PHYSICAL REVIEW D VOLUME 23, NUMBER 10 15 MAY 1981

Energy-momentum tensor in quantum field theory

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(Received 15 December 1980)

The definition of the energy-momentum tensor as a source current coupled to the background gravitational field receives an important modification in quantum theory. In the path-integral approach, the manifest covariance of the integral measure under general coordinate transformations dictates that field variables with weight 1/2 should be used as independent integration variables. An improved energy-momentum tensor is then generated by the variational derivative, and it gives rise to well-defined gravitational conformal (Weyl) anomalies. In the flat–space-time limit, all the Ward-Takahashi identities associated with space-time transformations including the global dilatation become free from anomalies in terms of this energy-momentum tensor, reflecting the general covariance of the integral measure; the trace of this tensor is thus finite at zero momentum transfer for renormalizable theories. The Jacobian for the local conformal transformation, however, becomes nontrivial, and it gives rise to an anomaly for the conformal identity. All the familiar anomalies are thus reduced to either chiral or conformal anomalies. The consistency of the dilatation and conformal identities at vanishing momentum transfer determines the trace anomaly of this energy-momentum tensor in terms of the renormalization-group β function and other parameters. In contrast, the trace of the conventional energy-momentum tensor generally diverges even at vanishing momentum transfer depending on the regularization scheme, and it is subtractively renormalized. We also explain how the apparently different renormalization properties of the chiral and trace anomalies arise.

I. INTRODUCTION

The Ward-Takahashi (WT) identities are formulated by means of the variational derivative in the path-integral formalism without referring to the equal-time commutator nor equations of motion. All the known anomalies are then identified as the nontrivial Jacobian factors in the path-integral measure.[1,2] The anomalies are thus related to the incompatibility of classical symmetry properties with the quantization procedure. The integral measure is generally defined by using a complete set of basis vectors[3,4]; we thus effectively introduce the notion of "representation" into the path integral, and it is formally shown that the measure is independent of the basis vectors chosen. The appearance of the anomaly, however, signals the failure of this *naive* unitary transformation among different sets of basis vectors (or different representations in the operator formalism), as the Jacobian factor is strongly dependent on the basis vectors chosen.[1] For example, the choice of the plane-wave basis corresponds to the interaction picture, and all the Jacobian factors become physically trivial in this basis. As the correct choice of basis vectors for nonlinear systems is not known in general, the explicit evaluation of the Jacobian factor is not possible except for several simple cases.

It is, however, important to recognize that *all* the known anomalies[5] have been identified as the Jacobian factors. In the path-integral method, therefore, we can anticipate all the possible anomalies in WT identities by keeping track of Jacobian factors. Moreover, those WT identities, such as the original vector WT identity in electrodynamics, which do *not* contain any Jacobian factor, are expected to be free from the anomaly: Those WT identities hold irrespective of the choice of basis vectors (or the representation) and, in particular, they hold in the interaction-picture perturbation theory without losing any information contained in the Jacobian.

In this paper we first define the path-integral measure which is invariant under the general coordinate transformation by using the weight-$\frac{1}{2}$ field variables.[2,3] We then show that the energy-momentum tensor, which is defined as a source current for the background gravitational field by the variational derivative, gives rise to a well-defined conformal anomaly in curved space-time. In the flat–space-time limit,[6] this tensor exhibits satisfactory high-energy behavior. In particular, the "dilatation identity" is free from the anomaly essentially due to the general covariance of the path-integral measure, and the trace of this energy-momentum tensor becomes finite at vanishing momentum transfer for any renormalizable theory. However, one cannot simultaneously remove the Jacobian factor corresponding to the conformal (Weyl) anomaly.[6-9] All the familiar anomalies are thus reduced to either chiral[10] or conformal anomalies, which share the interesting algebraic characterization.[2] It may be worth noting that both chiral and conformal symmetries are strongly dependent on the space-time dimensionality, while the general covariance is not.

The consistency of the dilatation and conformal identities in the flat–space-time limit determines the conformal anomaly at vanishing momentum transfer by means of the renormalization-group equation. The conformal anomaly thus determined is consistent with the direct estimate of the Jacobian factor.[2] This indicates the self-consistency of the path-integral formalism.

II. SPACE-TIME PROPERTIES OF THE PATH-INTEGRAL MEASURE

In the path-integral formalism, it is rather easy to incorporate manifest covariance under space-time transformations. We first recall that the path-integral measure for the fermion field $\psi(x)$ is defined in flat space-time as

$$d\mu \equiv \prod_x \mathcal{D}\bar{\psi}(x)\mathcal{D}\psi(x) \tag{2.1}$$

which differs from

$$d\mu' \equiv \prod_x \mathcal{D}\psi(x)^\dagger \mathcal{D}\psi(x) \tag{2.2}$$

by a Jacobian factor associated with the γ_0 matrix. Although (2.2) is more closely related to the canonical quantization scheme which suggests[11]

$$d\mu \sim \prod \mathcal{D}\pi(x)\mathcal{D}\psi(x)\,,$$

we prefer (2.1) on the basis of the manifest Lorentz covariance. Note that $\psi(x)^\dagger$ has rather complicated transformation properties under the Lorentz transformation. The difference between (2.1) and (2.2) is a trivial Jacobian factor in a *finite* theory, but (2.1) is better suited for maintaining *manifest* Lorentz covariance in the field theory with an infinite number of degrees of freedom. In fact the chiral anomaly, for example, can be derived in a natural manner[1] on the basis of (2.1).

In curved space-time, (2.1) is now replaced by[2,3]

$$d\mu \equiv \prod_x \mathcal{D}\tilde{\bar{\psi}}(x)\mathcal{D}\tilde{\psi}(x) \tag{2.3}$$

with the field variables with weight $\frac{1}{2}$

$$\tilde{\psi}(x) \equiv g^{1/4}\psi(x), \quad \tilde{\bar{\psi}}(x) \equiv g^{1/4}\bar{\psi}(x)\,. \tag{2.4}$$

The choice of (2.3) is dictated by the manifest covariance under the general coordinate transformation as can be understood as follows.

We first expand $\psi(x)$ and $\bar{\psi}(x)$ in terms of a complete set of spinor basis vectors (in Wick-rotated space R^4):

$$\begin{aligned} \psi(x) &= \sum_n a_n \varphi_n(x) = \sum_n \langle n|x\rangle a_n\,, \\ \bar{\psi}(x) &= \sum_n \bar{b}_n \varphi_n(x)^\dagger = \sum_n \langle x|n\rangle \bar{b}_n \end{aligned} \tag{2.5}$$

with[12]

$$\int \varphi_n(x)^\dagger \varphi_m(x)\sqrt{g}\,d^4x = \delta_{n,m} \tag{2.6}$$

and the coefficients a_n and $\bar{b}_n$ are the elements of the Grassmann algebra.[11] Then

$$\begin{aligned} d\mu &= \prod_x \mathcal{D}\tilde{\bar{\psi}}(x)\mathcal{D}\tilde{\psi}(x) \\ &= \frac{1}{\det[g^{1/4}\varphi_n(x)^\dagger]\det[g^{1/4}\varphi_m(x)]} \prod_n da_n d\bar{b}_n \end{aligned} \tag{2.7}$$

by noting that the translation-invariant integral over the elements of the Grassmann algebra is equivalent to the left derivative.[11] The Jacobian factor in (2.7) may be regarded as consisting of matrices whose rows and columns are specified by x and n, respectively. The Jacobian is then evaluated as

$$\begin{aligned} \det[g^{1/4}\varphi_n(x)^\dagger]\det[g^{1/4}\varphi_m(x)] &= \det\left[\sum_x \sqrt{g}\,\varphi_n(x)^\dagger \varphi_m(x)\right] \\ &= \det\left[\int dx\sqrt{g}\,\varphi_n(x)^\dagger\varphi_m(x)\right] \\ &= \det[\delta_{n,m}] = 1\,. \end{aligned} \tag{2.8}$$

The measure (2.7) thus becomes (up to a trivial normalization factor)

$$d\mu = \prod_n da_n d\bar{b}_n\,. \tag{2.9}$$

The general coordinate and internal SO(3,1) [or SO(4) in the present Euclidean theory] transformation properties are carried by $\varphi_n(x)$ in (2.5), and the coefficients a_n and $\bar{b}_n$ are scalars under these transformations.

This argument shows that we should use the fields with weight $\frac{1}{2}$ in (2.4) to define the manifestly covariant path-integral measure. This argument of manifest covariance is quite general and it applies to other fields such as the scalar field,[2,3] as the weight factor $g^{1/4}$ essentially arises from the covariance of the volume $\sqrt{g}\,d^4x$. (The manifestly covariant measure generally differs from the measure derived from the naive phasespace path integral by a Jacobian $\det[(g^{00}(x))^{1/2}\delta(x-y)]$ for each degree of freedom.) It is formally shown that the path-integral measure is independent of the choice of basis vectors in (2.5), and a change of basis vectors corresponds to a change of the representation in the operator formalism; we can thus specify the path integral more precisely by means of the expansion (2.5).

The important implication of the choice of (2.4) as integration variables is that the energy-momentum tensor defined as a source current for the background gravitational field becomes in quantum theory[13]

$$\sqrt{g}\,\tilde{T}_{\mu\nu}(x) \equiv h_{a\mu}(x)\frac{\delta}{\delta h^\nu_a(x)}\tilde{S}(h^\mu_a, \tilde{\psi}, \tilde{\bar{\psi}}), \tag{2.10}$$

which *differs* from the classical energy-momentum tensor

$$\sqrt{g}\,T_{\mu\nu}(x) \equiv h_{a\mu}(x)\frac{\delta}{\delta h^\nu_a(x)}S(h^\mu_a, \psi, \bar{\psi}). \tag{2.11}$$

Here $S(h,\psi,\bar\psi) \equiv \tilde{S}(h,\tilde\psi,\tilde{\bar\psi})$ is a classical Euclidean action and $h^\mu_a(x)$ the vierbein field $h^\mu_a(x)h^{a\nu}(x) \equiv g^{\mu\nu}(x)$. We note that the *quantized* energy-momentum tensor $\langle \tilde{T}_{\mu\nu}(x)\rangle$ transforms as a well-defined second-rank tensor. [At the classical level, where the equations of motion may be freely used, the difference between (2.10) and (2.11) is trivial. At the quantum level, however, the equations of motion when *locally* multiplied by other operators cannot simply be set to zero, and this gives rise to a difference, for example, between (4.19) and (4.20) below.]

