Three flavors		
	MOM	MS	$\overline{\text{MS}}$
MOM	1	6.55	2.46
MS	0.153	1	0.376
$\overline{\text{MS}}$	0.406	2.66	1
	Four flavors		
	MOM	MS	$\overline{\text{MS}}$
MOM	1	5.73	2.15
MS	0.175	1	0.376
$\overline{\text{MS}}$	0.464	2.66	1
	Five flavors		
	MOM	MS	$\overline{\text{MS}}$
MOM	1	4.91	1.85
MS	0.204	1	0.376
$\overline{\text{MS}}$	0.542	2.66	1

Table II.

The choice of $[4\pi\beta_0 \ln(\mu^2/\Lambda_i^2)]^{-1}$ instead of $\alpha_s{}^i(\mu)$ as an expansion parameter has occasionally been advocated. It may sometimes be convenient in the context of "leading-logarithm" and "next-to-leading-logarithm" summations. With the known scale dependence given by (2.34) there will obviously be significant differences in the expansion of a physical observable in these two different ways. This is a trivial example of the importance of a good expansion parameter. The results are mixed. Buras[11] points out that $[4\pi\beta_0 \ln(\mu^2/\Lambda_{\overline{\text{MS}}}{}^2)]^{-1}$ is better than $\alpha_{\overline{\text{MS}}}(\mu)$ for paraquarkonium decays, but Abbott[12] has observed that a similar choice by Moshe[8] leads to *large* third-order corrections in deep-inelastic scattering. When we compare renormalization prescriptions, we will compare expansions in $\alpha_s(\mu)$ and *not* in $[4\pi\beta_0 \ln(\mu^2/\Lambda_i^2)]^{-1}$.

F. Prescription dependence in QED

It is clear from the discussion above that the question of prescription dependence is not unique to QCD but exists also for QED. Here we describe the result of applying our three renormalization techniques to the primitive divergences of QED. We will also introduce a "new" subtraction procedure known as mass-shell subtraction. This, for historical reasons, is the method used by all QED practitioners.

In QED there are three primitive divergences. These are as shown in Fig. 1 where the counterterms Z_1^F, Z_2, and Z_3 are defined. Since most of the QED measurements discussed are at low energies where the electron mass cannot be ignored, we also must consider an additional divergence which has the counterterm Z_m. In dimensional regularization the Ward identities give the relation

$$Z_1(\text{pole}) = Z_2(\text{pole}). \tag{2.36}$$

G. The QED photon propagator

In the Feynman gauge and $4+\epsilon$ dimensions, the photon propagator can be written through one loop[13] as

$$\pi_\gamma^{\mu\nu}(q^2) = -i\left[\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{h(q^2)}{q^2} + \frac{q^\mu q^\nu}{q^4}\right]\mu^\epsilon, \tag{2.37}$$

where

$$\begin{aligned} h(q^2) - 1 = \frac{e^2}{12\pi^2}\Bigg\{&\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln\left(\frac{m^2}{\mu^2}\right) \\ &+ (1 + 2m^2/q^2)(1 - 4m^2/q^2)^{1/2} \\ &\times \ln\left[\frac{1 + (1 - 4m^2/q^2)^{1/2}}{-1 + (1 - 4m^2/q^2)^{1/2}}\right] \\ &- \frac{4m^2}{q^2} - \frac{5}{3}\Bigg\} - (Z_3\mu^{-\epsilon} - 1). \end{aligned} \tag{2.38}$$

In comparison with (2.7) we define minimal subtraction to remove only the ϵ pole,

$$Z_3^{\text{MS}}(\mu)\mu^{-\epsilon} - 1 = \frac{e^2}{6\pi^2}\frac{1}{\epsilon}, \tag{2.39}$$

and modified minimal subtraction to give

$$Z_3^{\overline{\text{MS}}}\mu^{-\epsilon} - 1 = \frac{e^2}{12\pi^2}\left(\frac{2}{\epsilon} - \ln 4\pi + \gamma_E\right). \tag{2.40}$$

The *usual* mass-shell subtraction in QED defines Z_3 so that in Feynman gauge

$$\lim_{q\to 0} q^2 \pi_\gamma^{\mu\nu}(q) = -ig^{\mu\nu}. \tag{2.41}$$

That is,

$$Z_3^{\text{mass}} = Z_3^{\text{MOM}}(0). \tag{2.42}$$

Expanding $h(q^2)$ around $q^2 = 0$ we find

$$\begin{aligned} \lim_{q^2\to 0} h(q^2) = 1 + \frac{e^2}{12\pi^2}&\left[\frac{2}{\epsilon} + \ln\left(\frac{m^2}{\mu^2}\right) - \ln(4\pi) + \gamma_E\right] \\ &- (Z_3\mu^{-\epsilon} - 1). \end{aligned} \tag{2.43}$$

Choosing Z_3^{mass} to cancel the entire $O(e^2)$ correction in (2.43) and choosing $\mu = m$ in (2.40) and (2.43), we see that

$$Z_3^{\text{mass}} = Z_3^{\overline{\text{MS}}}(m). \tag{2.44}$$

The identification (2.44) combined with the formal similarity of many QCD calculations with their QED counterparts may help explain the relative success of QCD calculations in the $\overline{\text{MS}}$ prescription. It should be noted that we can, as usual, define the momentum-space-subtraction procedure away from $q^2 = 0$ by

$$Z_3^{\rm MOM}(q)m^{-\epsilon}-1=\frac{e^2}{12\pi^2}\left\{\frac{2}{\epsilon}-\ln(4\pi)+\gamma_E+(1-4m^2/q^2)^{1/2}(1+2m^2/q^2)\ln\left[\frac{1+(1-4m^2/q^2)^{1/2}}{-1+(1-4m^2/q^2)^{1/2}}\right]-\frac{4m^2}{q^2}-\frac{5}{3}\right\},\tag{2.45}$$

which may be useful for processes with nontrivial momentum transfers. This will be discussed in more detail later.

H. QED electron propagator

We can write the expression through one-loop for the QED electron propagator[13,14] as

$$\begin{aligned}\Sigma(p)=&\frac{-e^2}{16\pi^2}\xi\left(-\frac{2}{\epsilon}-\gamma_E+2+\ln\frac{4\pi\mu^2}{m^2}+\frac{m^2-p^2}{p^2}+\frac{1-(p^2/m^2)^2}{(p^2/m^2)^2}\ln\frac{m^2-p^2}{m^2}\right)\not p\\&+\frac{e^2}{16\pi^2}\left[(3+\xi)\left(-\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}+\frac{m^2-p^2}{p^2}\ln\frac{m^2-p^2}{m^2}\right)+4+2\xi\right]m\\&-(Z_2\mu^{-\epsilon}-1)(\not p-m)+(Z_m\mu^{-\epsilon}-1)m\,,\end{aligned}\tag{2.46}$$

where $\Sigma(p)$ is given by

$$S_F(p)=\left[\frac{i}{\not p-m}+\frac{i\Sigma(p)}{(\not p-m)^2}\right]\mu^{\epsilon}.\tag{2.47}$$

Minimal subtraction and modified minimal subtraction are defined as before. The problem is a bit trickier due to the fact that we must unravel those divergences which are associated with mass counterterms and those which are associated with wave-function counterterms. We obtain

$$Z_2^{\rm MS}(\mu)\mu^{-\epsilon}-1=\frac{e^2}{16\pi^2}\xi\left(\frac{2}{\epsilon}\right)\tag{2.48}$$

and

$$Z_2^{\overline{\rm MS}}(\mu)\mu^{-\epsilon}-1=\frac{e^2}{16\pi^2}\xi\left(\frac{2}{\epsilon}+\gamma_E-\ln4\pi\right).\tag{2.49}$$

The mass-shell subtraction requires that we expand $\Sigma(p)$ as a series in $(\not p-m)$. (The usual technique for regulating infrared divergences in QED is to give the photon a small mass λ. This will be assumed in what follows.) For instance, $(p^2-m^2)=2m(\not p-m)+(\not p-m)^2$. Picking out the coefficients of $(\not p-m)$ in Σ we get

$$\begin{aligned}\lim_{\not p\to m}\Sigma(p)=&\,\text{constant}+O((\not p-m)^2)\\&-(\not p-m)\frac{e^2}{16\pi^2}\left[\xi\left(\frac{-2}{\epsilon}-\gamma_E+2+\ln\frac{4\pi\mu^2}{m^2}-2-2\ln\frac{\lambda}{m}\right)+6\ln\frac{\lambda}{m}\right],\end{aligned}\tag{2.50}$$

$$Z_2^{\rm mass}m^{-\epsilon}-1\equiv-\frac{e^2}{16\pi^2}\left\{\xi\left[\frac{-2}{\epsilon}-\gamma_E+\ln4\pi-2\ln\left(\frac{\lambda}{m}\right)\right]+6\ln\frac{\lambda}{m}\right\}.\tag{2.51}$$

$Z_2^{\overline{\rm MS}}$ and $Z_2^{\rm mass}$ differ only in the term with $\ln(\lambda/m)$. $Z_m^{\rm mass}$ is defined to cancel the constant term in (2.49). In any gauge the mass counterterms are given as

$$Z_m^{\rm mass}m^{-\epsilon}-1=-\frac{e^2}{16\pi^2}3\left(\frac{-2}{\epsilon}-\gamma_E+\tfrac{4}{3}+\ln4\pi\right).\tag{2.52}$$

Of course,

$$Z_m^{\rm MS}m^{-\epsilon}-1=\frac{e^2}{16\pi^2}3\left(\frac{2}{\epsilon}\right)$$

and

$$Z_m^{\overline{\rm MS}}m^{-\epsilon}-1=\frac{e^2}{16\pi^2}3\left(\frac{2}{\epsilon}+\gamma_E-\ln4\pi\right).$$

We now turn to momentum-space subtraction. Again, because of the fact that the electron is massive, one has some flexibility in defining the MOM prescription. We choose our counterterms, $Z_2^{\rm MOM}(\mu)$ and $Z_m^{\rm MOM}(\mu)$, so that the Georgi-Politzer[15] renormalization condition is satisfied:

$$S_F^{-1}(p^2=-\mu^2)=-i(\not p-m).\tag{2.53}$$

Then

$$\begin{aligned}Z_2^{\rm MOM}(\mu)m^{-\epsilon}-1=-\frac{e^2}{16\pi^2}\xi\Bigg(&-\frac{2}{\epsilon}-\gamma_E+1+\ln4\pi-\frac{m^2}{\mu^2}\\&+\frac{m^4-\mu^4}{\mu^4}\ln\frac{m^2+\mu^2}{m^2}\Bigg)\end{aligned}\tag{2.54}$$

and (in Feynman gauge)

$$Z_m^{\rm MOM}(m)m^{-\epsilon}-1=-\frac{e^2}{16\pi^2}3\left(-\frac{2}{\epsilon}-\gamma_E+\ln4\pi+2-\tfrac{8}{3}\ln2\right).\tag{2.55}$$

Similar definitions can be made for Z_1's. It follows from the Ward identities that $Z_1^{\rm MS} = Z_2^{\rm MS}$ and $Z_1^{\overline{\rm MS}} = Z_2^{\overline{\rm MS}}$. It also can be shown that $Z_1^{\rm mass} = Z_2^{\rm mass}$. However, we have some flexibility in defining $Z_1^{\rm MOM}$. To achieve an optimal prescription we would have to compute the massive-electron vertex at the symmetric point. In principle, this can be obtained from Karplus and Kroll[16] or Matsuki and Yamamoto,[13] but those results are expressed in terms of complicated integrals which presumably must be done numerically (also, see 't Hooft and Veltman[17]). We have not done this computation but will indicate, in a later section, how one might deduce an optimal definition for $Z_1^{\rm MOM}$ from QED calculations which have been done. For our present purposes here we will use our flexibility to *define* $Z_1^{\rm MOM} = Z_2^{\rm MOM}$.

I. Relating coupling constants in QED

The renormalized QED change is defined by

$$e = Z_3^{1/2} Z_2 Z_1^{-1} e_{\rm bare} .$$

Therefore, much as in QCD we can relate couplings in different prescriptions. In all the prescriptions above we have $Z_1 = Z_2$ so that

$$e = Z_3^{1/2} e_{\rm bare} .$$

We can therefore relate different prescriptions by

$$e = Z_3^{1/2} Z_3'^{-1/2} e' .$$

Take, for instance, $Z_3 = Z_3^{\rm mass}$ and $Z_3' = Z_3^{\rm MOM}(q)$:

$$\left(\frac{e^2}{4\pi}\right) = \left(\frac{e'^2}{4\pi}\right)\left(1 + a\frac{e'^2}{16\pi^2}\right),$$

where $a = 0.24$ if $q = m$, $a = 0.77$ if $q = 2m$, and $a = 0.06$ if $q = \frac{1}{2}m$. We now have a running coupling. As another example of this, take $Z_3'' = Z_3^{\rm MS}(m)$. Then

$$\left(\frac{e^2}{4\pi}\right) = \left(\frac{e''^2}{4\pi}\right)\left(1 - 2.6\frac{e''^2}{16\pi^2}\right).$$

In Sec. IV we will consider a process where we choose $Z_1^{\rm MOM} \neq Z_2^{\rm MOM}$. We will also see in that section how the use of momentum-space subtraction helps reduce the importance of higher-order corrections in QED.

