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Relativistic potential models as systems with constraints and their interpretation

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Recent proposals of classical relativistic two-particle systems with an interaction potential instead of a force field are studied. These are local in an evolution parameter and are formulated as generalized gauge theories within Dirac's formalism for constrained Hamiltonian systems simulating reparametrization invariance. It is shown that the solutions of the equations of motion in general are world sheets instead of world lines. Requiring world lines we derive a set of conditions generalizing the world-line conditions usually given in the literature. Only a limited set of gauge constraints is shown to satisfy these conditions, notably when the evolution parameter is chosen to be the time of the rest frame of the system. A property of this gauge is that the total momentum at laboratory time is in general not just the sum of the momenta of the particles; the missing part is normally viewed as a momentum carried by a force field.

I. INTRODUCTION

One of the outstanding problems in particle physics is the construction and understanding of relativistic particle bound states. In distinction to nonrelativistic theories it is generally thought to be necessary to introduce a field as force mediator. However, since the field in general has degrees of freedom of its own, any many-particle system can exchange energy and momentum with the field, and thus is expected to radiate at the classical level. One way to escape this conclusion is effectively to suppress the field degrees of freedom by imposing unconventional boundary conditions on the field such that it does not radiate. This leads to action-at-a-distance theories[1] which, however, are nonlocal in time and therefore almost impossible to quantize.[2] Stable bound states of particles can be expected to exist within a quantum-field-theory framework, in which case we have to rely on the Bethe-Salpeter equation; but this is difficult both to solve and to interpret physically.

Faced with this situation one may try to extend the idea of an interaction potential between particles from the nonrelativistic to the relativistic domain and then attack the bound-state problem. Now, however, one must contend with the well-known no-interaction theorem[3] which, under certain conditions, forbids the possibility of interactions in a relativistic many-particle system. It is worth recalling explicitly these conditions. They are as follows: (i) the theory is set up within the classical canonical formalism so that the effect of any element of the Poincaré group is represented by a canonical transformation; (ii) the set of three-dimensional position coordinates of all the particles at a common physical (or laboratory) time forms one half of a system of canonical variables in the phase space of the entire system; (iii) with respect to the elements of the Euclidean group these position variables behave as expected, with the geometrical and canonical rules of transformation coinciding; and (iv) the world lines of the particles in any given state of motion, when imagined drawn in space-time, have an objective reality. This last statement means that if one passes from the description of the system as given in one inertial frame to that in another frame by applying the appropriate canonical transformation, the world lines remain the same and only their numerical description changes in the manner dictated by the geometrical relationship between the two frames. This requirement is expressed through a set of so-called world-line conditions,[3,4] and these conditions together with the structure relations of the Poincaré group entailed by the canonical formalism combine to rule out any possibility of interaction.[3]

It is clear that the relativistic bound-state problem is part of the more basic problem of introducing any interactions at all between relativistic classical particles in an acceptable way. Recently, there have been several action-at-a-distance-type models which seem to solve this problem.[5-8] They are constructed to be local in a single "time" parameter and thus can be quantized.[8] In fact, their quantum counterparts have been considered for a long time.[9-14] These models are set up in the generalized Hamiltonian formalism devised by Dirac for the description of constrained dynamical systems.[15,16] Now Dirac's method has the property of bringing in several independent parameters each playing the role of a time variable[15] in contrast to the single-evolution parameter of a Lagrangian theory. Hence, the recently presented

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models appear to leave the questions of the existence and properties of particle world lines in a vague and unsatisfactory state. We will examine these models from a formal as well as a physical point of view by insisting that an acceptable classical relativistic particle theory must have the following feature: in any state of motion of the system, howsoever defined, there must be definite rules permitting the construction of definite world lines, one for each particle, and these must be objectively real. We shall show how in the constraint formalism based on a Hamiltonian approach one can develop equations that serve as world-line conditions, and how one may check in any specific case if they are obeyed. This will clarify the conceptual basis of the models under consideration. The results will be that these conditions impose severe restrictions on the presented models and a physical reason for this is that these models contain traces of a force field.

In Sec. II we remind ourselves of the description of a free relativistic particle. This section is inserted for pedagogical reasons as an introduction to Sec. III. In Sec. III we consider a class of models for two interacting particles previously given in the literature. These models are then thoroughly studied within Dirac's constraint formalism with special attention to the question of determining individual particle world lines. Section IV describes the relativistic harmonic oscillator as an illustrative example, and the final Sec. V contains a discussion and general remarks.