III. GRAVITATIONAL CONFORMAL ANOMALIES

The gravitational conformal anomaly associated with a quantized scalar field has been discussed elsewhere.[2,3] We here illustrate the path-integral method for two simple examples. The discussion in this section also serves to fix the notation for the discussion of spinor electrodynamics in the next section.

A. Fermion field

We consider the free fermion field inside the *background* gravitational field

$$S \equiv \int h(x)\mathcal{L}\,dx \equiv \int h(x)(\bar\psi i\gamma^\mu D_\mu\psi - m\bar\psi\psi)dx, \tag{3.1}$$

where[14]

$$\begin{aligned}
&\gamma^\mu(x) \equiv h^\mu_a(x)\gamma^a, \quad \{\gamma^a,\gamma^b\} = 2G^{ab},\\
&D_\mu \equiv \partial_\mu - \frac{i}{2}A_{mn\mu}S^{mn}, \quad S^{mn} = \frac{i}{4}[\gamma^m,\gamma^n],\\
&A_{mn\nu}(x) \equiv \tfrac{1}{2}h^\lambda_m h^\mu_n(c_{\lambda\mu\nu} - c_{\mu\lambda\nu} - c_{\nu\lambda\mu}),\\
&C^\lambda{}_{\mu\nu} \equiv h^\lambda_k(\partial_\mu h^k_\nu - \partial_\nu h^k_\mu),\\
&\{\gamma^\mu,\gamma^\nu\} = 2h^\mu_a h^\nu_b G^{ab} \equiv 2g^{\mu\nu}(x),\\
&h(x) \equiv \det[h^a_\mu(x)] = \sqrt{-g}\,.
\end{aligned} \tag{3.2}$$

The metric for the local Lorentz frame is $G^{ab} = (1,-1,-1,-1)$ which becomes $G^{ab} = (-1,-1,-1,-1)$ after a suitable Wick rotation in the *local Lorentz frame*, e.g., $h^\mu_0 \to ih^\mu_4$, $h \to -ih$ (thus $S \to -iS$), and $\gamma^0 \to -i\gamma^4$ with $\gamma_5 \equiv \gamma^4\gamma^1\gamma^2\gamma^3$.

The action (3.1) satisfies the local conformal (Weyl) symmetry property

$$\tilde{S}(e^\alpha h^\mu_a, e^{-\alpha/2}\tilde\psi, e^{-\alpha/2}\tilde{\bar\psi}) = \tilde{S}(h^\mu_a,\tilde\psi,\tilde{\bar\psi}) + \int dx\,\alpha(x)m\tilde{\bar\psi}\,\tilde\psi(x) \tag{3.3}$$

in terms of the variables (2.4), or equivalently,

$$h(x)\tilde{T}_\mu{}^\mu(x) = \tfrac{1}{2}\tilde{\bar\psi}(x)\frac{\delta}{\delta\tilde{\bar\psi}(x)}\tilde{S} + \tfrac{1}{2}\tilde\psi(x)\frac{\delta}{\delta\tilde\psi(x)}\tilde{S} + m\tilde{\bar\psi}\tilde\psi(x) \tag{3.4}$$

by remembering the defintion of $\tilde{T}_{\mu\nu}$ in (2.10).

We next derive the local identity associated with the change of integration variables

$$\begin{aligned}
&\tilde\psi(x) \to \tilde\psi'(x) \equiv e^{-\alpha(x)/2}\tilde\psi(x),\\
&\tilde{\bar\psi}(x) \to \tilde{\bar\psi}'(x) \equiv e^{-\alpha(x)/2}\tilde{\bar\psi}(x).
\end{aligned} \tag{3.5}$$

Under this transformation, the measure (2.9) is transformed as

$$d\mu \to d\mu\,\exp\left[\int \alpha(x)A_1(x)h(x)dx\right] \tag{3.6}$$

with

$$A_1(x) \equiv \sum_n \varphi_n(x)^\dagger\varphi_n(x) \tag{3.7}$$

if one remembers that the coefficients a_n in (2.5) are transformed as

$$\begin{aligned}
a'_n &= \sum_m \int \varphi_n(x)^\dagger e^{-\alpha(x)/2}\varphi_m(x)h(x)dx\,a_m\\
&\equiv \sum_m c_{n,m}a_m
\end{aligned} \tag{3.8}$$

under (3.5); the Jacobian in (3.6) corresponds to[1,2]

$$[\det c_{n,m}]^{-2} = \exp\left[\int \alpha(x)A_1(x)h(x)dx\right] \tag{3.9}$$

for infinitesimal $\alpha(x)$.

The variational derivative (the change of integration variables does not change the integral itself)

$$\frac{\delta}{\delta\alpha(x)}Z(\tilde\eta,\tilde{\bar\eta})\bigg|_{\alpha=0} \equiv 0 \tag{3.10}$$

with

$$Z(\tilde\eta,\tilde{\bar\eta}) = \frac{1}{N}\int d\mu\,\exp\left[\tilde{S} + \int(\tilde{\bar\eta}\tilde\psi + \tilde{\bar\psi}\,\tilde\eta)dx\right] \tag{3.11}$$

gives rise to (by discarding source terms)

$$\frac{1}{2}\left[\left\langle\tilde{\bar\psi}(x)\frac{\delta}{\delta\tilde{\bar\psi}(x)}\tilde{S}\right\rangle + \left\langle\tilde\psi(x)\frac{\delta}{\delta\tilde\psi(x)}\tilde{S}\right\rangle\right] = h(x)A_1(x). \tag{3.12}$$

From (3.4) and (3.12), we obtain the WT identity

$$h(x)\langle \tilde{T}_\mu{}^\mu(x)\rangle = h(x)A_1(x) + m\langle\tilde{\bar\psi}\tilde\psi(x)\rangle\,. \qquad (3.13)$$

It is formally shown that the path-integral measure is independent of the choice of basis vectors (2.5). But this *naive* unitary transformation among different sets of basis vectors fails in WT identities with the anomaly, as the anomaly factor (3.7) strongly depends on the basis vectors chosen. From our experience in the case of the topological chiral anomaly,[1,2] we evaluate $A_1(x)$ by means of the basis vectors for $\not{D} = \gamma^\mu D_\mu$ in (3.1) (the "Heisenberg representation"[12]):

$$\not{D}\varphi_n(x) = \lambda_n\varphi_n(x), \quad \int \varphi_m(x)^\dagger\varphi_n(x)h(x)dx = \delta_{n,m}\,. \qquad (3.14)$$

We sum the series in (3.7) starting from small eigenvalues $|\lambda_n| \lesssim M$ as

$$A_1(x) = \lim_{M\to\infty}\left[\sum_n \varphi_n(x)e^{-(\lambda_n/M)^2}\varphi_n(x)\right]. \qquad (3.15)$$

There are various ways[1,15] to evaluate (3.15); we quote here the result which appears in the intermediate stage of the ζ regularization[15]:

$$A_1(x) = \lim_{M\to\infty}\frac{1}{(4\pi)^2}\left[4M^4 + \tfrac{1}{3}M^2R + \left(\tfrac{1}{30}D^\mu D_\mu R + \tfrac{1}{72}R^2 - \tfrac{1}{45}R_{\mu\nu}R^{\mu\nu} - \tfrac{7}{360}R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}\right)\right]. \qquad (3.16)$$

The second term in (3.16) depends on how to sum the series in (3.7), and it is customarily eliminated by a suitable regularization scheme such as the ζ regularization.[15] We thus obtain the conformal identity for the connected components[8,9]

$$\langle\tilde{T}_\mu{}^\mu(x)\rangle_c = m\langle\bar\psi\psi(x)\rangle_c + \frac{1}{(4\pi)^2}\left(\tfrac{1}{30}D^\mu D_\mu R + \tfrac{1}{72}R^2 - \tfrac{1}{45}R_{\mu\nu}R^{\mu\nu} - \tfrac{7}{360}R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}\right). \qquad (3.17)$$

B. Electromagnetic field

We consider the Euclidean action

$$S \equiv \int dx\, h(x)\left(-\tfrac{1}{4}g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}\right) \qquad (3.18)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \qquad (3.19)$$

We obtain various energy-momentum tensors depending on the choice of independent variables in the action (3.18):

$$h(x)\theta_{\mu\nu}(x) \equiv h_{\mu a}(x)\frac{\delta}{\delta h_a^\nu(x)}\hat{S}(h_a^\mu, A_\nu) = h(x)\left(\frac{g_{\mu\nu}}{4}g^{\lambda\rho}g^{\alpha\beta}F_{\lambda\alpha}F_{\rho\beta} - g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}\right), \qquad (3.20)$$

$$h(x)T_{\mu\nu}(x) \equiv h_{\mu a}(x)\frac{\delta}{\delta h_a^\nu(x)}S(h_a^\mu, A_a) = h(x)\theta_{\mu\nu}(x) - h_{\mu a}h_\nu^b A_b(x)\frac{\delta}{\delta A_a(x)}S\,, \qquad (3.21)$$

$$h(x)\tilde{T}_{\mu\nu}(x) \equiv h_{\mu a}(x)\frac{\delta}{\delta h_a^\nu(x)}\tilde{S}(h_a^\mu, \tilde{A}_a) = h(x)T_{\mu\nu}(x) + \tfrac{1}{2}g_{\mu\nu}\tilde{A}_b(x)\frac{\delta}{\delta\tilde{A}_b(x)}\tilde{S}\,, \qquad (3.22)$$

where

$$\hat{S}(h, A_\nu) \equiv S(h, A_a) \equiv \tilde{S}(h, \tilde{A}_a),$$

and (3.23)

$$A_a \equiv h_a^\nu(x)A_\nu(x), \quad \tilde{A}_a(x) \equiv \sqrt{h(x)}\,A_a(x)\,.$$

We note that $\theta_{\mu\nu}$ is symmetric, whereas $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$ are not. We do *not* symmetrize $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$ at this moment, as the antisymmetric part of $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$ generates the "spin-rotation," and it simplifies the derivation of the basic identity (4.26) below.