III. CHOOSING A PRESCRIPTION

We have seen in Sec. II that the procedure for extracting numbers from a perturbation theory for comparison with experiment necessarily involves a prescription for handling infinities which occur in the calculation. The choice of prescription specifies the definition of an expansion parameter and influences the interpretation of the results. To understand this influence we must first consider what it means to compare theory and experiment.

A. Comparison of perturbative calculations with experiment

There apparently does not exist a general proof of the universal applicability of the perturbative approach to quantum field theory. The necessity of handling infinities through complicated renormalization procedures is one of the reasons why such a general proof is unlikely to exist. Feynman,[18] who has probably done more to further the cause of perturbative quantum field theory than any other single individual, has frequently expressed misgivings about the approach.

However, most physicists currently have a lot of faith in perturbation theory. In the absence of formal proofs this faith relies on quantitative experience with specific QED calculations and with the example of certain "toy" systems which can be solved both exactly and by a perturbation expansion. It is not always clear that these experiences are transferable to a new situation such as that presented by QCD. A large component of the belief in QCD perturbation theory can be attributed to the amount of formal similarity between QCD calculations and QED calculations. In the same vein, the skepticism which still greets perturbative QCD calculations can be partially attributed to the vastly different spectra in QCD and QED. It is possible to argue that a perturbative field theory of quarks and gluons is too remote to have anything to do with the properties of hadrons.[19] In this paper we take the view that it is possible to consistently separate a calculation into a piece which can be calculated perturbatively and some infrared sensitive pieces which cannot be calculated but can be measured.[1] We *assume* that it is possible to test QCD perturbation theory experimentally.

Experience with QED suggests that a perturbation series is valid as an asymptotic approximation in terms of the expansion parameter $\alpha_s = g_s^2/4\pi$. This means that if we make a prediction in the theory calculated through nth order in the expansion parameter, then

$$P^{[\rm exact]} - P^{(n)} = O(\alpha_s^{n+1}) , \tag{3.1}$$

where $P^{[\rm exact]}$ is the exact solution. Except for unusual cases where there exist nonperturbative estimates, we do not know $P^{[\rm exact]}$. We can make the identity

$$P^{[\rm exact]} = M \pm \Delta M , \tag{3.2}$$

where M is the experimental measurement and ΔM is the experimental error. We can express the comparison between theory and experiment by interpreting (3.1) and (3.2). There exists a neighborhood near $\alpha_s = 0$ and a constant C_M^n such that

$$|(M \pm \Delta M - P^{(n)})| \leq C_M^n \alpha_s^{n+1} . \tag{3.3}$$

The QID model discussed in the Introduction gives a simple example of how the right-hand side of (3.3) can be reduced by an appropriate choice of an expansion parameter. In Appendix A we give a more complete example of how the behavior of a familiar asymptotic expansion can be influenced by the choice of an expansion parameter.

From the form of (3.3) we can see that a perturbative prediction is not completely useful without an estimate of C_M^n. There are several possible ways to obtain such an estimate.

(a) Calculate the next-order term. For $\alpha_s < 1$ this is a good way to estimate the contribution of higher-order terms.

(b) Bound the next-order term using unitarity or some other property of the quantity to be calculated.

(c) Use semiclassical or other nonperturbative estimates for $P^{[\text{exact}]}$ in (3.1).

(d) Use guesses based on intuition and/or experience with similar problems.

(e) Inspect the diagrams which are likely to contribute to the higher-order corrections.

An estimate for C_M^n in (3.3) can give a feeling for why one prescription will be more useful than another. We will give some arguments based on (d) and (e) which suggest that α_{MOM} is the parameter most likely to minimize the effects of higher-order graphs.

B. Why momentum subtraction might work

Any perturbation-theory calculation which relies on the evaluation of Feynman diagrams is eventually hampered by the fact that the number of relevant diagrams grows factorially with the order of the perturbation expansion. Unless there are significant cancellations among different diagrams, the explosion of numbers makes it almost inevitable that the coefficients of high-order terms in the perturbation expansion will be large.

Because of the specific properties built into the Green's functions in the momentum-space-subtraction prescription, we can give some crude arguments which indicate that such cancellations can occur. (In Secs. IV and V explicit examples are shown where the contribution of high-order terms in momentum-space subtraction are, in fact, small.) The starting point for these arguments is the structure of high-order diagrams. Consider, for example, all those diagrams which have a topology like that shown in Fig. 3 where an internal gluon line contains an insertion with a primitive divergence. In order to make the contribution of each of these diagrams finite, we have to regularize and renormalize the $\int d^4(l)$ integration in Fig. 3 as discussed in Sec. II. Let us consider

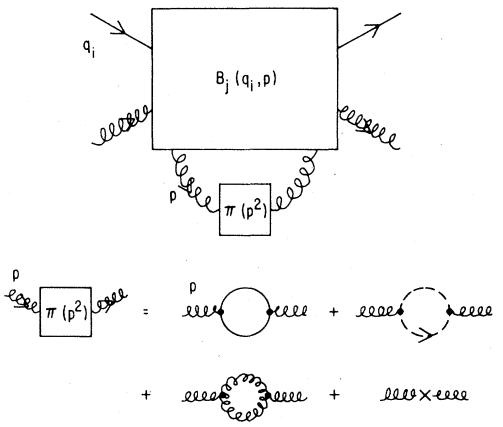

FIG. 3. Schematic of Feynman diagram where an internal line of momentum p is subjected to propagator renormalization.

the effect of the choice of prescription on the rest of the diagram. Assume, for simplicity, that the p integration in Fig. 3 is convergent. We calculate in the Landau gauge. In the momentum-space-subtraction prescription the contribution of a diagram such as shown in Fig. 3 is of the form (for three flavors)

$$A_j^{\text{MOM}}(q_i) = \int \frac{d^4p}{(2\pi)^4}\left[\frac{i\alpha_s^{\text{MOM}}(\mu^2)}{4\pi}\left(-\tfrac{9}{2}\right)\ln\left(\frac{-p^2}{\mu^2}\right)\right] \times B_j(q_i,p)\,, \tag{3.4}$$

where we have isolated in square brackets the factor associated with the d^4l integration. We still have to perform the d^4p integration. Because of the design of the momentum-subtraction prescription the factor in square brackets vanishes at $-p^2 = \mu^2$. If we are clever in choosing μ to be a *typical* magnitude for a spacelike momentum carried by an internal gluon in this process, then the contribution of the diagram will be small. An example of how this cancellation can suppress an integrand is sketched in Fig. 4. It is important to compare with the MS and $\overline{\text{MS}}$ schemes. In the $\overline{\text{MS}}$ prescription, the same diagram will give

$$A_j^{\overline{\text{MS}}}(q_i) = \int \frac{d^4p}{(2\pi)^4}\frac{i\alpha_s^{\overline{\text{MS}}}(\mu^2)}{4\pi}\left[\frac{-9}{2}\ln\left(\frac{-p^2}{\mu^2}\right)+\frac{19}{4}\right] \times B_j(q_i,p)\,. \tag{3.5}$$

In this prescription, the propagator factor does not have any special properties near $-p^2 = \mu^2$ and the integrand is not suppressed.

By considering the analogy between Feynman diagrams and electrical circuits, it is obvious that all internal lines do not carry the same mean momentum. However, if we isolate r diagrams at a given order with the configuration of Fig. 3 and find that the remaining factors in the integrands

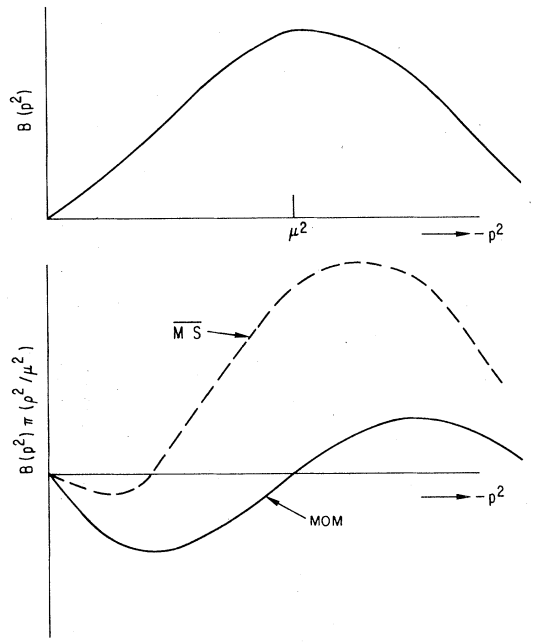

FIG. 4. If the d^4p integration is peaked at $p^2=-\mu^2$ as indicated in the top figure, the propagator renormalization in the MOM prescription will give the integrand shown as a solid curve in the bottom figure. Renormalization in the $\overline{\text{MS}}$ prescription will give the integrand shown as a dashed curve.

for each diagram are sharply peaked near $-p^2=\mu_j^2$.

$$B_j(q_i,p^2)\cong B_j(p^2)e^{-a_j^2(p^2+\mu_j^2)^2},\quad j=1,\ldots,r, \tag{3.6}$$

then we can sum this class of diagrams in the MOM prescription to get a factor proportional to

$$A=\sum_{j=1}^{r}\int p^2d(p^2)B_j(p^2)e^{-a_j^2(p^2+\mu_j^2)^2}[\ln(-p^2/\mu^2)]$$
$$\cong\pi^{1/2}\sum_{j=1}^{r}[-\mu_j^2B_j(-\mu_j^2)/a_j]\ln(\mu_j^2/\mu^2). \tag{3.7}$$

If we now choose the subtraction point $\mu^2\simeq\text{mean}(\mu_j^2)$ we find that the logarithmic factors are positive or negative depending on whether $\mu_j^2<\mu^2$ or $\mu_j^2>\mu^2$. In this case, the construction of the momentum-space-subtraction prescription tends to introduce cancellations among different diagrams as we sum at a given order.

We have simplified the argument above by pretending there is only one kind of primitive divergence to be renormalized. We can repeat the steps above on those diagrams with other types of divergences: the fermion propagator, the ghost propagator, the triple-gluon vertex, etc. From the renormalized Green's functions defined in Sec. II we can see that the momentum-space-subtraction prescription is designed to systematically introduce zeros into the integrands of the remaining momentum-space integrals at a "typical" point. A clever choice of μ can therefore make the contributions of diagrams small or can make diagrams cancel against each other. These arguments cannot be made rigorous because, in the final analysis, each perturbation-theory calculation is a special case. A given calculation may be so perverse that no obvious renormalization prescription will give a useful perturbation expansion.

In the calculation of a given set of diagrams, the value of μ^2 will be determined by the external momenta. We can therefore connect our considerations above with the usual renormalization-group equation. If Q^2 is some large invariant in the problem, the calculation of the mth correction to some observable can give factors[20]

$$d\sigma_{(m)}\propto\alpha_s{}^m(\mu^2)[b\ln(Q^2/\mu^2)+C+\cdots]^m. \tag{3.8}$$

For $Q^2/\mu^2\gg1$ this is unacceptable since the coefficient of a high-order correction can be huge. The renormalization-group analysis solves this problem by introducing the running coupling which has a well-defined behavior with Q^2. If we choose $\mu^2=aQ^2$ in (3.8) the bad behavior of the observable vanishes,

$$d\sigma_{(m)}\propto\alpha_s{}^m(aQ^2)[-b\ln(a)+C+\cdots]^m, \tag{3.9}$$

and the coefficients are now Q independent. What we have shown above is that in the MOM prescription a judicious choice of μ^2 should exist which makes the contribution of high-order diagrams not just Q independent, but small. It is therefore important in the context of the argument above that we understand a process well enough to guess what a typical internal momentum might be. If $\mu^2=Q^2$ is the subtraction point which minimizes the higher-order corrections, then choosing $\mu^2=2Q^2$ can introduce potentially large factors in (3.7).