II. FREE RELATIVISTIC PARTICLE

The purpose of this section is to illustrate, by using the constraint formalism to describe the familiar and elementary free particle, how one may maintain complete freedom in the choice of an evolution parameter; and how, for each given choice, one can develop a corresponding world-line condition that guarantees the objective reality of world lines. Of course for this system we shall find that the world-line condition is always obeyed.

We choose as Lagrangian the expression

$$L(\tau)=\frac{1}{2}\left(\frac{\dot{x}^{\mu}\dot{x}_{\mu}}{V}+m^2V\right), \tag{2.1}$$

with m the mass, x^μ the space-time position vector, and V the so-called einbein variable. (We use a timelike metric.) The dot denotes differentiation with respect to an independent parameter τ which is, at this stage, invariant under Poincaré transformations. That is, under a Poincaré transformation (Λ, a), $x^\mu(\tau)$ and $V(\tau)$ transform as

$$x'^{\mu}(\tau)=\Lambda^{\mu}{}_{\nu}x^{\nu}(\tau)+a^{\mu}\,,\quad V'(\tau)=V(\tau)\,. \tag{2.2}$$

The above Lagrangian avoids an expression with square roots but maintains invariance of the equations of motion under arbitrary changes in the choice of τ a property implying the unobservability of τ.

The Euler-Lagrange equation corresponding to a variation of V is a Lagrangian constraint as it involves no accelerations. In the Hamiltonian formulation the vanishing of the canonical momentum conjugate to V is a primary constraint, which by the equations of motion induces a secondary constraint, the mass-shell condition on the four-momentum p_μ conjugate to x^μ:

$$K=p_\mu p^\mu-m^2\approx 0\,. \tag{2.3}$$

(The symbol $\approx$ denotes an equality that is not necessarily satisfied inside the Poisson brackets below.) The intrinsic reason for this constraint is the above-mentioned reparametrization invariance of the equations of motion.

On eliminating V and its conjugate momentum by means of a gauge choice on V, one has an eight-dimensional phase space Γ, say, with canonical coordinates x^μ, p_μ obeying the usual Poisson bracket (PB) relations

$$\{x^\mu, x^\nu\}=\{p_\mu, p_\nu\}=0\,,\quad \{x^\mu, p_\nu\}=\delta^\mu_\nu\,. \tag{2.4}$$

On Γ the Poincaré transformations (2.2) are realized as canonical transformations $R(\Lambda, a)$ generated by

$$\begin{aligned} J_{\mu\nu}&=x_\mu p_\nu-x_\nu p_\mu\,,\\ P_\mu&=p_\mu\,, \end{aligned} \tag{2.5}$$

which satisfy the Poincaré algebra via their PB's.

The region in Γ where K vanishes defines a hypersurface, Σ say, of seven dimensions. (We restrict the following discussion to the branch of Σ where $p^0\geq m$.) It is obvious that the canonical transformations $R(\Lambda, a)$ map Σ onto itself. On the other hand, the one-parameter family of canonical transformations generated by K also has the property of mapping Σ onto itself. Starting with any point (x,p) in Σ and applying this family of canonical transformations to it, we generate a one-dimensional line in Σ, the orbit of (x,p) under this family of transformations. Σ is evidently the union of these orbits, which must form a six-parameter family; we shall denote a general orbit by L. These lines L are obtained as solutions to the generalized Hamiltonian equation of motion

$$\frac{dx^\mu}{d\tau}\approx v\{x^\mu, K\}\,,\quad \frac{dp^\mu}{d\tau}\approx v\{p^\mu, K\}=0\,. \tag{2.6}$$

By allowing v to be chosen at will, we have a situation in which the line L passing through any point (x,p) in Σ is determined, but the value of τ assigned to each point of L can be chosen differently, even though we may assign $\tau=0$ to the point (x,p)

itself. Since K is invariant under the canonical transformations $R(\Lambda, a)$, such a transformation will map a line L onto another line L' exactly according to a Poincaré transformation in spacetime. The canonical and geometrical rules of transformation for x^μ coincide, and a world line (2.6) is objectively real.