The tensor $\theta_{\mu\nu}$ is manifestly gauge invariant and satisfies the relation

$$\theta_\mu{}^\mu(x) = 0 \qquad (3.24)$$

which is a result of the invariance of the action (3.18) under the conformal (Weyl) transformation

$$h_a^\mu(x) \to e^{\alpha(x)}h_a^\mu(x)\,, \quad A_\mu(x) \to A_\mu(x)\,. \qquad (3.25)$$

The tensor $T_{\mu\nu}$ (3.21) corresponds to the canonical energy-momentum tensor, as $A_a(x) \to e^{\alpha(x)}A_a(x)$ under (3.25) and the conformal weight of $A_a(x)$ coincides with the naive canonical dimension: The choice of the variable $A_a(x)$ is also consistent with the description of the spin degrees of freedom with respect to the internal SO(3,1) [or SO(4) in R^4] symmetry. The tensor $\tilde{T}_{\mu\nu}$ (3.22) is dictated by the manifest covariance of the path-integral measure [see (3.34)], and $\tilde{A}_a$ is transformed as

$$\tilde{A}_a(x) \to e^{-\alpha(x)}\tilde{A}_a(x) \qquad (3.26)$$

under (3.25). Although $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$ are not manifestly gauge invariant, it is not a drawback in quantum theory where the gauge symmetry is replaced by the Becchi-Rouet-Stora (BRS) symmetry.[16]

To quantize the theory, we choose the Feynman gauge

$$S \equiv \int dx\, h(x)[-\tfrac{1}{4}g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \tfrac{1}{2}(D_\mu g^{\mu\nu}A_\nu)^2 + i\bar{c}(x)D^\mu\partial_\mu c(x)]$$

$$= \int dx\, h(x)\{\tfrac{1}{2}A_a(x)[(D^\mu D_\mu)^{ab} - R^{ab}]A_b(x) + i\bar{c}(x)D^\mu D_\mu c(x)\}\,, \qquad (3.27)$$

where D_μ acting on $A_b(x)$ is the full covariant derivative

$$D_\mu \equiv \partial_\mu + i\Gamma_{\beta\mu}{}^{\alpha}U_\alpha{}^\beta - \frac{i}{2}A_{mn\mu}S_1^{mn} \qquad (3.28)$$

with $U_\alpha{}^\beta$ an appropriate generator of GL(4, R) and S_1^{mn} a generator for the Lorentz vector representation

$$(S_1^{mn})_{ab} \equiv (-i)(\delta_a^m\delta_b^n - \delta_b^m\delta_a^n)\,. \qquad (3.29)$$

The scalar fermions $\bar{c}(x)$ and $c(x)$ are the Faddeev-Popov ghosts. The action (3.27) satisfies the *local* conformal-symmetry relation

$$\tilde{S}(e^\alpha h_a{}^\mu, e^{-\alpha}\tilde{A}_a; \tilde{\bar{c}}, e^{-2\alpha}\tilde{c}) = \tilde{S}(h_a^\mu, \tilde{A}_a, \tilde{\bar{c}}, \tilde{c}) + \int dx\, h(x)\partial_\mu\alpha(x)J^\mu(x)\,, \qquad (3.30)$$

where

$$J^\mu(x) \equiv 2[A^\mu(D_\lambda A^\lambda) - ig^{\mu\nu}\bar{c}\partial_\nu c] \qquad (3.31)$$

and $\tilde{\bar{c}}(x) = \sqrt{h}\,\bar{c}(x)$, $\tilde{c}(x) = \sqrt{h}\,c(x)$. The relation (3.30) gives rise to an identity after quantization (by discarding the source terms):

$$h(x)\langle \tilde{T}_\mu{}^\mu(x)\rangle + h(x)D_\mu\langle J^\mu(x)\rangle = \left\langle \tilde{A}_a(x)\frac{\delta}{\delta\tilde{A}_a(x)}\tilde{S}\right\rangle + 2\left\langle \tilde{c}(x)\frac{\delta}{\delta\tilde{c}(x)}\tilde{S}\right\rangle. \qquad (3.32)$$

The right-hand side of (3.32) is converted into the anomaly by the variational derivative (3.10) associated with the change of integration variables

$$\tilde{A}_a(x) \to \tilde{A}_a(x)' = e^{-\alpha(x)}\tilde{A}_a(x)\,,$$
$$\tilde{c}(x) \to \tilde{c}(x)' = e^{-2\alpha(x)}\tilde{c}(x)\,. \qquad (3.33)$$

We note that the covariant integral measure is defined by

$$d\mu \equiv \prod_{x,a} \mathfrak{D}\tilde{A}_a(x)\mathfrak{D}\tilde{c}(x)\mathfrak{D}\tilde{\bar{c}}(x) = \prod_n dc_n\, d\alpha_n\, d\beta_n\,, \qquad (3.34)$$

where

$$A_a(x) = \sum_n c_n V_{a,n}(x)\,,$$
$$c(x) = \sum_n \alpha_n S_n(x)\,, \qquad (3.35)$$
$$c(x) = \sum_n \beta_n S_n(x)\,,$$

with c_n the ordinary number, and α_n and β_n the elements of the Grassmann algebra. We thus obtain the conformal identity

$$\langle \tilde{T}_\mu{}^\mu(x)\rangle + D_\mu\langle J^\mu(x)\rangle = -A_2(x) + 2A_3(x) \qquad (3.36)$$

with the Jacobian factors [see Eqs. (3.8) and (3.9)]

$$A_2(x) \equiv \sum_n V_n(x)^T V_n(x)\,,$$
$$A_3(x) \equiv \sum_n S_n(x)S_n(x)\,. \qquad (3.37)$$

The sign difference on the right-hand side of (3.36) arises from the fermionic nature of $\tilde{c}(x)$.

In the explicit evaluation of anomaly factors (3.36), the basis vectors V_n are chosen to be the eigenvectors.

$$[(D^\mu D_\mu)_a{}^b - R_a{}^b]V_{b,n}(x) = \lambda_n V_{a,n}(x)\,,$$
$$\int dx\, h(x)V_n(x)^T V_m(x) = \delta_{n,m} \qquad (3.38)$$

with D_μ the full covariant derivative (3.28). Similarly, we use $D^\mu D_\mu S_n = \lambda_n S_n$ for the scalar basis. Following the standard procedure (3.15) and (3.16), we obtain (see Brown and Cassidy[9])

$$A_3(x)_c = \frac{1}{(4\pi)^2}\left(\tfrac{1}{72}R^2 + \tfrac{1}{30}D^\mu D_\mu R - \tfrac{1}{180}R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{180}R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right), \qquad (3.39)$$

$$A_2(x)_c = 4A_3(x)_c + \frac{1}{(4\pi)^2}\left(-\tfrac{1}{6}R^2 - \tfrac{1}{6}D^\mu D_\mu R + \tfrac{1}{2}R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{12}R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right), \qquad (3.40)$$

and finally the conformal identity for connected components[8,9]

$$\langle \tilde{T}_\mu{}^\mu(x)\rangle_c + D_\mu\langle J^\mu(x)\rangle_c = \frac{1}{(4\pi)^2}\left(\tfrac{5}{36}R^2 + \tfrac{1}{10}D^\mu D_\mu R - \tfrac{44}{90}R_{\mu\nu}R^{\mu\nu} + \tfrac{13}{180}R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right). \qquad (3.41)$$

IV. ENERGY-MOMENTUM TENSOR AND ASSOCIATED IDENTITIES IN RENORMALIZABLE THEORIES

The treatment of conformal anomalies in the preceding section crucially depends on the eigenvalue equations (3.14) and (3.38). For renormalizable theories in the flat-space-time limit, one deals with nonlinear systems for which the "correct" choice of basis vectors is not known in general. As a result, the explicit evaluation of the Jacobian factor is not possible except for several simple cases. Nevertheless we show here that the path-integral method is powerful enough to prove several fundamental renormalization properties of $\tilde{T}_{\mu\nu}(x)$ and other tensor quantities. The strategy we employ is to use suitable tensor quantities so that the resulting WT identities do not contain the (basically unknown) Jacobian factors. We illustrate this for spinor electrodynamics.