The potential advantages of using the momentum-space-subtraction procedure to create a well-behaved perturbation expansion are discussed in more detail in Sec. III C. It may be useful to consider briefly some of the obvious limitations of the technique. If the process under consideration has two distinct large-momentum scales, it may not be possible to find a single subtraction point for which the necessary large cancellations occur. Another special situation can occur if there are a large number of diagrams for which the typical momentum is timelike rather than spacelike. Then the factor

$$\ln(-p^2/\mu^2)^2\xrightarrow[p^2\ \text{timelike}]{}(\ln|p^2/\mu^2|+i\pi)^2 \tag{3.10}$$

introduces into the various Green's functions some factors of π^2 which are not small compared to 1.

In a calculation for this type of observable it may be useful to first calculate an amplitude for spacelike external momenta and consider afterwards the continuation to the timelike region.

C. How important is a good prescription?

We have introduced three different prescriptions for doing QCD calculations. The advantages of the MS or $\overline{\text{MS}}$ schemes are that the Z's are easy to calculate and that the constraints implied by the Ward identities are automatically satisfied. One must keep in mind, however, that their usefulness is tied to the technique of dimensional regularization—a formal theoretical procedure—and not to any intrinsic relationship to experimental measurement. In defining the momentum-space-subtraction prescription we made several decisions designed to minimize the importance of higher-order corrections to physical observables at a given momentum. Since this technique is more complicated, the question naturally arises "How important is it to choose a good prescription?" In Appendix A we give a simple mathematical example of how the choice of an expansion parameter can be important in the asymptotic expansion for $\Gamma(x)$ (Stirling's approximation). In this section we will attempt to answer the question in a way which has phenomenological meaning, and so, we must consider some physical observables.

One specific requirement for a physical theory is that we can use it in calculations relating different processes. However, in making comparisons, it is important to keep in mind that the sensitivity of a calculation to the specific choice of prescription can vary from process to process. We can demonstrate this dependence quite easily by hypothesizing the existence of a "good" prescription and considering the calculation of various observables in the framework of the "QCD-augmented" parton model. The elements of any calculation in the extended parton model are the QCD coupling, parton distribution functions, and parton decay functions. We have already discussed in Sec. II the existence of a relationship between the couplings in two different prescriptions for α_s (renormalization-prescription dependence):

$$\alpha_{(1)}=\alpha_{(2)}[1+a_1\alpha_{(2)}+a_2(\alpha_{(2)})^2+\cdots]. \tag{3.11}$$

Similarly, the parton distribution and decay functions are quantities related to physical observables and these relationships are also subject to the prescription used to specify singularities. This we call factorization-prescription dependence and it is discussed in detail in the sequel to this paper (Ref. 21).

We will restrict our attention to the effects of renormalization prescription on the size of QCD corrections. Consider a physical process, to be denoted P, whose leading behavior (in QCD perturbation theory) is

$$P=\alpha_s{}^A P_0. \tag{3.12}$$

P_0 might be, for instance, some convolution such as

$$\int Q(x)Q(x')d\sigma^0(x,x'), \tag{3.13}$$

where $Q(x)$ is a parton distribution function. In Secs. IV and V some explicit examples of P will be given. The perturbation expansion for P is then written [in scheme (1)] as

$$P=\alpha_{(1)}{}^A P_0\left[1+N_1\frac{\alpha_{(1)}}{\pi}+N_2\left(\frac{\alpha_{(1)}}{\pi}\right)^2+\cdots\right]. \tag{3.14}$$

If we have chosen a good prescription, the coefficients N_1 and N_2 of the higher-order terms should be small—say of order 1 or 2. Notice that we are following fashion (and some hints from high-order expansions[22]) by choosing the expansion parameter to be (α/π) rather than α. Let us now reexpand by expressing the formula in terms of the quantities defined in prescription (2):

$$P=\alpha_{(2)}{}^A P_0\left[1+(N_1+a_1A\pi)\left(\frac{\alpha_{(2)}}{\pi}\right)+\left(N_2+\frac{A(A-1)}{2}a_1{}^2\pi^2+a_2A\pi^2+(A+1)N_1a_1\pi\right)\left(\frac{\alpha_{(2)}}{\pi}\right)^2+\cdots\right]. \tag{3.15}$$

A bad prescription where $a_1, a_2, \ldots$ are the same sign as $N_1, N_2, \ldots$ can give a very different looking expansion; for instance, if $a_i=2$, $A=3$ then

$$\alpha_{(1)}{}^3\left[1+1\left(\frac{\alpha_{(1)}}{\pi}\right)+1\left(\frac{\alpha_{(1)}}{\pi}\right)^2+\cdots\right] \Rightarrow \alpha_{(2)}{}^3\left[1+19.8\left(\frac{\alpha_{(2)}}{\pi}\right)+204\left(\frac{\alpha_{(2)}}{\pi}\right)^2+\cdots\right]. \tag{3.16}$$

The meaning of the two different expansions for the same physical measurement has been discussed above. One important fact to remember is that the comparison of experimental data and calculations usually involves the implicit assumption that corrections of higher order than those explicitly displayed are negligible. If the a_i and A are large, this cannot be simultaneously true for both (3.14) and (3.15). We now turn to an examination of specific calculations where we will attempt to demonstrate the assumption that momentum-space subtraction is indeed a useful prescription.

IV. EXAMPLES FROM QED

In spite of the possibility for different renormalization prescriptions, results in QED have (for historical reasons) always been presented using mass-shell subtraction. In this prescription the low-energy limit of Thomson scattering is equal, through all orders in α, to the Born term. That is, as $q \to 0$, the QED asymptotic expansion to Thomson scattering becomes the ultimate in a convergent series with all higher-order terms vanishing. This fact suggests the possibility that calculations for *similar low-energy* QED processes in the mass-shell-subtraction prescription will result in well-behaved perturbation expansions. So far, this expectation has been realized.

The reason why a discussion of prescription dependence in QCD is so pertinent and the reason why so many prescriptions are found in the literature is related to the absence of any theorems analogous to that for low-energy Thomson scattering. However, because of the large amount of formal similarity in the two theories, a study of prescription dependence in QED can be expected to provide some guidelines for choosing a good prescription in QCD. We now examine two examples of high-order calculations in QED.

A. The anomalous magnetic moment of the electron

One of the most accurately predicted and measured quantities in QED (indeed, in *any* quantum field theory) is the anomalous magnetic moment of the electron. This process involves the scattering of a zero-energy photon from an electron. To be specific, consider the photon-electron vertex $\Gamma^\mu(k)$, where the off-mass-shell photon has momentum k and the electrons are on shell. $\Gamma^\mu(k)$ can be decomposed into two invariant functions,

$$\Gamma^\mu(k) = e\bar{u}'\gamma^\mu u F_1(k^2) + \frac{ie}{2m}\bar{u}'\sigma^{\mu\nu}k_\nu u F_2(k^2). \quad (4.1)$$

The anomalous magnetic moment is $F_2(0)$. Mass-shell subtraction defines the charge so that $F_1(0) = 1$. We might expect that with this subtraction method, $F_2(0)$ would be a very well-behaved perturbation expansion. Indeed it is[9]:

$$a_e \equiv F_2(0) = \frac{1}{2}\frac{\alpha}{\pi} - 0.328\,478\,445\left(\frac{\alpha}{\pi}\right)^2 + 1.183(11)\left(\frac{\alpha}{\pi}\right)^3 + \cdots . \quad (4.2)$$

From the Josephson effect the value for α is found experimentally to be[23]

$$\alpha^{-1}_{\text{Josephson}} = 137.035\,987(29). \quad (4.3)$$

Inserting this value into Eq. (4.2), the theoretical prediction becomes

$$a_e^{\text{th}} = 1.159\,652\,359(282) \times 10^{-3} \quad (4.4)$$

compared with the experimental value[24] of

$$a_e^{\text{exp}} = 1.159\,652\,410(200) \times 10^{-3}. \quad (4.5)$$

The splendid agreement between theory and experiment is now taken for granted, but as was shown in the QID example of Sec. I, such agreement should only be expected if the expansion parameter is known to be "optimal." Mass-shell subtraction is (as explained in Sec. II) just $\alpha_{\text{MOM}}(0)$ and our arguments of Sec. III suggest that $\alpha_{\text{MOM}}(0)$ is indeed the best expansion parameter for $F_2(0)$. That, then, is the true reason for the excellent agreement between (4.4) and (4.5). In order to see that a different QED parameter could lead to a worse result,[25] we redefine α by

$$\alpha' = \alpha\left(1 - 10\frac{\alpha}{\pi}\right). \quad (4.6)$$

From (4.3) the value $(\alpha')^{-1}$ is determined to be

$$(\alpha')^{-1} = 140.294\,78. \quad (4.7)$$

The expansion of a_e in terms of α'/π is

$$a_e = \frac{1}{2}\frac{\alpha'}{\pi} + 4.671\,521\,5\left(\frac{\alpha'}{\pi}\right)^2 + 94.613\,54\left(\frac{\alpha'}{\pi}\right)^3 + \cdots . \quad (4.8)$$

Upon substitution of (4.7) into (4.8) the new prediction is

$$a_e' = 1.159\,585 \times 10^{-3}. \quad (4.9)$$

This is not at all in agreement with the experimental value (4.5). The large coefficients in (4.8) give a hint that the expansion is inadequate.

It should also be pointed out that the issue of "reliability" of a series is evidently *not* necessarily connected to the smallness (or largeness) of the coupling constant. In the high-precision measurement just discussed, the poor choice of expansion parameter leads to disagreement in the fourth decimal place, whereas the good parameter leads to disagreement only in the eighth place.

B. The decay of orthopositronium

It is interesting to examine a QED process where the typical momentum transfer is *not* zero and therefore where one might expect mass-shell subtraction *not* to be optimal. The decay of orthopositronium is such a process. Orthopositronium decays into three photons. The most recent calculation of this process was done by Caswell *et al.*[26] They obtain

$$\Gamma(o\text{-}Ps\to 3\gamma)=\frac{2\alpha^6 m_e}{9\pi}(\pi^2-9)\left[1-\frac{\alpha}{\pi}(10.35\pm 0.07)\right]. \tag{4.10}$$

The leading order is called Γ_0. In order to change prescriptions a decision must be made on what are the optimal Z_i's. In Sec. II, $Z_m^{\rm MOM}$, $Z_3^{\rm MOM}$, and $Z_2^{\rm MOM}$ were defined. However, $Z_1^{\rm MOM}$ was not. In order to decide on a best definition for $Z_1^{\rm MOM}$, consider the vertex-insertion diagrams calculated by Caswell *et al.* [see Fig. 5(a)]. If the configuration in which they appear at this order is typical of that for higher orders, then it is reasonable to define $Z_1^{\rm MOM}$ so that the vertex-insertion diagrams add up to 0. That will be the approach taken here.

The Z_1 counterterm appears at each of the three vertices shown in Fig. 5. The counterterms used in Caswell *et al.* are just the QED counterterms

$$Z_i^{\rm mass}\equiv 1+X_i^{\rm mass}. \tag{4.11}$$

Then the vertex contribution, including the $Z^{\rm mass}$ counterterm, is[26]

$$V^{\rm mass}=[-2.868\pm 0.003-6\ln(\lambda/m)]\frac{\alpha}{\pi}\Gamma_0. \tag{4.12}$$

$Z_1^{\rm MOM}\equiv 1+X_1^{\rm MOM}$ will be defined by demanding that the counterterm be chosen so that $V=0$. By substituting $X_1^{\rm MOM}$ for $X_1^{\rm mass}$ in Fig. 5(a) it is easy to derive that with this definition of $X_1^{\rm MOM}$.

$$X_1^{\rm MOM}=X_1^{\rm mass}+\left(\frac{2.87}{6}+\ln(\lambda/m)\right)\frac{\alpha}{\pi}. \tag{4.13}$$

Now,

$$\alpha_{\rm MOM}=\alpha_{\rm mass}[1+(X_3^{\rm MOM}-X_3^{\rm mass}+2X_2^{\rm MOM}-2X_1^{\rm MOM})], \tag{4.14}$$

where we have used the fact that $X_1^{\rm mass}=X_2^{\rm mass}$ (Sec. II).

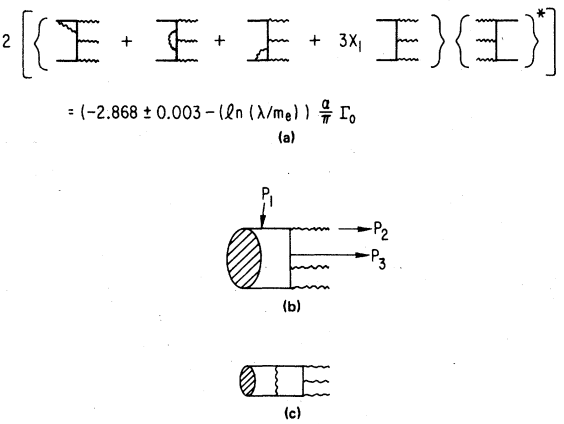

FIG. 5. (a) Diagrams for positronium decay from Caswell *et al.*, Ref. 26. (b) Typical routing of momentum, as discussed in the text. (c) Specific diagram which gives a large contribution in Feynman gauge.

From Eqs. (2.42), (2.45), (2.51), and (2.54) we can find $X_3^{\rm MOM}$, $X_3^{\rm mass}$, $X_2^{\rm MOM}$, and $X_2^{\rm mass}$. Since $X_1^{\rm MOM}$ is related by (4.13) to $X_1^{\rm mass}=X_2^{\rm mass}$, we also now know the value of $X_1^{\rm MOM}$. All that is left to be determined is the subtraction scale used to define $X_3^{\rm MOM}$ and $X_2^{\rm MOM}$. We choose this to be at the Euclidean point $p^2=-m_e^2$ [so in (2.53), $\mu=m_e$ and in (2.44), $q^2=-m_e^2$]. This choice of scale is best explained by referring to Fig. 5(b). The typical values for P_1 and P_2 are $P_1=(m_e,0,0,0)$ and $P_2=(m_e,m_e,0,0)$. Then $P_3=(0,-m_e,0,0)$. The reason for guessing that P_2 has half of the photon momentum is that it seems likely that the two remaining photons will "clump" together. Since $Z_1^{\rm MOM}$ was determined from the Feynman-gauge calculation of Caswell *et al.*, the other $Z_i^{\rm MOM}$ are also taken to be in Feynman gauge. The final result from (4.14) is that

$$\alpha^{\rm mass}=\alpha^{\rm MOM}\left(1+1.02\frac{\alpha}{\pi}\right). \tag{4.15}$$

The definition of mass also depends on prescription through $m=Z_m^{-1}m_B$. From (2.51) and (2.54) (using Feynman gauge and $\mu=m_e$),

$$m^{\rm mass}=m^{\rm MOM}\left(1+\frac{3}{2}\frac{\alpha}{\pi}\ln(2)\right). \tag{4.16}$$

Finally, the expression for positronium decay is ready to be rewritten using (4.15) and (4.16) and the fact that α and m_e in (4.10) are actually $\alpha^{\rm mass}$ and $m_e^{\rm mass}$:

$$\Gamma(o\text{-Ps}\to 3\gamma)=\Gamma_0^{\rm MOM}\left[1-\frac{\alpha}{\pi}(3.26\pm 0.07)\right], \tag{4.17}$$

where

$$\Gamma_0^{\rm MOM}=\frac{2(\alpha^{\rm MOM})^6 m_e^{\rm MOM}}{9\pi}(\pi^2-9). \tag{4.18}$$

The coefficient in (4.17) has been greatly reduced from that in (4.10). Some remarks are in order. First of all, Γ_0 is proportional to α^6, thus any *small* change of prescription can result in a very noticeable change in the size of the coefficient. Secondly, the largeness of the coefficient in (4.10) is actually due[26] to the graph shown in Fig. 5(c). This graph is not one of those obviously affected by our vertex and propagator modifications. However, the motivation for finding a best prescription is that this reduces the size of higher-order corrections. It is not necessary that it reduces low-order coefficients. The fact that the technique *does*, in many cases (including the present one), make next to leading orders small is, of course, pleasing.

One final exercise that can be done with this positronium-decay result is to compute the predicted value of Γ based on both (4.10) and (4.17). The value of $\alpha_{\text{Josephson}}$ in (4.3) is used in (4.10). In (4.17), $\alpha_{\text{Josephson}} \to \alpha^{\text{MOM}}$ via (4.15). Similarly, $m_e \to m_{\text{MOM}}$ via (4.16). The results are

$$\Gamma(\text{theoretical})\ [\text{Eq. (4.10)}] = (7.0379 \pm 0.0012) \times 10^6\ \mu\text{sec}^{-1},$$
$$\Gamma(\text{theoretical})\ [\text{Eq. (4.17)}] = (7.0396 \pm 0.0012) \times 10^6\ \mu\text{sec}^{-1}. \tag{4.19}$$

Unfortunately, experimental data[27] is not yet sufficiently reliable to distinguish between these two predictions.

V. EXAMPLES FROM QCD

By now, a number of calculations for physical observables have been done using perturbative QCD beyond the leading order. This enables us to explicitly test the ideas discussed previously about prescription dependence. In all the cases we have found so far, momentum-space subtraction achieves a reduction in the size of the leading correction. This is somewhat surprising since the idea behind the MOM scheme is the suppression of typical high-order diagrams. We must not expect that α_{MOM} will always lead to small coefficients in leading or next to leading order. If there are instances where it does not, we expect that the results will be both reliable *and* measurable. In the examples we have looked at, the $\overline{\text{MS}}$ prescription frequently leads to small corrections although we have no good reason why this should occur. The MS scheme always leads to very large corrections. Perturbative calculations in the MS scheme should therefore be considered a very unreliable guide to the underlying physics.

A. e^+e^- annihilation

One of the cleanest calculations which can be performed in perturbative QCD is that for the quantity

$$R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-). \tag{5.1}$$

Away from physical thresholds and resonances, the process involves no small energy scale and, hence, it is not necessary to separate short-range and long-range effects. The QCD expression for the quantity R can be written

$$R(s) = 3\Sigma e_Q^2 \left[1 + \left(\frac{\alpha_s^i(s)}{\pi}\right) + B_i\left(\frac{\alpha_s^i(s)}{\pi}\right)^2 + \cdots\right], \tag{5.2}$$

where the coefficient B_i depends on the renormalization prescription and $s = (P_{e^+} + P_{e^-})^2$. In the five-flavor energy range we have[28,29]

$$B_{\text{MOM}} = -0.94,$$
$$B_{\overline{\text{MS}}} = 1.41, \tag{5.3}$$
$$B_{\text{MS}} = 5.16.$$

There is no basis in the example for choosing between the MOM and $\overline{\text{MS}}$ prescriptions. The coefficient in the MS scheme is already uncomfortably large. The answers through second order in all three schemes are close (see Fig. 6). However, MOM and $\overline{\text{MS}}$ are closer to each other than to MS.

B. Paraquarkonium decay[30]

The leading corrections to the decay $\eta_Q \to$ hadrons, where $\eta_Q = 0^{-+}$ paraquarkonium state, have been calculated by Barbieri, d'Emilio, Curci, and Remiddi.[7] They give the result in the form

$$\frac{\Gamma(\eta_B \to gX)}{\Gamma(\eta_B \to \gamma\gamma)} = \frac{2}{9e_B^4}\left(\frac{\alpha_s^{\text{MS}}(2m_B)}{\alpha_{\text{QED}}}\right)^2 \times \left[1 + 22.14\left(\frac{\alpha_s^{\text{MS}}(2m_B)}{\pi}\right) + \cdots\right]. \tag{5.4}$$

Inspection of the diagrams in Fig. 7 suggests that a typical internal momentum is $|P|^2 \cong -m_q^2 \cong t_{\max}$, so that a candidate for a good expansion parameter is $\alpha^{\text{MOM}}(m_B)$. Because the lowest-order expression starts out proportional to α_s^2, this calculation is very sensitive to the prescription. The coefficient of the leading correction for the three different prescriptions is given in Table III both for $\mu = m$ and for $\mu = 2m$. We can see that with momentum subtraction at $\mu = m$, the coefficient, 1.78, is substantially smaller than with MS($2m$), which was quoted by Barbieri *et al.*[7] In fact, the coefficient 22.14 of the MS($2m$) scheme has prompted some reviewers[1] to regard this as a "debacle" of perturbation theory. Clearly, all of this apprehension was caused simply by using a bad prescription.

We can pursue the speculation that the MS prescription is ill suited to the calculation by considering the possible range of coefficients for the terms proportional to $(\alpha_s/\pi)^2$ in the two schemes. We assume temporarily that the basic design of the

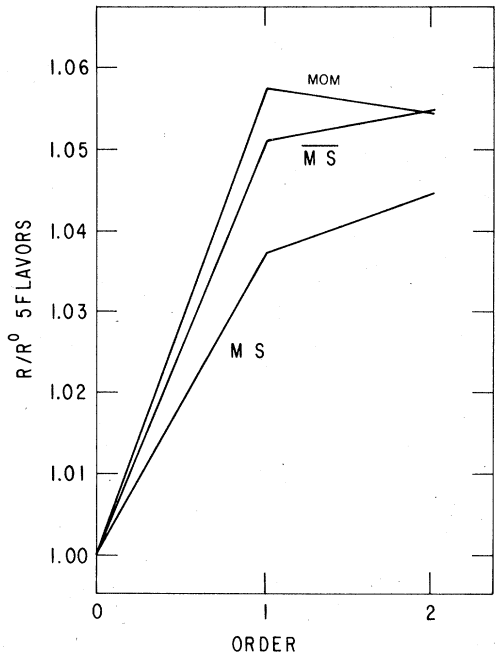

FIG. 6. The value of R/R_0 for five flavors through second order in the three prescriptions.

MOM scheme is successful and that the procedures for cancellation and suppression of diagrams in the second-order calculation described in Sec. III actually work to produce a small coefficient in the MOM scheme at scale m_B. We can then parametrize the relation

$$\alpha_s^{\mathrm{MOM}}(m)=\alpha_s^{\mathrm{MS}}(2m)\left[1+10.18\left(\frac{\alpha_s^{\mathrm{MS}}(2m)}{\pi}\right)+B\left(\frac{\alpha_s^{\mathrm{MS}}(2m)}{\pi}\right)^2+\cdots\right], \tag{5.5}$$

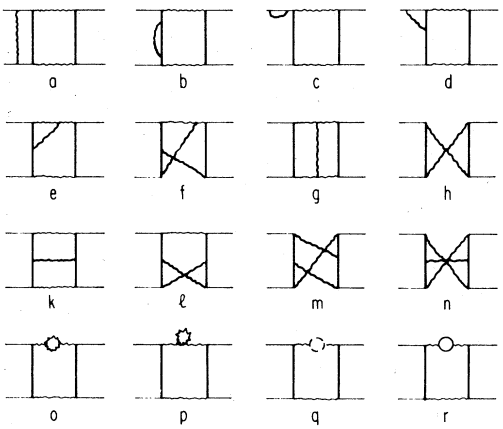

FIG. 7. Diagrams for quarkonium decay. From Barbieri *et al.*, Ref. 7.

TABLE III. Coefficient of leading correction to Eq. (5.4) for paraquarkonium decay (four flavors).

Scale \ Prescription	MS	$\overline{\mathrm{MS}}$	MOM
$2m$	22.14	14.00	7.56
m	16.36	8.22	1.78

where the number B depends on the two-loop corrections to the Z's (Ref. 9) as well as the β function and is not yet known. The expansion for momentum-space subtraction is transformed,

$$[\alpha_s^{\mathrm{MOM}}(m)]^2\left[1+1.78\left(\frac{\alpha_s^{\mathrm{MOM}}(m)}{\pi}\right)+C\left(\frac{\alpha_s^{\mathrm{MOM}}(m)}{\pi}\right)^2+\cdots\right]$$
$$=[\alpha_s^{\mathrm{MS}}(2m)]^2\left[1+22.14\left(\frac{\alpha_s^{\mathrm{MS}}(2m)}{\pi}\right)+(158.0+2B+C)\left(\frac{\alpha_s^{\mathrm{MS}}(2m)}{\pi}\right)^2+\cdots\right]. \tag{5.6}$$

Clearly there needs to be a substantial conspiracy with B and C *both* large and negative in order for the perturbation expansion in $\alpha_s^{\mathrm{MS}}(2m)$ to be well behaved at this order. Theoretical estimates for the convergence of the two expressions for Γ_{η_B} are compared in Fig. 8.

This simple exercise quickly demonstrates the importance of choosing a renormalization prescription for which there is some indication that the contribution of higher-order diagrams is

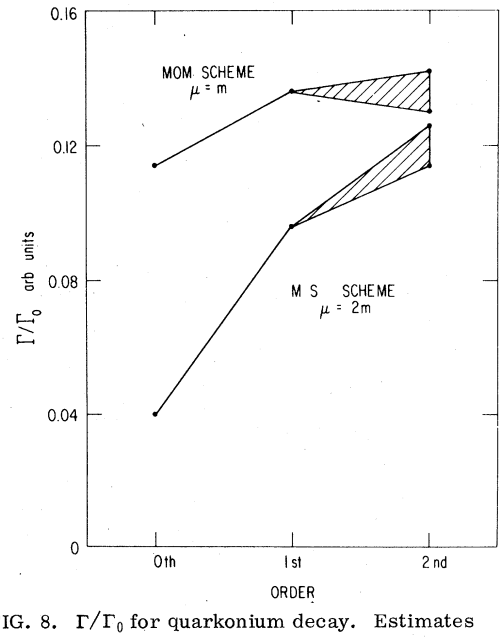

FIG. 8. Γ/Γ_0 for quarkonium decay. Estimates through second order in the MOM scheme (scale m) and in the MS scheme (scale $2m$).

small. Note that the calculation for the hadronic decay of the $^3S_1$ ($J^{PC}=1^{--}$) $\Upsilon(9.4)$ (Ref. 31) which starts out proportional to α_s^3 will be even more sensitive to the choice of the expansion parameter.