A. Spinor electrodynamics

We start with the Euclidean action defined by the Feynman-type gauge in the background gravitational field

$$S=\int dx\, h(x)[\bar\psi i\gamma^a h_a^\mu D_\mu\psi - m_0\bar\psi\psi - \tfrac{1}{4}g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{2\xi_0}(D_\mu A^\mu)^2], \tag{4.1}$$

where D_μ acting on ψ is defined by

$$D_\mu \equiv \partial_\mu + ie_0A_\mu - \frac{i}{2}A_{mn\mu}S^{mn}. \tag{4.2}$$

See also Eqs. (3.2) and (3.27). As we are eventually interested in the flat-space-time limit in this section, we ignore the Faddeev-Popov ghosts in (4.1). Various energy-momentum tensors, corresponding to (3.20)–(3.22), are defined from the action (4.1) by[13]

$$\begin{aligned}\theta_{\mu\nu}(x) &\equiv \frac{1}{h(x)}h_{\mu a}(x)\frac{\delta}{\delta h_a^\nu(x)}\hat S(h_a^\mu,\psi,\bar\psi,A_\nu)\\ &=\tfrac{1}{2}\bar\psi i(\gamma_\mu\overrightarrow{D}_\nu - \overleftarrow{D}_\nu\gamma_\mu)\psi - g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}\\ &\quad+\tfrac{1}{4}h_{k\mu}h_{l\nu}h_{m\rho}\epsilon^{klmn}D^\rho(\bar\psi\gamma_n\gamma_5\psi) - g_{\mu\nu}\mathcal{L}(x)\\ &\quad+\frac{1}{\xi_0}(A_\mu\partial_\nu + A_\nu\partial_\mu)(D_\alpha A^\alpha) - \frac{g_{\mu\nu}}{\xi_0}D_\alpha(A^\alpha D_\beta A^\beta),\end{aligned} \tag{4.3}$$

$$\begin{aligned}h(x)T_{\mu\nu}(x) &\equiv h_{\mu a}\frac{\delta}{\delta h_a^\nu(x)}S(h_a^\mu,\psi,\bar\psi,A_a)\\ &=h(x)\theta_{\mu\nu}(x) - h_{\mu a}h_\nu^b A_b(x)\frac{\delta}{\delta A_a(x)}S,\end{aligned} \tag{4.4}$$

$$\begin{aligned}h(x)\tilde T_{\mu\nu}(x) &\equiv h_{\mu a}\frac{\delta}{\delta h_a^\nu(x)}\tilde S(h_a^\mu,\tilde\psi,\tilde{\bar\psi},\tilde A_a)\\ &=h(x)T_{\mu\nu}(x) + \tfrac{1}{2}g_{\mu\nu}\Phi(x)\frac{\delta}{\delta\Phi(x)}\tilde S,\end{aligned} \tag{4.5}$$

where $\epsilon^{1230}=\epsilon^{1234}=1$ and Φ stands collectively for weight-$\frac{1}{2}$ fields [see Eqs. (2.4) and (3.23)]:

$$\Phi(x)\equiv(\tilde\psi(x),\tilde{\bar\psi}(x),\tilde A_a(x)). \tag{4.6}$$

The tensor $\tilde T_{\mu\nu}$ (4.5) gives the quantized source current for the background gravitational field in the covariant path-integral formalism. The explicit form of $\tilde T_{\mu\nu}$ in the flat-space-time limit is given in Appendix A.

The general covariance of the action (4.1) gives rise to an identity (in the flat-space-time limit)

$$\int dx\,\partial_\mu\epsilon^\nu(x)\tilde T^\mu{}_\nu(x)=\int dx\left\{\epsilon^\alpha(x)[\partial_\alpha\Phi(x)]\frac{\delta\tilde S}{\delta\Phi(x)} + \tfrac{1}{2}[\partial_\alpha\epsilon^\alpha(x)]\Phi(x)\frac{\delta\tilde S}{\delta\Phi(x)}\right\} \tag{4.7}$$

by considering the variations

$$\begin{aligned}\delta h_a^\mu(x) &= -\epsilon^\alpha(x)\partial_\alpha h_a^\mu(x) + [\partial_\lambda\epsilon^\mu(x)]h_a^\lambda(x),\\ \delta\Phi(x) &= -\epsilon^\alpha(x)\partial_\alpha\Phi(x)\\ &\quad-\tfrac{1}{4}g_{\mu\nu}[\partial_\lambda\epsilon^\mu)g^{\lambda\nu} + (\partial_\lambda\epsilon^\nu)g^{\mu\lambda}]\Phi(x)\end{aligned} \tag{4.8}$$

corresponding to the general coordinate transformation

$$x^\mu \to x'^\mu \equiv x^\mu + \epsilon^\mu(x). \tag{4.9}$$

We next derive two local identities associated with the change of integration variables

$$\Phi(x)\to[1+\alpha(x)]\Phi(x) \tag{4.10}$$

and

$$\Phi(x)\to[1+\alpha^\mu(x)\partial_\mu]\Phi(x) \tag{4.11}$$

by means of the variational derivative (the change of integration variables does not change the integral itself)

$$\frac{\delta}{\delta\alpha(x)}Z(\tilde J)\Big|_{\alpha=0}\equiv 0 \tag{4.12}$$

with

$$Z(\tilde J)\equiv\frac{1}{N}\int\prod_x \mathcal{D}\Phi(x)\exp\left(\tilde S + \int\tilde J(x)\Phi(x)\,dx\right). \tag{4.13}$$

We thus obtain, respectively,

$$\left\langle\left[\Phi(x)\frac{\delta\tilde S}{\delta\Phi(x)}\Phi(x_1)\cdots\Phi(x_N)\right]_+\right\rangle + \langle[A(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle + \sum_{l=1}^{N}\delta(x-x_l)\langle[\Phi(x_1)\cdots\Phi(x_l)\cdots\Phi(x_N)]_+\rangle = 0 \tag{4.14}$$

and

$$\left\langle \left[[\partial_\mu \Phi(x)] \frac{\delta \tilde{S}}{\delta \Phi(x)} \Phi(x_1) \cdots \Phi(x_N) \right]_+ \right\rangle + \tfrac{1}{2}\partial_\mu \langle [A(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle$$
$$+ \sum_{l=1}^{N} \delta(x-x_l)\partial_\mu^l \langle [\Phi(x_1)\cdots\Phi(x_l)\cdots\Phi(x_N)]_+\rangle = 0 \,. \quad (4.15)$$

Here $[\cdots]_+$ corresponds to the covariant T^* product in the operator formalism.[6] The Jacobian factor (anomaly) is defined by

$$A(x) \equiv -2A_1(x) + A_2(x) \quad (4.16)$$

with [cf., Eqs. (3.7)–(3.9)]

$$A_1(x) \equiv \sum_n \varphi_n(x)^\dagger \varphi_n(x) \,, \quad (4.17)$$

$$A_2(x) \equiv \sum_n V_n(x)^T V_n(x) \quad (4.18)$$

in terms of the complete sets of basis vectors introduced to define the path integral [see Eqs. (2.5) and (3.35)]. The minus sign in front of $A_1(x)$ in (4.16) arises from the fact that a_n and $\bar{b}_n$ in (2.5) are the elements of the Grassmann algebra.

Combining (4.7) with (4.14) and (4.15), we obtain the *local* WT identity

$$\partial_\mu \langle [\tilde{T}^\mu{}_\nu(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle = \sum_{l=1}^{N} [\delta(x-x_l)\partial_\nu^l - \tfrac{1}{2}\partial_\nu\delta(x-x_l)]\langle [\Phi(x_1)\cdots\Phi(x_l)\cdots\Phi(x_N)]_+\rangle \,. \quad (4.19)$$

We emphasize that the possible anomaly terms completely dropped out of this final result, reflecting the general covariance of the path-integral measure (i.e., the quantization procedure is compatible with classical space-time symmetries). This cancellation of the anomaly for $\tilde{T}_{\mu\nu}$ is therefore quite general and it takes place for any theory such as gauge theory. The left-hand side of (4.19) is thus *finite* up to the ordinary wave-function renormalization factors (for any regularization scheme) in renormalizable theories.[17] In contrast, the tensor $T_{\mu\nu}(x)$ (4.4) in the flat-space-time limit satisfies an identity with the anomaly (4.16):

$$\partial_\mu \langle [T^\mu{}_\nu(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle = \sum_{l=1}^{N} \delta(x-x_l)\partial_\nu^l \langle [\Phi(x_1)\cdots\Phi(x_l)\cdots\Phi(x_N)]_+\rangle + \tfrac{1}{2}\partial_\nu \langle [A(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle \quad (4.20)$$

and the finiteness of the left-hand side cannot be proved from this identity.

We next derive identities associated with the invariance of (4.1) under local Lorentz transformations

$$\Phi(x) \to \left[1 + \frac{i}{2}\omega_{kl}(x) S^{kl}\right]\Phi(x) \,, \quad (4.21)$$

$$h_a^\mu(x) \to \left[\left(1 + \frac{i}{2}\omega_{kl}(x) S^{kl}\right) h^\mu(x)\right]_a \quad (4.22)$$

with S^{kl} the appropriate generators of SO(3, 1) [or SO(4) in R^4] for the Lorentz spinor $\psi(x)$ and the Lorentz vectors $A_a(x)$ and $h_a^\mu(x)$. We have the symmetry relation

$$(S^{kl}h^\mu(x))_a \frac{\delta \tilde{S}}{\delta h_a^\mu(x)} + (S^{kl}\Phi(x))_a \frac{\delta \tilde{S}}{\delta \Phi_a(x)} = 0 \,. \quad (4.23)$$

This relation when combined with the variational derivative associated with the change of variables (4.21) gives rise to the local WT identity

$$(S^{kl})_a{}^b \langle [\tilde{T}_b{}^a(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle = \sum_{m=1}^{N} \delta(x-x_m)\langle [\Phi(x_1)\cdots(S^{kl}\Phi(x_m))\cdots\Phi(x_N)]_+\rangle \quad (4.24)$$

with

$$\tilde{T}_b{}^a(x) = h_b^\mu(x) \frac{\delta \tilde{S}}{\delta h_a^\mu(x)} \,. \quad (4.25)$$

The Jacobian (anomaly) for the non-Abelian symmetry (4.21) identically vanishes; $\mathfrak{D}\tilde{\psi}$ and $\mathfrak{D}\bar{\tilde{\psi}}$ give rise to Jacobian factors with opposite signs, and the Jacobian for $\mathfrak{D}\tilde{A}_a$ vanishes due to the antisymmetry property of S_1^{kl} (3.29). In the flat–space-time limit $\tilde{T}_{ab}$ is identified with $\tilde{T}_{\mu\nu}$, and (4.24) together with (3.29) shows that the antisymmetric part of $\tilde{T}_{\mu\nu}$ is a *finite* operator for renormalizable theories. By decomposing $\tilde{T}_{\mu\nu}$ into symmetric and antisymmetric parts by $\tilde{T}^{\mu}{}_{\nu} \equiv \tilde{T}^{\mu}_{S\nu} + \tilde{T}^{\mu}_{A\nu}$ in (4.19) and using (4.24) for the antisymmetric part, we obtain

$$\partial_\mu \langle [\tilde{T}^{\mu}_{S\nu}(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle = \sum_{l=1}^{N} [\delta(x-x_l)\partial^l_\nu + \tfrac{1}{2}i\partial_\mu\delta(x-x_l)S^{\mu}{}_{\nu}(l) - \tfrac{1}{2}\partial_\nu\delta(x-x_l)] \times \langle [\Phi(x_1)\cdots\Phi(x_l)\cdots\Phi(x_N)]_+\rangle . \tag{4.26}$$

B. Global space-time symmetries[18]

By integrating (4.19), we obtain

$$\sum_{l=1}^{N} \partial^l_\mu \langle [\Phi(x_1)\cdots\Phi(x_l)\cdots\Phi(x_N)]_+\rangle = 0 , \tag{4.27}$$

which is a statement of *global* translation invariance. By multiplying (4.26) by x^λ in an antisymmetric manner and integrating over x, we have

$$\sum_{l=1}^{N}[(x_{l\lambda}\partial^l_\nu - x_{l\nu}\partial^l_\lambda) - iS_{\lambda\nu}(l)] \times \langle [\Phi(x_1)\cdots\Phi(x_l)\cdots\Phi(x_N)]_+\rangle = 0 , \tag{4.28}$$

which is a statement of the *global* Lorentz invariance in the flat–space-time limit. We can derive the same set of identities (4.27) and (4.28) starting from $T_{\mu\nu}$ in (4.20), as the second terms in (4.19) and (4.20) do not contribute to these identities; in other words, the Poincaré invariance alone *cannot* exclude the anomaly in (4.20).

Multiplying (4.19) by x^ν and integrating over x, we obtain[18]

$$\int dx \langle [\tilde{T}_\mu{}^\mu(x)_c \Phi(x_1)\cdots\Phi(x_N)]_+\rangle = \left(-\sum_{l=1}^{N} x_l^\mu \partial^l_\mu - 2N\right)\langle [\Phi(x_1)\cdots\Phi(x_l)\cdots\Phi(x_N)]_+\rangle . \tag{4.29}$$

We call this the global "dilatation identity" associated with $x \to e^\rho x$ in the following. The dilatation identity expressed in terms of $\tilde{T}_{\mu\nu}$ is thus free from the anomaly, whereas $T_{\mu\nu}$ in (4.4) gives rise to an identity with an anomaly due to the second term in (4.20). Equation (4.29) shows that the trace $\int dx\, \tilde{T}_\mu{}^\mu(x)_c$ is a *finite* operator in renormalizable theories.

C. Conformal anomaly

We next derive the "conformal identity" associated with the local conformal (Weyl) transformation

$$h^\mu_a(x) \to e^{\alpha(x)} h^\mu_a(x) ,\quad \tilde{\psi}(x) \to e^{-\alpha(x)/2}\tilde{\psi}(x),\quad \tilde{\bar{\psi}}(x) \to e^{-\alpha(x)/2}\tilde{\bar{\psi}}(x) ,\quad \tilde{A}_a(x) \to e^{-\alpha(x)}\tilde{A}_a(x) . \tag{4.30}$$

The action (4.1) under (4.30) gives rise to the relation (in the flat–space-time limit)

$$\int dx\, \alpha(x)\, \tilde{T}_\mu{}^\mu(x) = \int dx\, \alpha(x) \left[\tfrac{1}{2}\tilde{\bar{\psi}}(x)\frac{\delta \tilde{S}}{\delta\tilde{\bar{\psi}}(x)} + \tfrac{1}{2}\tilde{\psi}(x)\frac{\delta\tilde{S}}{\delta\tilde{\psi}(x)} + \tilde{A}_a(x)\frac{\delta\tilde{S}}{\delta\tilde{A}_a(x)} + m_0\tilde{\bar{\psi}}\tilde{\psi}(x)\right] + \int dx\, \partial_\mu\alpha(x) J^\mu(x) \tag{4.31}$$

with [see Eq. (3.31)]

$$J^\mu(x) \equiv (2/\xi_0) A^\mu(\partial_\lambda A^\lambda) . \tag{4.32}$$

Equation (4.31) combined with local identities similar to (4.14) gives rise to the local "conformal identity"

$$\begin{aligned}\langle [\tilde{T}_\mu{}^\mu(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle = &-\sum_{l=\text{fermions}} \tfrac{1}{2}\delta(x-x_l)\langle [\Phi(x_1)\cdots\Phi(x_l)\cdots\Phi(x_N)]_+\rangle \\ &-\sum_{l=\text{photons}} \delta(x-x_l)\langle [\Phi(x_1)\cdots\Phi(x_l)\cdots\Phi(x_N)]_+\rangle \\ &+ m_0\langle [\tilde{\bar{\psi}}\tilde{\psi}(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle + \langle\{[A_1(x) - A_2(x)]\Phi(x_1)\cdots\Phi(x_N)\}_+\rangle \\ &- \partial_\mu\langle [J^\mu(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle ,\end{aligned} \tag{4.33}$$

where the anomaly factors are defined in (4.17) and (4.18).

Although we cannot evaluate $A_2(x)$ explicitly [$A_1(x)$ corresponds to the familiar trace anomaly[7] as will be explained in the next section], we can determine the combination $A_1(x) - A_2(x)$ at the *zero-momentum-transfer* limit by the consistency of (4.29) and (4.33):

$$\int dx \langle \{[A_1(x) - A_2(x) + m_0\bar{\psi}\psi(x)]_c \Phi(x_1)\cdots\Phi(x_N)\}_+\rangle = \left(-\tfrac{3}{2}N_f - N_A - \sum_{l=1}^{N} x_l^\mu \partial_\mu^l\right)\langle[\Phi(x_1)\cdots\Phi(x_N)]_+\rangle \tag{4.34}$$

or in terms of the renormalized Green's function

$$G_r\left(x_1,\ldots,x_N;\int dx[A_1(x) - A_2(x) + m_0\bar{\psi}\psi(x)]_c\right) = \left(-\tfrac{3}{2}N_f - N_A - \sum_{l=1}^{N} x_l^\mu\partial_\mu^l\right)G_r(x_1,\ldots,x_N), \tag{4.35}$$

where N_f and N_A stand for the numbers of fermions and photons appearing in the Green's function, respectively; $N_f + N_A = N$ with N_f an even integer. Equation (4.35) shows that the connected component

$$\int dx[A_1(x) - A_2(x) + m_0\bar{\psi}\psi(x)]_c$$

is a finite operator.

We next recall the dimensional structure of $G_r(x_1,\ldots,x_N)$:

$$G_r(x_1,\ldots,x_N) = x^{-3N_f/2-N_A}f(\mu x, m/\mu) \tag{4.36}$$

which suggests

$$\sum_l x_l^\mu\partial_\mu^l G_r(x_1,\ldots,x_N) = \left(-\tfrac{3}{2}N_f - N_A + \mu\frac{\partial}{\partial\mu} + m\frac{\partial}{\partial m}\right)G_r(x_1,\ldots,x_N), \tag{4.37}$$

where μ and m stand for the renormalization point and the renormalized fermion mass, respectively. The dimensional identity (4.37) combined with a particular form of the renormalization-group equation[19]

$$\left(\mu\frac{\partial}{\partial\mu} + D_N\right)G_r(x_1,\ldots,x_N) = 0, \tag{4.38}$$

$$D_N \equiv \beta(e)\frac{\partial}{\partial e} + \delta(e)m\frac{\partial}{\partial m} + \tfrac{1}{2}[N_f\gamma_\psi(e) + N_A\gamma_A(e)] + \epsilon(e)\xi\frac{\partial}{\partial\xi} \tag{4.39}$$

fixes the right-hand side of (4.35) as

$$G_r\left(x_1,\ldots,x_N;\int dx[A_1(x) - A_2(x) + m_0\bar{\psi}\psi(x)]_c\right) = \left(-m\frac{\partial}{\partial m} + D_N\right)G_r(x_1,\ldots,x_N). \tag{4.40}$$

This relation operationally determines the conformal anomaly $A_1(x) - A_2(x)$ at the vanishing momentum transfer. By combining (4.40) with (4.29) or (4.33), we obtain (in Euclidean metric)

$$G_r\left(x_1,\ldots,x_N;\int dx\,\tilde{T}_\mu{}^\mu(x)_c\right) = \left(D_N - m\frac{\partial}{\partial m} - \tfrac{1}{2}N_f - N_A\right)G_r(x_1,\ldots,x_N), \tag{4.41}$$

where the conformal weights in (4.30), $-\frac{1}{2}$ and -1 for $\tilde{\psi}$ and $\tilde{A}_a$, respectively, explicitly appear.

From our derivation, it is clear that the identity (4.41) is quite general and we can derive a corresponding identity for any renormalizable theory including gauge theory. It is gratifying that the quantized source current $\tilde{T}_{\mu\nu}$ for the background gravitational field has satisfactory high-energy behavior.