C. Moments of deep-inelastic nonsinglet structure functions

We now turn to one of the classical applications of QCD—the calculation of the nonsinglet structure functions in deep-inelastic lepton-hadron scattering. Consider the moment

$$M(n,Q^2)=\int_0^1 dx\, x^{n-2}[F_2^{ep}(x,Q^2)-F_2^{en}(x,Q^2)]. \tag{5.7}$$

From QCD perturbation theory we obtain the Q^2 evolution of $M(n,Q^2)$ in the form[32]

$$M(n,Q^2)=A_n[\alpha_s^i(Q^2)]^{a_n}\left[1+b_n\left(\frac{\alpha_s^i(Q^2)}{\pi}\right)+\cdots\right], \tag{5.8}$$

where A_n is not calculated and $a_n=-\gamma_n^0/2\beta_0$ is the usual leading-order expression for the exponent of α_s. In (5.8) we use the complete two-loop expression for the momentum dependence of the QCD coupling given by (2.30) and *do not* reexpand in terms of the parameter

$$\bar{\alpha}_i=\frac{1}{4\pi\beta_0\ln(Q^2/\Lambda_i^2)}.$$

As was pointed out by Abbott,[12] large estimates of the higher-order corrections to (5.8) found by Moshe[8] can largely be attributed to the expansion of $\alpha^{\overline{\rm MS}}(Q^2)$ in terms of $\bar{\alpha}$. Our point of view is that, since we are able to solve the renormalization-group equation, the two-loop corrections to $\alpha(Q^2)$ (in any renormalization scheme) should be treated as part of the leading-order expression. With this convention, the values[32] for the corrections b_n in Eq. (5.8) in the three different prescriptions are given in Table IV.

Once again, momentum-space subtraction leads to small second-order coefficients. Minimal subtraction gives second-order coefficients which are systematically large while the $\overline{\rm MS}$ prescription gives coefficients of intermediate size. In all three prescriptions, the coefficients will become big at large n where it is also expected that corrections from higher-twist operators will be significant since these higher moments are sensitive to the elastic and resonance regions.

TABLE IV. Coefficients of corrections, b_n, in Eq. (5.8) for nonsinglet moments in deep-inelastic scattering.

n	MOM	$\overline{\rm MS}$	MS
2	−0.85	0.52	2.26
3	−0.85	1.29	3.99
4	−0.66	2.03	5.43
5	−0.41	2.71	6.60
6	−0.13	3.34	7.73
7	+0.16	3.91	8.70
8	+0.44	4.45	9.55

D. Photon-photon deep-inelastic scattering

It was observed by Witten[33] that it is possible, within the context of perturbative QCD, to predict the structure functions of deep-inelastic scattering of a photon off a photon. This prediction is valid in the limit of $Q^2\to\infty$ whereas experiments have at present been unable to achieve $Q^2>1.5$ GeV^2 for $\gamma\gamma$ scattering.[34] Nevertheless, the theoretical prediction is interesting. $F_{2,n}^{\gamma\gamma}(Q^2)$ is the nth moment (for precise definitions of the structure function, see Witten[33] and Bardeen *et al.*[5]):

$$F_{2,n}^{\gamma\gamma}(Q^2)=\frac{\alpha_{\rm QED}d_n}{\alpha_s(Q)}\left[1+B_s^{(n)}\left(\frac{\alpha_s(Q)}{\pi}\right)\right]. \tag{5.9}$$

As in the case of deep-inelastic scattering we absorb all of the two-loop β-function effects into $\alpha_s(Q)$. d_n in Eq. (5.9) is just the leading-order result and all of the rest of the higher-order terms are absorbed in $B_s^{(n)}$. The values[5] of $B_s^{(n)}$ are given in Table V. Momentum subtraction here seems significantly more reliable than the other two schemes presented.

It is worth mentioning here that it is not possible to make a prediction for the next-to-leading corrections to the second moment. (The leading-order d_2 *can* be predicted.) Briefly, the reason for this is that the anomalous dimension of the $n=2$ operator is zero (by energy-momentum conservation) and therefore the unknown matrix element $\langle\gamma|O_2|\gamma\rangle$ does not die off as $Q^2\to\infty$. (For $n>2$, the matrix elements die logarithmically in Q^2.) [Associated with this is the fact that the calculation of the second moment also involves $g(Q_0)$, where Q_0 is arbitrary (and obviously related to how O_2 is renormalized).[35]]

The fact that corrections to the second moment are not calculated tells us to avoid making the pre-

TABLE V. Coefficients of corrections, $B_s^{(n)}$, in Eq. (5.9) for deep-inelastic scattering off a photon.

n	MOM	$\overline{\rm MS}$	MS
4	−1.04	−4.26	−8.33
6	−1.47	−4.69	−8.76
8	−1.85	−5.07	−9.14
10	−2.17	−5.39	−9.46
12	−2.42	−5.64	−9.71

dictions in terms of distribution functions. The low-x behavior of these is strongly influenced by $F_{2,2}^{\gamma\gamma}$.

VI. THE ART OF CHOOSING A PRESCRIPTION

We have described a particular approach to reducing the effect of higher-order diagrams. The approach led to the definition of $\alpha_{\rm MOM}$. In the examples of Secs. IV and V we saw that this approach successfully improved the perturbation expansions for some physical processes. It now makes sense for us to ask whether it is possible to improve our analysis by becoming more systematic and quantitative in our quest for the "ultimate" parameter. That is, can we find a technique which will enable us to absorb the maximal amount of physics into low-order predictions?

In any given process one could, in principle, more carefully scrutinize the nature of high-order diagrams and perhaps improve upon the $\alpha_{\rm MOM}$ prescription. Furthermore, such scrutiny might enable one to guess at the "most typical" momentum flow. It would of course be desirable to find some example of a QCD process where a theorem could be proven about the nature of higher-order terms. This would be the analog of the low-energy theorem for Thomson scattering in QED.

Even without such a theorem it is probably worthwhile to try to extend the concept of momentum-space subtraction beyond what we have done in Sec. II. In this section we will describe several possible methods for doing this. It is not our aim here to answer all the questions that we pose or to completely develop all the techniques which we describe. Rather, we wish to set up guidelines for the in-depth investigation of prescription dependence. The subject requires a detailed qualitative analysis of high-order Feynman diagrams and attention must be focused in this direction if prescription dependence is to become a "science."

A. The general principles

There are at least two possible approaches to the problem of improving perturbation theory. The first of these is well known and involves the identification of logarithms in high-order diagrams. Various summation and renormalization-group techniques exist for doing this and the resulting improved series are written in terms of new improved parameters such as the "running coupling," the "running mass," etc.[1] The second approach is more difficult and has been the subject of most of our discussion in this paper. In that approach, which we will refer to as "choosing a prescription" (though strictly speaking, the identification of logarithms can be included as part of this subject), one attempts to identify certain terms which are common to many high-order diagrams and which can be "summed" by absorbing them into the definition of the coupling. Such terms typically are associated with subdiagrams such as propagator and vertex insertions.

We described in Sec. III how we try to choose the renormalization constants so that the above-mentioned subdiagrams have very small values. It is interesting to consider the effect of choosing a renormalization constant which differs from the "good choice" by a value V. Consider, for example, the gluon propagator Π. Then

$$\Pi_B(Q)=\Pi_G(Q)+V, \tag{6.1}$$

where Π_G is the value of the propagator renormalized using a good renormalization constant (so that typical high-order diagrams are small) and Π_B is the propagator in the bad prescription. Since the number of diagrams in QCD grows factorially,[36] it is not unreasonable to imagine that the number of propagator insertions grows factorially with the order of the calculation. If one assumes that the diagrams are uncorrelated and that their contributions are statistically distributed around zero with a standard deviation which is smallest when $V=0$, then the "expected" magnitude of the nth coefficient will be increased by $\sim BVn!$, where B is some fixed constant. Clearly, unless $V\sim 0$, high orders will rapidly develop large coefficients and it is therefore crucial to find a scheme (the "$V\sim 0$ scheme") which assures that typical subdiagrams are small.

We again emphasize that we are attempting to control the *high orders* of the expansion. We do this by studying high-order diagrams which have subdiagrams whose values depend directly on the prescription choice we make. Our choice is aimed at reducing the size of a "typical" such high-order diagram. It is important to point out that we do *not* attempt to choose a prescription by reducing the magnitude of low-order—already calculated—quantities. Doing so would be pointless for it could be at the expense of increasing the size of uncalculated high-order coefficients. For instance, we did not choose (in Sec. V) a coupling constant by trying to eliminate the first-order correction to $\Gamma(\eta_c\to 2g)$. Instead, we made a choice which is likely to give small high-order corrections to that process. It *happens* that this choice also reduces the first-order coefficient. In fact, for all our examples of Sec. V, when $\alpha_{\rm MOM}$ was chosen to reduce high-order corrections, the next-to-leading-order coefficients were also reduced.

A specific, reasonable goal would be to inspect each process individually and choose, *a priori*, an α_s most suited *to that process*. Expansions can be

made in that α_s and for purposes of comparing experiments the different α_s's can be related to one another. This is similar to the way in which scale dependence is conventionally handled.

How well can we hope to do by using a judicious prescription? The usual folklore[36] is that eventually the coefficients will grow combinatorially and the series will diverge. This estimate is based on diagram counting. However, it is easy to imagine that with certain renormalization prescriptions, the combinatorial growth can be avoided. This has been conjectured by Cvitanovic[37] for the case of the anomalous magnetic moment in QED. He proposes that gauge sets, rather than diagrams, should be counted (these sets are bounded in magnitude). This series would grow like n rather than the $n!$ which one gets from diagram counting. Of course, individual nth-order gauge sets become increasingly complicated and it is highly desirable to find some quantitative way of bounding these. That is where it becomes crucial to have made an appropriate renormalization prescription. In the QED mass-shell prescription Cvitanovic[37] finds that gauge sets appear to be bounded by $\frac{1}{2}(\alpha/\pi)^n$.

The possible convergence of a perturbation series in QED (and QCD) seems to contradict Dyson's argument[38] about the breakdown of the physical vacuum signaling the nonanalyticity of Green's functions. Landau pole diagrams may, in fact, be the signal of eventual divergence of the expansion. However, this may not happen until a much larger order than is indicated by simple diagram counting.

B. Gauge dependence of $\alpha_{\rm MOM}$

The definition of $\alpha_{\rm MOM}$ which we have presented in Sec. II was obtained by subtracting the values of propagators and vertices in the Landau gauge. It is clear from Eqs. (2.6) and (2.16) that other gauge choices will lead to other definitions of the coupling. Of course, this does not mean that predictions will become gauge dependent. The gauge enters the final result *only* in the definition of the coupling; that is, it enters as part of the prescription dependence and so does not affect the physics. Another way to see this is to notice that in massless QCD there is only one physical parameter. In principle, the measurement of one physical quantity will completely determine all other physical quantities. True "gauge dependence" would introduce a second parameter and two measurements would be needed to determine the physics. Off-shell Green's functions indeed *do* have this two-parameter dependence (hence the options of gauge in $\alpha_{\rm MOM}$) but for on-shell processes only one parameter remains.

The gauge dependence of $\alpha_{\rm MOM}$ is derived to be[3]

TABLE VI. $A(\xi_i,\xi_j)$ where $\alpha_{\rm MOM}(\xi_i)=\alpha_{\rm MOM}(\xi_j)[1+A(\xi_i,\xi_j)\alpha_{\rm MOM}(\xi_j)/\pi]$ as in Eq. (6.3).

ξ_i \ ξ_j	0	1	3	−2
0	0	0.79	1.32	−0.66
1	−0.79	0	0.53	−1.45
3	−1.32	−0.53	0	−1.98
−2	0.66	1.45	+1.98	0

$$\alpha_{\rm MOM}(\xi)=\alpha_{\rm MOM}(0)\left(1+A(\xi,0)\frac{\alpha_{\rm MOM}}{\pi}\right), \qquad (6.2)$$

where

$$A(\xi,0)=\tfrac{3}{2}[(\tfrac{3}{8}-\tfrac{3}{3}I)+\xi^2(-\tfrac{1}{4}+\tfrac{1}{12}I)+\xi^3(\tfrac{1}{24})], \qquad (6.3)$$

where $I=2.343\,907\,2\cdots$ and ξ is the gauge parameter. [$\alpha_{\rm MOM}(0)$ is what we have been calling $\alpha_{\rm MOM}$.] In Table VI and Fig. 9 (see Ref. 3) we have tabulated and graphed this gauge dependence. Evidently, for small ξ the dependence is quite weak. Nevertheless, one must address the question of "which gauge can be expected to lead to the optimal definition of the coupling?"

In order to answer this question we recall some remarks made in the previous section on "general principles." There it was noted that in QED, gauge-set counting has been conjectured to provide a reasonable estimate of nth-order terms. All of this depends critically on the use of mass-shell renormalization. Similar rules might be imagined for QCD but first it is necessary to find an analog for "gauge sets." One promising idea is that we should use gauge invariance to reduce the degrees of freedom in the intermediate steps of the calculation. On-shell renormalization effectively eliminates unphysical degrees of freedom from the

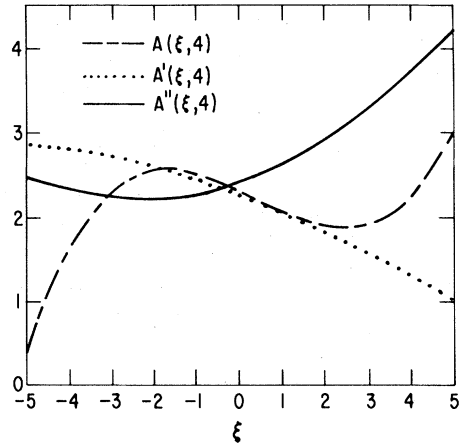

FIG. 9. Gauge dependence of the QCD couplings defined by subtracting the three-gluon $A(\xi,4)$, quark-gluon $A'(\xi,4)$, and ghost-gluon $A''(\xi,4)$ vertices. The Landau gauge has $\xi=0$.

definition of the coupling and also allows for simple classification of QED gauge sets. The closest we can come to this in covariant gauge QCD is to use the renormalization scheme represented by $\alpha_{\rm MOM}(\xi=0)$. The use of the Landau gauge eliminates the unphysical longitudinal degree of freedom present in the propagator. This extra degree of freedom would presumably proliferate in nth order, causing the coefficients to grow as $n!$

More precisely, we imagine doing the calculation in Landau gauge. (In practice, because the physical answer is gauge-independent, we do the calculation in Feynman gauge and then convert prescriptions.) We subtract our propagators and vertices (in Landau gauge) so that, as usual, high-order diagrams are small. This then gives us a result expressed in terms of $\alpha_{\rm MOM}(\xi=0)$. We see that the diagrams involved did not carry the extra propagator longitudinal degree of freedom.

Following this line of reasoning we might expect that an even better choice of gauge would be an axial gauge, where the ghost degrees of freedom are absent. In these gauges the lowest-order gluon propagator is

$$\pi^{\mu\nu}_{ab}(p) = -\frac{i\delta_{ab}}{p^2}\left(g_{\mu\nu} - \frac{\eta_\mu p_\nu + \eta_\nu p_\mu}{\eta\cdot p} + \frac{\eta^2 p_\mu p_\nu}{(\eta\cdot p)^2}\right). \tag{6.4}$$

The gauge parameter η^μ represents an arbitrary choice of direction and presumably one must choose η^μ to be aligned along a typical direction of momentum flow. This may require an extra degree of insight which may be forthcoming from those presently engaged in axial-gauge calculations. If we are to relate $\alpha_{\rm MOM}$ to $\alpha_{\rm MOM}(\eta)$, the axial-gauge-subtracted coupling, we must compute the axial propagator and vertex. This is considerably more involved than the computations whose results are given in Sec. II. To date, those calculations have not been done.

C. Other vertices and propagators

In Sec. II we chose to define the counterterms Z_1 and Z_3. By the Ward identities,

$$Z_{1F} = (Z_1/Z_3)Z_2 .$$

Therefore, if we *also* specify Z_2 then the quark-gluon vertex is completely specified. In particular, if Z_1, Z_2, and Z_3 are chosen to minimize high-order diagrams containing gluon propagators, vertices, and/or fermion propagators (momentum-space subtraction), there is no guarantee that we have minimized diagrams containing a quark-gluon vertex. $Z_1^{\rm MOM}$ might, in principle, be very different from $Z_{1F}^{\rm MOM}$. In fact, since quark propagators are attached to quark-gluon vertices it can be shown that any physical result must depend only on the ratio $Z_{1F}/Z_2 = Z_1/Z_3$. Therefore, there is nothing to be gained by doing a momentum-space subtraction of Z_2.

However, if we suspect that high-order corrections are dominated by diagrams with quark-gluon subgraphs, then we could define a new renormalization prescription based on $Z_{1F}^{\rm MOM}$, $Z_2^{\rm MOM}$, and $Z_3^{\rm MOM}$. In this technique, Z_1 would be the unadjustable renormalization constant.

To define $Z_{1F}^{\rm MOM}$ we must compute the one-loop corrections to the fermion-fermion-gluon vertex shown in Fig. 1. The result of this calculation can be written

$$\Gamma^\mu_{ij,\rho\tau}(\mu) = (-ig)L^a_{\rho\tau}(\gamma^\mu a_1 + \not{p}p^\mu a_2 + \not{p}r^\mu a_3 + \not{r}p^\mu a_4 + \not{r}r^\mu a_5 + \gamma_5\gamma_\nu \epsilon^{\nu\mu\alpha\beta}p_\alpha r_\beta a_6)_{ij}, \tag{6.5}$$

where, in Landau gauge,[3]

$$a_1 = -\frac{g^2}{16\pi^2}\left[\frac{9}{2\epsilon} - \frac{85}{12} + \frac{85}{72}I + \frac{9}{4}(\gamma_E - \ln 4\pi)\right] + (Z_{1F} - \mu^\epsilon) \tag{6.6}$$

and a_2–a_6 are invariant functions of the momenta (at the symmetric point they are functions only of μ^2). $L^a_{\beta\gamma}$ are the SU(3) generators of the fundamental representation. In analogy with Sec. II, we define $Z_{1F}^{\rm MOM}$ to be (in Landau gauge)

$$Z_{1F}^{\rm MOM} = \left\{1 + \frac{g^2}{16\pi^2}\left[\frac{9}{2\epsilon} - \frac{85}{12} + \frac{85}{72}I + \frac{9}{4}(\gamma_E - \ln 4\pi)\right]\right\}\mu^\epsilon . \tag{6.7}$$

It is also necessary to use $Z_2^{\rm MOM}$ as defined in Eqs. (2.18) and (2.19). In the Landau gauge, the fermion propagator can be written

$$S_{ij}(p) = \frac{i\not{p}}{p^2}\delta_{ij}(1 - Z_2\mu^{-\epsilon} - 1)\mu^\epsilon . \tag{6.8}$$

We see that in this gauge there are no $O(g^2)$ corrections to Z_2. This is a result familiar from QED. To be precise, $Z_2^{\rm MOM}$ is chosen so

$$Z_2^{\rm MOM} = \mu^\epsilon . \tag{6.9}$$

As in Sec. II we can now define a coupling constant $g'_{\rm MOM}(\mu)$ which is related to $g_{\rm MOM}(\mu)$ by

$$g'_{\rm MOM}(\mu) = (Z_3^{\rm MOM})^{-1}Z_2^{\rm MOM}(Z_{1F}^{\rm MOM})^{-1}Z_1^{\rm MOM}g_{\rm MOM}(\mu) . \tag{6.10}$$

The numerical result is (weakly) flavor dependent. For four flavors,

$$\alpha'_{\rm MOM}(\mu) = \alpha_{\rm MOM}(\mu)\left[1 - 0.12\frac{\alpha_{\rm MOM}(\mu)}{\pi}\right]. \tag{6.11}$$

It is indeed heartening that $\alpha'_{\rm MOM}$ and $\alpha_{\rm MOM}$ are

numerically so close to one another. The practical consequence of this fact is that, in next to leading order, results depend very little on whether they are expressed in terms of $\alpha_{\rm MOM}$ or in terms of $\alpha'_{\rm MOM}$.

Eventually, for high-enough orders, the diagrams involving only gluons can be expected to dominate over those containing fermions. This follows from looking at SU(3) group factors or, put another way, it is a result of the $1/N$ expansion. For this reason we believe it best to do our subtractions by defining Z_1 and Z_3 rather than Z_1^F, Z_2, and Z_3.

There are, of course, other possibilities that remain to be investigated. One of these would be to specify the counterterm Z_4 associated with the four-gluon vertex. We have not computed this vertex but it would be interesting to see whether the resulting $\alpha''_{\rm MOM}$ is numerically similar to $\alpha_{\rm MOM}$. As another alternative we can define $\tilde{Z}_1^{\rm MOM}$ and $\tilde{Z}_3^{\rm MOM}$. These are the ghost counterterms. That was done in Ref. 3 and leads to

$$\alpha_{\rm MOM}^{\rm ghost}(\mu) = \alpha_{\rm MOM}(\mu)\left[1+0.31\frac{\alpha_{\rm MOM}(\mu)}{\pi}\right]. \quad (6.12)$$

Again, the two prescriptions give very similar results.

In deciding which vertices to use in defining the optimal coupling for a given calculation, we must study in detail which subdiagrams are important at high order. This may be process dependent, or it may turn out that two primitive divergences share their importance, in which case α might be defined as some average of the different versions of $\alpha_{\rm MOM}$.

D. Other combinations of the invariants

In Sec. II we defined $Z_1^{\rm MOM}$ by subtracting the term proportional to the tensor $[g_{\mu\nu}(p-q)_\omega + g_{\nu\omega}(q-r)_\mu + g_{\omega\mu}(r-p)_\nu]$ in the decomposition of the three-gluon vertex (2.11). However, this need not have been how we chose to make the momentum-space subtractions of that vertex. Here we will illustrate one of many possible alternatives. Define three tensors

$$\begin{aligned}\Gamma_0^{\mu\nu\omega} &= g^{\mu\nu}(p-q)^\omega + g^{\nu\omega}(q-r)^\mu + g^{\omega\mu}(r-p)^\nu,\\ \Gamma_1^{\mu\nu\omega} &= (q-r)^\mu (r-p)^\nu (p-q)^\omega,\\ \Gamma_2^{\mu\nu\omega} &= r^\mu p^\nu q^\omega - r^\nu p^\omega q^\mu.\end{aligned} \quad (6.13)$$

The three-gluon vertex at the symmetric point has the general structure [see (2.10)]

$$\Gamma_{abc}^{\mu\nu\omega} \propto (a_1\Gamma_0^{\mu\nu\omega} + a_2\Gamma_1^{\mu\nu\omega} + a_3\Gamma_2^{\mu\nu\omega})f_{abc}, \quad (6.14)$$

and a_1 includes the counterterm so we write

$$a_1 = 1 + \hat{a}_1 + (Z_1\mu^{-\epsilon} - 1). \quad (6.15)$$

Now notice that

$$p_\mu \Gamma_{abc}^{\mu\nu\omega} = [a_1 - (p\cdot r)a_3](p^\nu r^\omega + p^\nu p^\omega + r^\nu p^\omega) f_{abc} \quad (6.16)$$

at the symmetric point (this formula can be checked by remembering that $p+q=-r$ and $p^2=q^2=r^2$). If $Z_1^{\rm MOM\prime}$ is defined so that

$$\hat{a}_1 - (p\cdot r)a_3 + (Z_1\mu^{-\epsilon} - 1) = 0, \quad (6.17)$$

then $p_\mu \Gamma_{abc}^{\mu\nu\omega}$ is equal to its "bare vertex value." This way of defining Z_1 could also legitimately be called a "momentum-subtraction" definition but it is easy to see from (2.11) and (2.16) that $Z_1^{\rm MOM\prime} \neq Z^{\rm MOM}$.

From this example we see that in choosing a prescription we must decide which invariants or combination of invariants ought to be subtracted. There seems to be a reason to prefer the prescription ($Z_1^{\rm MOM}$) which subtracts the term proportional to $\Gamma_0^{\mu\nu\omega}$: the vector $\eta_\mu = \epsilon_{\mu\nu\omega\alpha}p^\nu q^\omega s^\alpha$, where s^α is some arbitrary vector (chosen so $\eta_\mu \neq 0$), has the property $\eta_\mu \Gamma_1^{\mu\nu\omega} = 0 = \eta_\mu \Gamma_2^{\mu\nu\omega}$. However, there is no vector k^μ satisfying $k_\mu \Gamma_0^{\mu\nu\omega} = 0$. This is reminiscent of our discussion of gauge dependence. We preferred the Landau gauge because it eliminates the longitudinal degree of freedom from the propagator. Phrased another way, only in Landau gauge can one find a vector k so that $\Pi^{\mu\nu}k_\nu = 0$. We see that by subtracting a_1 in (6.14) we leave only terms $\Gamma_i^{\mu\nu\omega}$ with the property $\Gamma_i^{\mu\nu\omega}\eta_\mu = 0$. This effectively reduces the "degrees of freedom" in the integrals and so, just as in the case of Landau gauge, we expect that convergence of the series has been improved.