D. Connection with the conventional perturbation theory

In the *flat-space-time* limit, the ordinary field variables and the weight-$\frac{1}{2}$ variables (4.6) become indistinguishable in the path-integral measure. We can thus convert (4.19) and (4.33) into WT identities for $\theta_{\mu\nu}$ (4.3) by remembering the definitions (4.3)–(4.5) and the identities similar to (4.14). We thus obtain

$$\partial_\mu\langle[\Theta^\mu{}_\nu(x)\Phi(x_1)\cdots A_\alpha(x_k)\cdots\Phi(x_N)]_+\rangle = \sum_{l=1}^{N}\delta(x-x_l)\partial_\nu^l\langle[\Phi(x_1)\cdots A_\alpha(x_k)\cdots\Phi(x_N)]_+\rangle - \sum_{k=\text{photons}}\partial_\alpha\delta(x-x_k)\langle[\Phi(x_1)\cdots A_\nu(x_k)\cdots\Phi(x_N)]_+\rangle \tag{4.42}$$

and

$$\begin{aligned}\langle[\Theta_\mu{}^\mu(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle = &\sum_{l=\text{fermions}}\tfrac{3}{2}\delta(x-x_l)\langle[\Phi(x_1)\cdots\Phi(x_l)\cdots\Phi(x_r)]_+\rangle\\ &+ m_0\langle[\bar{\psi}\psi(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle + \langle\{[A_1(x) - A_2(x)]\Phi(x_1)\cdots\Phi(x_N)\}_+\rangle\\ &- \partial_\mu\langle[J^\mu(x)\Phi(x_1)\cdots\Phi(x_N)]_+\rangle,\end{aligned} \tag{4.43}$$

where

$$\Theta_{\mu\nu}(x) \equiv \theta_{\mu\nu}(x) - \frac{g_{\mu\nu}}{2}A(x) + A_{\mu\nu}(x) \tag{4.44}$$

with a new (symmetric tensor) anomaly

$$A_{\mu\nu}(x) \equiv h_\mu^a h_\nu^b \left[\sum_n V_{a,n}(x) V_{b,n}(x)\right] \tag{4.45}$$

in terms of the basis vectors (3.35). We also have an identity corresponding to (4.24). In this rearrangement,[20] it is important to notice that all the possible anomalies are *local* objects. In (4.42) and (4.43), Φ is regarded to stand for $(\psi, \overline{\psi}, A_\mu)$ instead of (4.6), and we specified the index of the representative A_α in (4.42).

From (4.42), we have the dilatation identity[18]

$$\int dx \langle [\Theta_\mu{}^\mu(x)_c \Phi(x_1) \cdots \Phi(x_N)]_+\rangle = \left(-N_A - \sum_{l=1}^{N} x_l^\mu \partial_\mu^l\right) \langle [\Phi(x_1)\cdots\Phi(x_N)]_+\rangle \,, \tag{4.46}$$

where the factor N_A [and also $2N$ in (4.29)] arises to compensate for the difference between the conformal weight and the naive canonical dimension of independent field variables appearing in the definition of the energy-momentum tensors in (4.3)–(4.5). The consistency between (4.46) and (4.43) gives rise to the same condition (4.34) as before, and we finally obtain

$$G_r\left(x_1, \ldots, x_N; \int dx\, \Theta_\mu{}^\mu(x)_c\right) = \left(D_N - m\frac{\partial}{\partial m} + \tfrac{3}{2}N_f\right) G_r(x_1, \ldots, x_N) \,, \tag{4.47}$$

where the conformal weight factors of ψ and A_μ, $\frac{3}{2}$ and 0, respectively, instead of the naive canonical dimensions appear. This is the result of the conventional perturbation theory[21] written in the Euclidean metric starting from (a suitably symmetrized version of) $\theta_{\mu\nu}$.

Although the above procedure appears to be somewhat unorthodox, we emphasize that this *is* in fact the procedure taken in conventional perturbation theory. To understand this point, we note that WT identities such as (4.19) are *derived* from the symmetry property of the theory. Once WT identities are fixed, all the perturbative calculations are now performed by *imposing* these identities.[6] Note that there is no "anomalous" WT identity as such in the present formulation. The WT identity (4.42) is trivially true in the tree level where $\Theta_{\mu\nu}$ is reduced to $\theta_{\mu\nu}$ as the anomaly factors vanish to this order. Therefore, it is immaterial whether one started with $\theta_{\mu\nu}$ or $\Theta_{\mu\nu}$ in perturbation theory, once one adopted the identity (4.42) as a guiding principle at each stage of perturbation theory[21]; one always ends up with the tensor quantity which satisfies (4.42), namely $\Theta_{\mu\nu}$ (4.44), in our approach. When one imposes (4.42), the trace of $\Theta_{\mu\nu}$ at vanishing momentum transfer is uniquely determined by the renormalization-group (or Callan-Symanzik) equation by means of the dilatation identity (4.46). The consistency of this result with the conformal identity (4.43) then uniquely fixes the anomaly factor appearing in (4.43) at vanishing momentum transfer.

It is significant that the path-integral method provides a proof of the existence of $\Theta_{\mu\nu}$ which satisfies (4.42) consistently. In the path integral, it is also possible to evaluate some of the Jacobian factors such as $A_1(x)$ in (4.43) directly,[2] as will be explained in the next section. The agreement of this estimate with (4.47) indicates that the path-integral method is self-consistent. The advantage of $\tilde{T}_{\mu\nu}$ over $\Theta_{\mu\nu}$ is that one can explicitly evaluate $\tilde{T}_{\mu\nu}$ by (4.5). See Appendix A.

From the renormalization-theory point of view, the quantity $\int dx\, \tilde{T}_\mu{}^\mu(x)_c$ in (4.29) is finite and not renormalized, whereas $\int dx\, \theta_\mu{}^\mu(x)_c$ is *subtractively* renormalized as is understood from the relation (4.44) and the fact that $\int dx[-2A(x) + A_\mu{}^\mu(x)] \neq 0$. We shall explain this point in the next section.

V. RENORMALIZATION PROPERTIES OF CHIRAL AND TRACE ANOMALIES

The chiral anomaly[10] is derived in the path integral by considering the variation

$$\psi(x) \to e^{i\alpha(x)\gamma_5}\psi(x) \,,$$
$$\overline{\psi}(x) \to \overline{\psi}(x) e^{i\alpha(x)\gamma_5} \tag{5.1}$$

with *other variables fixed* in the action (4.1) in the flat-space-time limit. The result is,[1] for example,

$$\partial_\mu \langle [j_5^\mu(x)\psi(y)\overline{\psi}(z)]_+\rangle = 2m_0 i \langle [j_5(x)\psi(y)\overline{\psi}(z)]_+\rangle - i\delta(x-y)\langle [\gamma_5\psi(y)\overline{\psi}(z)]_+\rangle - i\delta(x-z)\langle [\psi(y)\overline{\psi}(z)\gamma_5]_+\rangle + i\langle [A_5(x)\psi(y)\overline{\psi}(z)]_+\rangle \,, \tag{5.2}$$

where the gauge-invariant currents are

$$j_5^\mu(x) \equiv \overline{\psi}(x)\gamma^\mu\gamma_5\psi(x) \,,$$
$$j_5(x) \equiv \overline{\psi}(x)\gamma_5\psi(x) \tag{5.3}$$

and the anomaly factor

$$A_5(x) \equiv \sum_n \varphi_n(x)^\dagger \gamma_5 \varphi_n(x) . \tag{5.4}$$

The corresponding identities can be derived by considering

$$\psi(x) \to e^{-\alpha(x)/2}\psi(x) ,$$
$$\bar\psi(x) \to \bar\psi(x) e^{-\alpha(x)/2} \tag{5.5}$$

with *other variables fixed.* We then obtain, for example,

$$\langle[\hat\theta(x)\psi(y)\bar\psi(z)]_+\rangle = m_0\langle[j(x)\psi(y)\bar\psi(z)]_+\rangle - \tfrac{1}{2}\delta(x-y)\langle[\psi(y)\bar\psi(z)]_+\rangle - \tfrac{1}{2}\delta(x-z)\langle[\psi(y)\bar\psi(z)]_+\rangle + \langle[A_W(x)\psi(y)\bar\psi(z)]_+\rangle \tag{5.6}$$

where

$$\hat\theta(x) \equiv \tfrac{1}{2}\bar\psi(x)\frac{\delta S}{\delta\bar\psi(x)} + \tfrac{1}{2}\psi(x)\frac{\delta S}{\delta\psi(x)}$$
$$= \tfrac{1}{2}\bar\psi(x) i\overleftrightarrow{\not{D}}\psi(x) ,$$
$$j(x) \equiv \bar\psi(x)\psi(x) \tag{5.7}$$

and

$$A_W(x) \equiv \sum_n \varphi_n(x)^\dagger \varphi_n(x) \tag{5.8}$$

which coincides with $A_1(x)$ in (4.17).

A careful evaluation[1,2] of (5.4) and (5.8) gives, respectively,

$$A_5(x) = \frac{e_0^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} , \tag{5.9}$$

$$A_W(x)_c = \frac{e_0^2}{24\pi^2}F^{\mu\nu}F_{\mu\nu} . \tag{5.10}$$

See also Appendix B. Identifying $\hat\theta(x)$ as the trace[22] of $\hat\theta_{\mu\nu}(x)$,

$$\hat\theta_{\mu\nu}(x) \equiv \tfrac{1}{2}\bar\psi(x) i\gamma_\mu \overleftrightarrow{D}_\nu \psi(x) , \tag{5.11}$$

Eqs. (5.2) and (5.6) correspond to the well-known chiral[10] and "trace"[7] identities, respectively.

As is well known, (5.2) is *exact* although (5.2) as it stands corresponds to $\infty = \infty$ for nonvanishing momentum transfer (the Adler-Bardeen theorem[23]). Incidentally this theorem provides a partial support for the path-integral manipulation. The derivations of (5.2) and (5.6) and the evaluations of (5.9) and (5.10) are completely on an equal footing. We therefore expect that (the connected component of) (5.6) is also *exact*, although it again becomes $\infty = \infty$. (In the lowest nontrivial order, this can be confirmed by following the procedure in Ref. 23.)