Unfortunately, the situation is apparently more obscure for the quark-gluon vertex. In the previous section we defined $Z_{1F}^{\rm MOM}$ by subtracting the term proportional to γ^μ in the decomposition of the fermion-gluon vertex (6.5). However, this decomposition is not unique and could have been given instead by

$$\begin{aligned}\Gamma_{ij,\beta\gamma}^{\mu a}(\mu) = (-ig)L_{\beta\gamma}^a(&\gamma^\mu b_1 + \not{p}p^\mu b_2 + \not{p}r^\mu b_3\\ &+ \not{r}p^\mu b_4 + \not{r}r^\mu b_5 + \not{p}\not{r}\gamma^\mu b_6)_{ij}.\end{aligned} \quad (6.18)$$

The difference between this decomposition and that of Eq. (6.5) is that $\gamma_5\gamma_\nu\epsilon^{\nu\mu\alpha\beta}p_\alpha r_\beta$ has been replaced by $\not{p}\not{r}\gamma^\mu$. Yet another alternative would be to replace $\not{p}\not{r}\gamma^\mu$ by $\not{r}\not{p}\gamma^\mu$. Noting that $\not{p}\not{r}\gamma^\mu = -\not{r}\not{p}\gamma^\mu + 2p\cdot r\gamma^\mu$ we see that in this new decomposition, b_2, b_3, b_4 would remain unchanged, and

$$\gamma^\mu b_1 \to \gamma^\mu(b_1 + 2p\cdot r) \text{ and } \not{p}\not{r}\gamma^\mu b_6 \to \not{r}\not{p}\gamma^\mu(-b_6).$$

Clearly, the coefficient of γ^μ depends on which choice of decomposition is made for the vertex.

Z_1^F is proportional to γ^μ and therefore its definition seems rather arbitrary. Ross[6] has suggested a momentum-subtraction method for the quark-gluon vertex which takes account of the fact that in matrix elements fermions always appear in loops. He considers the fermion-gluon vertex (at the symmetric point $-\mu^2$) with a gluon of momentum q^μ and polarization ϵ_μ (for off-shell gluons that simply means replacing $\epsilon_\mu \epsilon_\nu$ by $g_{\mu\nu}$ in the equation below). Squaring the vertex and summing over fermion spins,

$$\sum_{\text{spins}} |\mathfrak{M}|^2 \propto \epsilon_\mu \epsilon_\nu \left[\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \mu^2 F_1(\mu^2) - F_2(\mu^2) \left(p_\mu - \frac{p\cdot q}{q^2} q_\mu \right) \left(p_\nu - \frac{p\cdot q}{q^2} q_\nu \right) \right] . \tag{6.19}$$

If we define, following Ross,[6]

$$H_1(\mu^2) = \tfrac{1}{2}[F_2(\mu^2) - 4F_1(\mu^2)]$$

and (6.20)

$$H_2(\mu^2) = \tfrac{1}{2}[F_2(\mu^2) + 4F_1(\mu^2)],$$

then in the tree-graph approximation (no loop diagrams), $H_2(\mu^2) = 0$ and $H_1(\mu^2) = 1$. In higher orders, only H_1 is affected by the counterterm, and Ross defines this counterterm $Z_{1F}^{\text{MOM}'}$ so that $H_1(\mu^2) = 1$, to all orders. The resulting α_s (called α_{sym} by Goldman and Ross[39]) is related to α'_{MOM} of the previous section by

$$\alpha_{\text{sym}}(\mu) = \alpha'_{\text{MOM}}(\mu) \left(1 + 3.4 \frac{\alpha'_{\text{MOM}}(\mu)}{\pi} \right) . \tag{6.21}$$

These ambiguities of the fermion-gluon decomposition give us a further reason for choosing to define Z_1 rather than Z_{1F}. Even though there is some possible question of which choice of invariants to use in defining Z_1^{MOM}, we have seen that a "natural" choice can be made—one in which a degree of freedom is apparently removed by the subtraction procedure. Owing to spin complexities, a similar natural choice has not yet been identified for the fermion-gluon vertex. Fortunately, since gluons generally dominate over fermions in high orders, there may be no need to worry about Z_{1F}^{MOM}.

E. What to do when there is more than one momentum scale

In the absence of other information we expect that the typical insertions in higher-order diagrams involve a momentum distribution so that $p^2 = q^2 = r^2 = -Q^2$. There are presumably processes where this symmetric distribution might not be expected. In that case it would be necessary (in order to determine α_{MOM}) to compute the value of the vertex away from the symmetric point. The calculation is considerably more difficult than the one giving Eq. (2.17). In principle, this computation can be done following 't Hooft and Veltman's method for determining the value of *any* one-loop diagram.[17] An explicit formula for the three-gluon vertex (with massless quarks) can be found in Ball and Chiu.[40]

A related issue is the problem of what to do when the process of interest involves several large mass scales. An explicit example of this can be found in the calculation of high-p_T hadron-hadron scattering. Several "masses" appear there—for instance, S and p_T^2. It may be possible to determine that dominant contributions could be expected to come from one mass scale—as opposed to the other. If it should happen that one mass scale is not easily preferred over another, then the Z_i ought to be defined as an "average." For instance, if the two mass scales are μ_1 and μ_2 then

$$Z_i = \tfrac{1}{2}[Z_i(\mu_1) + Z_i(\mu_2)].$$

QCD corrections of such multimass-scale processes can be expected to converge less well than for single-mass processes (e.g., deep-inelastic scattering). That is because no vertex (or subprocess) has been subtracted at an optimal spot—instead, a "compromise" subtraction has been performed.

F. Running masses

The entire question of prescription dependence really amounts to "resummation" of the perturbation expansion. However, it is certainly possible to conceive of resummations that do not relate (in any obvious way) to a reexpansion of the coupling constant. The most obvious of these involves the identification of logarithms and double logarithms. Some are readily absorbed into the definition of the coupling constant (yielding the familiar "running" coupling) and are therefore just the usual prescription-dependent quantities. Others may be related to quark masses or multiple scales (of the kind described in the preceding section).

When there are particle masses in the theory these act as parameters and so are subject to prescription dependence. Much of what we have already said about choosing a prescription can be carried over directly to the problem of defining the mass parameters. Of course the introduction of masses simply aggravates the effects of making a bad prescription choice. The gluon propagator in Landau gauge with massive quarks (quarks have masses m_q) is

$$\Pi_{ab}^{\mu\nu}(p) = -i\delta ab \left[\left(g^{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{1}{p^2} \right] h(p^2) \mu^\epsilon ,$$

where

$$h(p^2) = 1 + \frac{3g^2}{16\pi^2}\left\{-\frac{13}{6}\left[\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln\left(-\frac{p^2}{\mu^2}\right)\right] + \frac{97}{36}\right\}$$
$$- \frac{g^2}{16\pi^2}\frac{2}{3}\sum_{\text{quarks}}\left\{-\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) + \ln\left(\frac{\mu^2}{m_q^2}\right)\right.$$
$$\left. - 4\frac{m_q{}^2}{\mu^2} + \frac{5}{3} - \frac{2m_q{}^2 - \mu^2}{\mu^2}\left(\frac{4m_q{}^2 + \mu^2}{\mu^2}\right)^{1/2}\ln\left[\frac{(1+4m_q{}^2/\mu^2)^{1/2} - 1}{(1+4m_q{}^2/\mu^2)^{1/2} + 1}\right]\right\}. \quad (6.22)$$

The mass corrections have complicated this propagator considerably but what is of most interest is the fact that $\ln(\mu^2/m_q{}^2)$ appears. If we perform a minimal subtraction so that only the pole is canceled from the propagator, then all propagator insertions will contain the quantity $\ln(\mu^2/m_q{}^2)$. When μ is chosen to be very large, this logarithm will also be large. In higher orders the mass effects will cause the appearance of terms $\ln^n(\mu^2/m_q{}^2)$. It is also worth noting that the QCD β function is, in the minimal-subtraction scheme, proportional to the coefficient of the ϵ pole which, in turn, depends linearly on the number of quarks. Even if some of those quarks are extremely massive, they contribute equally to the β function. This certainly contradicts the notion that heavy-particle masses should decouple from low-energy phenomena.[41] A way of correcting this would be, presumably, to define the masses and coupling by a momentum-space subtraction. Georgi and Politzer[15] have done this. They chose the renormalization condition (on the fermion self-energy)

$$S_F{}^{-1}|_{p^2=-\mu^2} = -i(\not{p} - m).$$

This definition, combined with the momentum-subtraction definition of g (based on the quark-antiquark-gluon vertex in Landau gauge), allows them to sum the leading logarithms associated with masses. They do this by solving the coupled equation given by

$$-\frac{g^3}{16\pi^2}\left\{11 - \tfrac{2}{3}\sum_{\text{quarks}}\left[1 - 6\frac{m_i{}^2}{\mu^2} + \frac{12m_i{}^4/\mu^4}{(1+4m_i{}^2/\mu^2)^{1/2}}\right.\right.$$
$$\left.\left.\times\ln\frac{(1+4m_i{}^2/\mu^2)^{1/2}+1}{(1+4m^2/\mu^2)^{1/2}-1}\right]\right\} = \mu\frac{dg}{d\mu} \quad (6.23a)$$

and

$$-8\frac{g^2}{16\pi^2}\left[1 - \frac{m_i{}^2}{\mu^2}\ln\left(1 + \frac{\mu^2}{m_i{}^2}\right)\right] = \frac{\mu}{m_i}\frac{dm_i}{d\mu}. \quad (6.23b)$$

The left-hand side of (6.23a) is just the m-dependent β function and is approximately given by

$$\beta \approx -\frac{g^3}{16\pi^2}\left[11 - \tfrac{2}{3}\sum_{\text{quarks}}\left(\frac{1}{1+5m_i{}^2/\mu^2}\right)\right]. \quad (6.24)$$

This is seen to cross thresholds as expected by the Appelquist-Carrazone theorem[41]—that is, large-mass quarks barely affect it.

A slightly modified version of the above equations is presented by Nachtmann and Wetzel,[42] who do a momentum-space subtraction of the three-gluon vertex. A comparison of methods is discussed in detail both by those authors and by Politzer.[43]

Prescription dependence of masses has been of *particular* interest to those who study grand unified theories (GUT's). In GUT's it is necessary to relate interactions which take place at relatively low energies (1–150 GeV) to parameters which involve mass scales $\sim 10^{15}$ GeV. (Examples are the calculation of the Weinberg angle and the prediction of the proton decay rate.) The mass logarithms are typically $\sim \ln(m/10^{15})$ and it is therefore appropriate to sum those logarithms by means of the renormalization-group equations and Eqs. (6.23). Furthermore, as we have seen, it is important to do a momentum-space subtraction as opposed to a minimal subtraction. For grand unified theories, these methods are somewhat difficult to implement and recently, an alternate scheme has been suggested by Weinberg[44] and by Ovrut and Schnitzer.[45] These authors propose using effective gauge theories in which the effect of heavy particles is totally absorbed into the renormalization scale of the strong and weak couplings.

A simple explicit example of the effect of running masses can be found in Braaten and Leveille.[46] These authors compute the QCD corrections to the decay rate of the Higgs boson into quarks as

$$\Gamma = \frac{3}{8\pi}m^2 G_F\sqrt{2}M\left(1 - \frac{4m^2}{M^2}\right)^{3/2}(1+\alpha)$$
$$\equiv \Gamma_0(1+\alpha), \quad (6.25)$$

where M is the Higgs-boson mass. α is first computed by renormalizing the fermion on mass shell and is found to be (in the limit $m^2/M^2 \to 0$)

$$\alpha(M) = \frac{2\alpha_s(M)}{3\pi}\left[\tfrac{9}{2} - 6\ln\left(\frac{M}{m}\right)\right]. \quad (6.26)$$

For large M it is easy to see that $\alpha(M)$ can become large and negative. However, it is shown that the

leading logarithms in m/M can be summed, leading to

$$\Gamma = \Gamma_0\left(\frac{\ln(2m/\Lambda_{\rm QCD})}{\ln(M/\Lambda_{\rm QCD})}\right)^{24/33-2N_F} + \text{nonleading logarithms} \quad (6.27)$$

which does *not* cause Γ to become negative. The authors point out that this result can be written as

$$\Gamma = \Gamma_0\left(\frac{\tilde{m}(M)}{\tilde{m}(2m)}\right)^2 + \text{nonleading logarithms}, \quad (6.28)$$

where $\tilde{m}(M)$ is the running mass. Further remarks on this example can be found in a recent paper by Sakai.[47]

VII. SUMMARY AND CONCLUSIONS

The definition of an expansion parameter in perturbation theory involves certain choices. These choices should be used to optimize the reliability of the perturbation expansion. If we look at the counterterms in three different renormalization prescriptions for QCD we can see how the choices used in defining $\alpha_{\rm MOM}(\mu)$ introduce zeros into renormalized Green's functions at $p^2 = -\mu^2$. To the extent that it is possible to estimate the typical momentum flow in Feynman diagrams, these zeros can be used to systematically suppress high-order corrections to physical observables.

Because of high-order effects, the difference between a good and a bad prescription can have important phenomenological consequences. We suggest that theoretical predictions in QCD be formulated in terms of $\alpha_{\rm MOM}(\mu)$. Since calculations are often performed in terms of $\alpha_{\rm MS}(\mu)$ or $\alpha_{\overline{\rm MS}}(\mu)$ we have provided tables which convert these to expansions in $\alpha_{\rm MOM}(\mu)$.

A number of explicit calculations in QCD have been examined in all three renormalization prescriptions. These examples tend to support the arguments for using momentum-space subtraction. In the cases studied, the expansions in $\alpha_{\rm MOM}$ have the smallest coefficients. We can, on the other hand, find several cases where the coefficients for the expansion in $\alpha_{\rm MS}$ are so large as to make that series completely unreliable. To further test the ideas behind momentum-space subtraction, we have examined an example in QED where it is possible to improve on the usual expansion in terms of a mass-shell-subtracted coupling.

In view of the importance of obtaining reliable predictions in perturbation theory we have investigated some of the options for refining the definition of momentum-space subtraction. This enables us to examine critically the arguments favoring one prescription over another. However, it is important to keep in mind that we have only addressed part of the problem in doing reliable QCD calculations. For those processes involving hadrons, calculations must be factored so that certain divergences are absorbed into parton distribution functions. This introduces a factorization-prescription dependence or a distribution-convention dependence in addition to the effects we have discussed here. In a separate paper we plan to examine the combined effects of renormalization-prescription dependence and factorization-prescription dependence. We hope that this will provide the framework for a critical evaluation of QCD phenomenology.

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APPENDIX A: ASYMPTOTIC EXPANSIONS AND STIRLING'S FORMULA

In order to clarify some of the points presented earlier concerning the interpretation of perturbative expansions in quantum field theory, it is instructive to consider the properties of a well-known asymptotic expansion. For example, Stirling's approximation for the Euler Γ function can

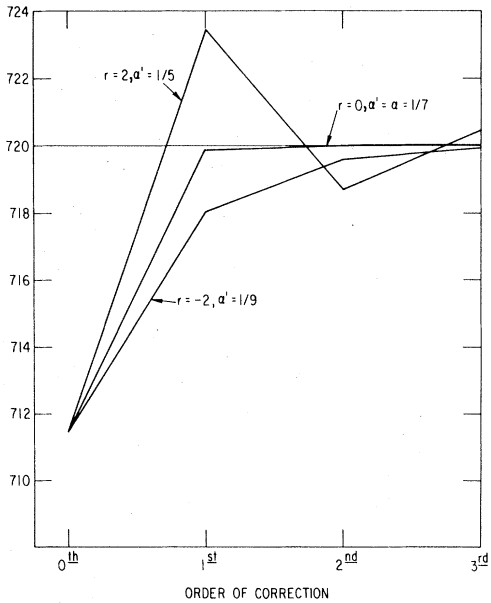

FIG. 10. Stirling's approximation for $\Gamma(7) = 720$ in Eq. (A5) as a function of order for different values of r and $\alpha' = (r + 1/\alpha)^{-1}$.

be written[48]

$$\Gamma(1/\alpha) = \Gamma_0(1/\alpha)\left(1+\frac{\alpha}{12}+\frac{\alpha^2}{288} - \frac{139}{51840}\alpha^3+\cdots\right), \tag{A1}$$

with

$$\Gamma_0(1/\alpha) = e^{-1/\alpha}e^{-(1/\alpha-1/2)\ln(\alpha)}(2\pi)^{1/2}. \tag{A2}$$

This is *not* a convergent series. It is obtained in the limit $\alpha \to 0$ by a saddle-point approximation to the integral which defines the Γ function. The expansion is known to be very good. For $\alpha < \frac{1}{10}$, it is possible to achieve better than 1% accuracy by neglecting all the terms in (A1) except the first. The applicability of (A1) and (A2) can be seen to depend on the "natural" choice of α as an expansion parameter. If we write the expansion for $\Gamma(1/\alpha)$ in terms of the variable α' where

$$\alpha = \alpha'/(1+\alpha' r), \tag{A3}$$

the expansion (A1) becomes

$$\begin{aligned}\Gamma(1/\alpha) &= \Gamma(r+1/\alpha') \\ &= \left(\frac{1+\alpha' r}{\alpha'}\right)^r \Gamma_0(1/\alpha')\left(1-\frac{(6r^2+6r-1)}{12}\alpha' \right. \\ &\quad + \frac{(36r^4+168r^3+96r^2-36r+1)}{288}\alpha'^2 \\ &\quad \left. - \frac{(1080r^6+11\,880r^5+30\,780r^4+11\,520r^3-8010r^2+450r+139)}{51\,840}\alpha'^3+\cdots\right).\end{aligned} \tag{A4}$$

For even modest sizes of r, the expansion (A4) is not very well behaved in that the contributions of the higher-order terms make sizable corrections for all but miniscule values of α. This merely reflects the fact that we are expanding in an awkward expansion parameter. The expansion can be improved somewhat by noting that some of the terms in (A4) can be summed to give

$$\Gamma(r+1/\alpha') = \Gamma_0(r+1/\alpha')\left(1+\frac{\alpha'}{12}-\frac{(24r-1)}{288}\alpha'^2+\frac{(4320r^2-360r-139)}{51\,840}\alpha'^3+\cdots\right), \tag{A5}$$

which is better behaved. Some numerical comparisons of the three asymptotic series for $\Gamma(1/\alpha)$ are given in Figs. 10 and 11.

The differences between the Euler Γ function and a field-theory perturbation expansion are also important. In ordinary field-theory calculations we are not granted the knowledge of the (functional) path integral which would enable us to decide what expansion parameter is natural. Instead, once we have decided on a definition for α_s we have a complicated set of procedures based on the Feynman rules *and* the renormalization and regularization prescription for calculating the coefficient of α_s^n in an expansion for a physical observable. We can change the definition of α_s provided we reexpand everything in the new parameter. Our criterion for a good expansion is simply that it provides a good estimate of a physical effect.

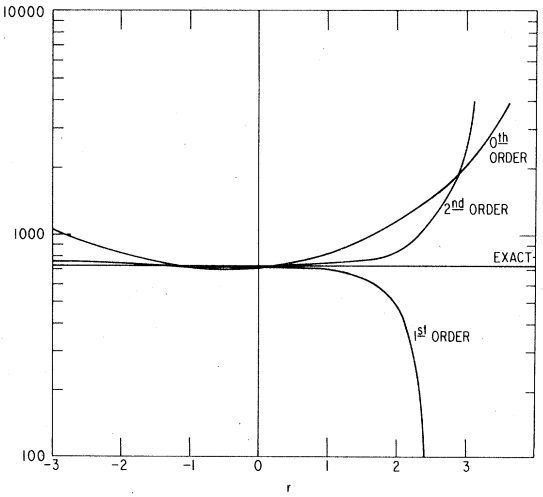

FIG. 11. Dependence on the parameter r of different orders in Eq. (A4) for $\Gamma(7) = 720$.

It is also instructive to compare the three expansions for $\Gamma(1/\alpha)$ in the case when α, α', and r are given by

$$\begin{aligned}\frac{1}{\alpha} &= c\ln t, \\ \frac{1}{\alpha'} &= (c-1)\ln t, \\ r &= \ln t.\end{aligned} \tag{A6}$$

The expansion (A4) has correction terms which behave as

$$C^{(n)} \propto r^{2n}\alpha'^{n} \to O(\ln t)^n \tag{A7}$$

and the expansion gets worse as t grows. After resumming to get (A5) we see the nth correction term now behaves as

$$C^{(n)} \propto r^{n-1}\alpha'^{n} \to O\left(\frac{1}{\ln t}\right) \tag{A8}$$

and, hence, gets better as $t \to \infty$. However, the ratio of the n and $n+1$ terms remains constant as $t \to \infty$. The change from (A4) to (A5) is an example of the procedure known as "summing leading logarithms." In terms of the "natural" expansion parameter α, the series (A1) has correction terms

$$C^{(n)} \propto \alpha^n \to O\left(\frac{1}{\ln t}\right)^n \tag{A9}$$

which become progressively less important at large t.

It is often convenient to multiply and divide asymptotic expansions, but care must be taken when doing so. For example, using the recursion relations for the Γ function we can write directly

$$\frac{\Gamma(r+1/\alpha)}{\Gamma(1/\alpha)} = \alpha^{-r}[1+\tfrac{1}{2}\alpha r(r-1)+\cdots]. \tag{A10}$$

This can also be recovered by evaluating (A4) at $\alpha' = \alpha$ and dividing by (A1). It is also important to realize that the analytic structure of an asymptotic expansion is not necessarily compatible with the original formula. The Γ function is meromorphic with poles at the negative integers $1/\alpha = -N$, $N = 0, 1, 2, \ldots$. The expression (A2) does not have these poles but does have oscillatory behavior with $\alpha < 0$. The expansion (A1) can be seen to be valid for $|\arg 1/\alpha| < \pi$. We can observe the following relation by multiplying (A1):

$$\Gamma(1/\alpha)\Gamma(-1/\alpha)|_{\text{Stirling}} = 2\pi i\alpha e^{-i\pi/\alpha}[1+O(\alpha^4)+\cdots] \tag{A11}$$

[where $1/\alpha = |z|(1-i\epsilon)$ defines the branch of the logarithm in (A2)]. We also have an exact expression

$$\Gamma(1/\alpha)\Gamma(-1/\alpha) = -\frac{-\pi\alpha}{\sin(\pi/\alpha)} = 2\pi i\alpha e^{-i\pi/\alpha}(1-e^{2i\pi/\alpha})^{-1}. \tag{A12}$$

The difference between (A11) and (A12) is highly nonuniform in ϵ and (A11) misses the poles which occur when

$$1 - e^{2i\pi z}e^{-2\pi\epsilon} = 0. \tag{A13}$$

In doing perturbation theory for particle physics we often have to continue from spacelike regions (where amplitudes are well behaved and free from singularities) to timelike regions. We have often lost information about the analytic structure of physical observables in making our approximations. A continuation such as (A12) which takes into account the proper singularity structure (A11) is bound to be better than one which does not.

APPENDIX B: PRESCRIPTION DEPENDENCE OF THE β FUNCTION

The perturbative expansion of the β function is defined by

$$\frac{dg}{dt} = -\beta_0 g^3 - \beta_1 g^5 - \beta_2 g^7 + \cdots. \tag{B1}$$

If we define g' by

$$g' = g + ag^3 + bg^5 + \cdots \left(\frac{d}{dt}a = 0 = \frac{d}{dt}b\right) \tag{B2}$$

and the β' function by

$$\frac{dg'}{dt} \equiv -\beta_0' g'^3 - \beta_1' g'^5 - \beta_2' g'^7 + \cdots,$$

then by (B2)

$$\frac{dg'}{dt} = -\beta_0' g^3 - 3\beta_0' a g^5 - 3\beta_0' b g^7 - 3a^2\beta_0' g^7 - \beta_1' g^5 - 5\beta_1' a g^7 - \beta_2' g^7. \tag{B3}$$

However,

$$\frac{dg'}{dt} = \frac{d}{dt}(g + ag^3 + bg^5 + \cdots) = (1 + 3ag^2 + 5bg^4)\frac{dg}{dt} = -\beta_0 g^3 - \beta_1 g^5 - \beta_2 g^7 - 3a\beta_0 g^5 - 3a\beta_1 g^7 - 5b\beta_0 g^7 + \cdots. \tag{B4}$$

Equating (B3) and (B4) we have the relationship between β functions,

$$\beta_0 = \beta_0', \quad \beta_1 = \beta_1', \tag{B5}$$
$$\beta_2' = \beta_2 + 3a\beta_1 + 5b\beta_0 - 3\beta_0 b - 3a^2\beta_0 - 5\beta_1 a = \beta_2 - 2a\beta_1 + 2b\beta_0 - 3a^2\beta_0.$$

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