There is, however, an important difference between (5.2) and (5.6). The left-hand side of (5.2) vanishes at the zero-momentum-transfer limit in the presence of the explicit chiral-breaking mass term. This observation when combined with the identity

$$\partial_\mu\langle[j_5^\mu(x)A_\alpha(y)A_\beta(z)]_+\rangle = 2m_0 i\langle[j_5(x)A_\alpha(y)A_\beta(z)]_+\rangle + i\langle[A_5(x)A_\alpha(y)A_\beta(z)]_+\rangle \tag{5.12}$$

written for the isovector current gives rise to the well-known formula for $\pi^0 \to \gamma\gamma$ via the partial conservation of axial-vector current (PCAC) relation.[23,24]

In contrast, the left-hand side of (5.6) does not vanish even at vanishing momentum transfer, and one cannot extract useful information from the relation among divergent operators. Fortunately, the identity (4.33) at vanishing momentum transfer provides a finite WT identity which includes $A_W(x)$ [$= A_1(x)$], and it illustrates the validity of (5.10). As we cannot evaluate $A_2(x)$ in (4.33) directly, we have to rely on a suitable regularization scheme[25] to learn the role of $A_2(x)$.

A. Pauli-Villars regularization

We introduce a negative-metric vector field $B_\mu(x)$ to regulate the photon propagator: We add

$$\tfrac{1}{4}g^{\mu\nu}g^{\alpha\beta}(\partial_\mu B_\alpha - \partial_\alpha B_\mu)(\partial_\nu B_\beta - \partial_\beta B_\nu) - \frac{M^2}{2}g^{\mu\nu}B_\mu B_\nu + \frac{1}{2\xi_0}(D_\mu B^\mu)^2 \tag{5.13}$$

to the Lagrangian (4.1) and replace A_μ by $A_\mu + B_\mu$ inside the interaction term. (The one-loop vacuum polarization diagrams for vector fields are treated in the customary way.[21]) Based on the experience in perturbation theory, we *assume*[26] that the interaction picture (i.e., the use of plane-wave basis) is well defined for *gauge fields* A_μ and B_μ after this regularization. We thus neglect the anomaly $A_2(x)$ in (4.33) as the Jacobian becomes physically trivial in the plane-wave basis, but we obtain an extra contribution on the right-hand side of (4.33) from the regulator mass term. [The current (4.32) is also modified.] Thus the anomaly factor $[A_1(x) - A_2(x)]_c$ in (4.33) is effectively replaced by [we use Eq. (5.10)]

$$\frac{e_0^2}{24\pi^2}[\partial_\mu(A_\nu+B_\nu)-\partial_\nu(A_\mu+B_\mu)]^2+M^2B_\mu B^\mu\,. \tag{5.14}$$

As has been analyzed in detail by Adler, Collins, and Duncan,[21] (5.14) defines a *subtracted* (finite) operator and it gives rise to $\beta(e)\partial/\partial e$ and other factors in (4.40) in the zero-momentum-transfer limit. We thus see that the Adler-Bardeen–type theorem for (5.2) and (5.6) and the appearance of the renormalized (subtracted) anomaly in (4.41) are consistent with each other. We note that mass terms (including the regulator mass) do *not* appear in $\tilde{T}_{\mu\nu}$ (4.5); this partly accounts for the good high-energy behavior of $\tilde{T}_{\mu\nu}$ regardless of the regularization scheme. (See also Appendix A.)

This exercise also shows that $A_2(x)$ in (4.40) cannot be neglected and, in fact, it is a "hard" operator, as $A_1(x)_c$ is a divergent operator and the fermion mass term is known to be finite[27] in (4.40). The subtraction term in (4.44) gives rise to

$$[-2A(x)+A_2(x)]_c=\{[A_1(x)-A_2(x)]+3A_1(x)\}_c \tag{5.15}$$

in the trace $\Theta_\mu{}^\mu(x)_c$; this combination (5.15) gives a divergent operator. We thus conclude that the trace $\int dx\theta_\mu{}^\mu(x)_c$ is divergent in this regularization and it is *subtractively* renormalized[21] to give rise to the finite $\int dx\Theta_\mu{}^\mu(x)_c$ in (4.47).

B. Dimensional regularization

In dimensional regularization,[28] all the Jacobian factors are expected to be discarded: A partial justification of this is given in Appendix B. The identity (4.26) holds as before, as it is independent of dimensionality. The dilatation identity (4.29) is thus valid with $2N$ replaced by $nN/2$. The conformal symmetry (4.31) is, however, spoiled by the dimensional continuation just as the chiral symmetry, and we pick up an extra term

$$\left(\frac{4-n}{2}\right)e_0\int\alpha(x)\bar{\psi}A\!\!\!/\psi(x)dx \tag{5.16}$$

on the right-hand side of (4.31). Note that the conformal weights in (4.30) are independent of the space-time dimensionality n. This extra term (5.16) replaces the anomaly factor $A_1(x)-A_2(x)$ in (4.33), as the Jacobian is absent in this regularization. [The current (4.32) is also slightly modified.] It is known[29] that (5.16) together with the mass term in (4.33) give rise to the correct anomalous factors in (4.41) as is expected from the consistency with (4.29). The path-integral manipulation is thus self-consistent.

There is one peculiar aspect with dimensional regularization, however. As the Jacobian factor is absent, $\theta_{\mu\nu}$ in (4.3) satisfies the same set of identities (4.42) and (4.43) as $\Theta_{\mu\nu}$ with the conformal anomaly factor in (4.43) replaced by (5.16). In other words, the dilatation anomalies in (4.44) are automatically subtracted away by the *assumption* of the smooth dimensional continuation,[30] whereas the conformal anomaly is not. This is presumably related to the fact that the general covariance, which includes dilatation, has a natural dimensional continuation but the conformal (and also chiral) symmetry is generally spoiled by the continuation.[31] We note here that the Poincaré invariance alone cannot remove the possible anomalies in, e.g., (4.20) as was explained in Sec. IV B.

VI. CONCLUSION

We illustrated the advantage of the formal treatment of the path integral with a specification of the "representation" by means of basis vectors. The anomalous behavior of WT identities is always anticipated by the nonvanishing Jacobian factors in the integral measure. In this sense, there is no "anomalous" WT identity in the present formulation, although the explicit evaluation of the Jacobian is not always possible; this may restore our confidence in WT identities as a guiding principle for perturbative calculations.

The definition of the energy-momentum tensor as a source current for the background gravitational field receives an important modification in the quantized theory because of the choice of the general covariant path-integral measure. An improved energy-momentum tensor is thus obtained, and it exhibits satisfactory high-energy behavior. By using this energy-momentum tensor, we were able to identify the trace anomaly as the conformal anomaly. All the familiar anomalies are thus reduced to either chiral or conformal anomalies, which share the interesting algebraic characterization.[2]

APPENDIX A

The explicit form of $\tilde{T}_{\mu\nu}$ (4.5) in the flat–space-time limit is given by

$$\tilde{T}_{\mu\nu}(x)=\tfrac{1}{2}\bar{\psi}i\gamma_\mu\overleftrightarrow{D}_\nu\psi-A_\nu\left[-e_0\bar{\psi}\gamma_\mu\psi+\partial_\alpha F^\alpha{}_\mu+\frac{1}{\xi_0}\partial_\mu(\partial_\alpha A^\alpha)\right]-F_{\alpha\nu}F^\alpha{}_\mu+\frac{1}{\xi_0}[A_\mu\partial_\nu(\partial_\alpha A^\alpha)+A_\nu\partial_\mu(\partial_\alpha A^\alpha)]$$
$$-\frac{g_{\mu\nu}}{2}\left[e_0\bar{\psi}A\!\!\!/\psi-\partial_\alpha(A^\beta F^\alpha{}_\beta)+\frac{1}{\xi_0}\partial_\alpha(A^\alpha\partial_\beta A^\beta)\right]+\tfrac{1}{4}\epsilon_{\mu\nu\lambda\rho}\partial^\lambda(\bar{\psi}\gamma^\rho\gamma_5\psi)$$

with $D_\mu \equiv \partial_\mu + ie_0 A_\mu$. In practical applications, it is sufficient to consider the symmetric part of $\tilde{T}_{\mu\nu}$ which satisfies the identity (4.26).

APPENDIX B

The Jacobian factor $A_1(x)$ $[=A_W(x)]$ in (4.17) is evaluated in Euclidean space by using the eigenvectors (the "Heisenberg representation" [12])

$$\not{D}\varphi_k(x) = \lambda_k \varphi_k(x)\,, \quad \int \varphi_k(x)^\dagger \varphi_l(x) d^n x = \delta_{k,l} \tag{B1}$$

with $\not{D} \equiv \gamma^\mu(\partial_\mu + ie_0 A_\mu)$. We sum the series in (4.17) starting from small eigenvalues ($|\lambda_l| \lesssim M$) in *n dimensions* as

$$\begin{aligned} A_1(x) &= \lim_{M\to\infty} \sum_l \varphi_l(x)^\dagger e^{-(\lambda_l/M)^2} \varphi_l(x) = \lim_{M\to\infty} \sum_l \varphi_l(x)^\dagger e^{-(\not{D}/M)^2} \varphi_l(x) \\ &= \lim_{M\to\infty} \mathrm{Tr} \int \frac{d^n k}{(2\pi)^n} e^{-ikx} \exp\left[-\left(D^\mu D_\mu + \frac{ie_0}{4}[\gamma^\mu, \gamma^\nu] F_{\mu\nu}\right)\Big/ M^2\right] e^{ikx} \\ &= \lim_{M\to\infty} M^n \,\mathrm{Tr} \int \frac{d^n k}{(2\pi)^n} \exp\left[-\left((ik^\mu + D^\mu/M)^2 + \frac{ie_0}{4M^2}[\gamma^\mu, \gamma^\nu] F_{\mu\nu}\right)\right], \end{aligned} \tag{B2}$$

where we transformed the basis vectors to plane waves for the well-defined operator $\exp[-(\not{D}/M)^2]$, and the trace runs over Dirac matrices. By expanding the exponential factor in (B2) in powers of $1/M$, we obtain

$$A_1(x) = \lim_{M\to\infty} M^n 2^{n/2} \left(1 + \frac{e_0^2}{6M^4} F^{\mu\nu} F_{\mu\nu}\right) \int \frac{d^n k}{(2\pi)^n} e^{-k^\mu k^\mu}\,, \tag{B3}$$

where the higher-order terms in $1/M$ are neglected. Evaluating (B3) at sufficiently small n, we may conclude that $A_1(x) = 0$ (by discarding the disconnected constant term) in dimensional regularization. The chiral Jacobian factor (5.4) also vanishes in this scheme. The same conclusion holds even if $\exp[-(\lambda_l/M)^2]$ is replaced by *any* smooth function[1,2] $f(\lambda_l^2/M^2)$ in (B2) with $f(0) = 1$ and $f(+\infty) = f'(+\infty) = 0$.

If one sets $n = 4$ in (B3), one recovers the familiar result (5.10). The Jacobian factors generally have the nonanalytic dependence on the dimensionality such as $\delta_{n,4}$.

[1] K. Fujikawa, Phys. Rev. Lett. 42, 1195 (1979); Phys. Rev. D 21, 2848 (1980); 22, 1499(E) (1980).

[2] K. Fujikawa, Phys. Rev. Lett. 44, 1733 (1980).

[3] S. Hawking, Commun. Math. Phys. 55, 133 (1977).

[4] S. Coleman, in *The Whys of Subnuclear Physics*, proceedings of the International School of Subnuclear Physics, Erice, 1977, edited by A. Zichichi (Plenum, New York, 1979). See also B. Davison, Proc. R. Soc. London A225, 252 (1954); W. Burton and A. De Borde, Nuovo Cimento 2, 197 (1955).

[5] The anomaly in Hodge theory, for example, also shares the same algebraic characterization noted in Ref. 2 above, and it can be identified as the Jacobian. See N. Nielsen, H. Römer, and B. Schroer, Nucl. Phys. B136, 445 (1978).

[6] A comprehensive account of *conventional* energy-momentum tensors and the trace anomaly is found in S. Coleman and R. Jackiw, Ann. Phys. (N. Y.) 67, 552 (1971). The dilatation symmetry in this reference corresponds to a suitable combination of the "dilatation" and conformal (Weyl) symmetries in this paper, just as the conventional Lorentz transformation corresponds to a combination of general coordinate and local Lorentz transformations in this paper. Our improved tensor $\tilde{T}_{\mu\nu}$ has no connection with the tensor proposed in C. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N. Y.) 59, 42 (1970). Our tensor $\tilde{T}_{\mu\nu}$ is expected to be free from the criticism by K. Symanzik quoted in this second reference concerning the regulator mass, as $\tilde{T}_{\mu\nu}$ does not contain any (including regulator) mass term.

[7] R. Crewther, Phys. Rev. Lett. 28, 1421 (1972); M. Chanowitz and J. Ellis, Phys. Lett. 40B, 397 (1972); Phys. Rev. D 7, 2490 (1973).

[8] D. Capper and M. Duff, Nuovo Cimento 23A, 173 (1974); M. Duff, S. Deser, and C. Isham, Nucl. Phys. B111, 45 (1976); J. Dowker and R. Critchley, Phys. Rev. D 13, 3224 (1976); 16, 3390 (1977).

[9] L. Brown and J. Cassidy, Phys. Rev. D 15, 2810 (1977); H. Tsao, Phys. Lett. 68B, 79 (1977); T. Yoneya, Phys. Rev. D 17, 2567 (1978).

[10] S. Adler, Phys. Rev. 177, 2426 (1969); J. Bell and

R. Jackiw, Nuovo Cimento 60A, 47 (1969); W. Bardeen, Phys. Rev. 184, 1848 (1969); J. Wess and B. Zumino, Phys. Lett. 37B, 95 (1971).

[11]F. Berezin, *The Method in Second Quantization* (Academic, New York, 1966); L. Faddeev, in *Methods in Field Theory*, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976); Y. Ohnuki and T. Kashiwa, Prog. Theor. Phys. 60, 548 (1978) and references therein.

[12]One may normalize the basis vectors inside a box with a periodic boundary condition, for example. The WT identity, which is based on the *local* symmetry transformation, is not sensitive to the precise boundary condition at space-time infinity. The global limit of the transformation, however, requires a careful treatment.

[13]The tensor $T_{\mu\nu}$ in Ref. 2 corresponds to $\tilde{T}_{\mu\nu}$ in this paper, and $T_{\mu\nu}$ in this paper corresponds to the canonical energy-momentum tensor. We define the energy-momentum tensor as a source current for $h_a^\mu(x)$ which differs in sign from the source current for $h_\mu^{\ a}(x)$.

[14]R. Utiyama, Phys. Rev. 101, 1597 (1956).

[15]B. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965); Phys. Rep. 19C, 295 (1975).

[16]C. Becchi, A. Rouet, and R. Stora, Ann. Phys. (N. Y.) 98, 28 (1976). It can be confirmed that the use of weight-$\frac{1}{2}$ fields does not spoil the BRS invariance of the path-integral measure.

[17]In the present scheme without any regularization specified, the ordinary scaling $\Phi(x) \to \sqrt{Z}\,\Phi_r(x)$ of integration variables is not allowed because of the Jacobian. The convenient renormalization scheme is then the perturbative expansion in terms of *bare* quantities instead of the expansion in renormalized quantities.

[18]In obtaining global identities from the local WT identity (4.19), we assume that boundary terms arising from the space-time infinity can be safely neglected. To remove the possible disconnected constant term in $\tilde{T}_{\mu\nu}$, which does not contribute to (4.19), we specify the connected component of $\tilde{T}_\mu^{\ \mu}(x)$ by $\tilde{T}_\mu^{\ \mu}(x)_c$; the same notation applies to other composite operators in the present paper. Incidentally, the kinematically conserved quantity such as $(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)O(x)$, even if it existed in $\tilde{T}_{\mu\nu}$, does not contribute to (4.10); finiteness of $\tilde{T}_{\mu\nu}$ in general, therefore, is not concluded from the finite WT identity (4.19) alone.

[19]G. 't Hooft, Nucl. Phys. B61, 455 (1973); S. Weinberg, Phys. Rev. D 8, 3497 (1973). One can also use the conventional form of the renormalization-group equation or the Callan-Symanzik equation as well.

[20]Essentially this possibility of rearrangement (4.44) has been recently noted by H. Hata [Kyoto report No. KUNS 561, 1980 (unpublished)] although the bosonic Jacobian factors are unjustifiably neglected.

[21]S. Adler, J. Collins, and A. Duncan, Phys. Rev. D 15, 1712 (1977). The tensor $\theta_{\mu\nu}$ in this reference corresponds to a symmetrized version of $\theta_{\mu\nu}$ in this paper; our $\theta_{\mu\nu}$, however, contains the contributions from the gauge-fixing term, whereas $\theta_{\mu\nu}$ in this reference does not. In the proof of the finiteness of $\theta_{\mu\nu}$ in this reference, they implicitly assume the conservation property of their $\theta_{\mu\nu}$. As is explained in Sec. V of this paper, this is correct only in a specific regularization scheme such as the dimensional regulator.

[22]The chiral identity can also be written as a "trace identity" by noting $\partial_\mu j_5^\mu = g^{\mu\nu}\bar{\psi}(\gamma_\mu\gamma_5\vec{D}_\nu + \overleftarrow{D}_\mu\gamma_\nu\gamma_5)\psi$.

[23]S. Adler and W. Bardeen, Phys. Rev. 182, 1517 (1969). See also A. Zee, Phys. Rev. Lett. 29, 1198 (1972); J. Lowenstein and B. Schroer, Phys. Rev. D 7, 1929 (1973); K. Nishijima, Prog. Theor. Phys. 57, 1409 (1977).

[24]The proof of the nonrenormalization of the low-energy theorem in spinor electrodynamics requires a careful analysis of the photon-photon scattering amplitude. See Ref. 23 above.

[25]In the present context, the regularization not only regulates Feynman diagrams but also justifies the naive unitary transformation to the interaction picture by eliminating the Jacobian factor; the physical contents of the Jacobian, however, reappear in a different sector of the theory, if the regularization does not distort the theory. It should be noted that the regularization itself is not the real cause of the anomaly. See K. Fujikawa, in *High-Energy Physics—1980*, proceedings of the XXth International Conference, Madison, Wisconsin, 1980, edited by L. Durand and L. G. Pondrom (AIP, New York, 1981).

[26]The treatment of the negative-metric bosonic fields in the Euclidean path integral is not well established.

[27]S. Adler and W. Bardeen, Phys. Rev. D 4, 3045 (1972); 6, 734(E) (1972).

[28]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189. (1972); C. Bollini and J. Giambiagi, Phys. Lett. 40B, 566 (1972); G. Cicuta and E. Montaldi, Lett. Nuovo Cimento 4, 329 (1972).

[29]See Eqs. (3.7) and (3.13) in Ref. 21 above.

[30]In dimensional regularization, the operator $\hat{\theta}(x)$ in (5.7) is effectively replaced by $\hat{\theta}(x) = [\hat{\theta}(x)] + [e_0^2/(24\pi^2)]F^{\mu\nu}F_{\mu\nu}$ with $[\hat{\theta}(x)]$ the regularized quantity, and the identity (5.6) is reduced to a trivial counting identity. The advantage of the Pauli-Villars regulator is that we can write the *bare* form of divergent identities such as (5.4) and (5.6).

[31]As for further discussions of the trace identity see, for example, N. Nielsen, Nucl. Phys. B120, 212 (1977); J. Collins, A. Duncan, and S. Jogleker, Phys. Rev. D 16, 438 (1980); L. Brown, Institute for Advanced Study report, 1980 (unpublished) and earlier references therein.