The concept of a world-line condition arises if one now imposes a gauge constraint

$$\chi(x,p,\tau) \approx 0 \tag{2.7}$$

intended to assign a definite value of τ to each point on a line L. To achieve this it is evident that χ must be explicitly τ dependent. The imposition of this constraint together with Eq. (2.3) enables us to reduce the phase space to a six-dimensional one. The appropriate PB on this "physical" phase space is then given by the Dirac bracket[15,16] (DB) ($\{\chi, K\} \neq 0$ is required):

$$\{f,g\}^* \equiv \{f,g\} - \frac{1}{\{\chi,K\}}(\{f,K\}\{\chi,g\} - \{f,\chi\}\{K,g\}) \tag{2.8}$$

for any two phase-space functions f and g. Since both K and χ have vanishing DB with all functions, they may be set to zero inside the DB. Since, furthermore, K has vanishing PB with the ten quantities $J_{\mu\nu}$, P_μ, it is clear that these quantities give a realization of the Poincaré algebra through their DB's just as they did through the PB. One thus has another realization of the Poincaré group, by transformations $R^*(\Lambda, a)$, say, which are canonical with respect to the DB. It can be seen that, acting on a line L in Σ, $R(\Lambda, a)$ and $R^*(\Lambda, a)$ yield the same new line L'. However, a given point (x,p) on L is generally carried by $R(\Lambda, a)$ and $R^*(\Lambda, a)$ to different points on L'. Since χ is unchanged by a Dirac canonical transformation, it follows that $R^*(\Lambda, a)$ carries a point (x,p) on a line L, with a value of τ determined by (2.7), to that point on L' that is assigned the same value of τ. On making the gauge choice (2.7), we switch to $R^*(\Lambda, a)$ as representing the change of inertial frame $O \to O'$ determined by the element (Λ, a) of the Poincaré group.

In order to set up a world-line condition within the canonical formalism we must pick six independent functions of x, p, τ which may serve as canonical coordinates for the DB (2.8). Any function of x, p, τ can be rewritten, when both constraints (2.3) and (2.7) hold, as some function of the six new variables and τ; clearly the meaning of the phrase "explicit τ dependence" can change in this process. Denote partial derivatives with respect to τ in the new sense by $\partial'/\partial\tau$. Before imposing (2.7) the general equation of motion for $f(x,p,\tau)$ is

$$\frac{df}{d\tau} \approx \frac{\partial f}{\partial \tau} + v\{f,K\} \tag{2.9}$$

with v arbitrary. When (2.7) is imposed, v is determined by the condition $\dot\chi = 0$, i.e.,

$$v = -\frac{\partial\chi}{\partial\tau}\frac{1}{\{\chi,K\}} \tag{2.10}$$

since (2.7) must be maintained for all τ. The general theory assures us then[17] that a new Hamiltonian $\mathcal{H}$ can always be found such that in place of (2.9) we have

$$\frac{df}{d\tau} \approx \frac{\partial f}{\partial\tau} - \frac{\partial\chi}{\partial\tau}\frac{\{f,K\}}{\{\chi,K\}} \approx \frac{\partial' f}{\partial\tau} + \{f,\mathcal{H}\}^* . \tag{2.11}$$

This is all we need to formulate the world-line condition. Let O and O' be two inertial frames related by an infinitesimal element (Λ, a) of the Poincaré group, with Λ differing from unity by an infinitesimal antisymmetric tensor ω. Let the phase-space point $x(\tau)$, $p(\tau)$ on a line L lead to the world point with coordinates $x^\mu(\tau)$ in O. $R^*(\Lambda, a)$ applied to $x(\tau)$, $p(\tau)$ yields a point $x'(\tau)$, $p'(\tau)$ on a line L', which then leads to a world point with coordinates $x'^\mu(\tau)$ in O':

$$x'^\mu(\tau) = x^\mu(\tau) + \{G, x^\mu(\tau)\}^* ,$$
$$G = \tfrac{1}{2}\omega^{\lambda\rho}J_{\lambda\rho} - a^\lambda P_\lambda . \tag{2.12}$$

The descriptions in O and O' refer to the same objectively real world line if there is some point on the world line as plotted in O, say, $x^\mu(\tau+\delta\tau)$, whose (geometrical) Lorentz transform is $x'^\mu(\tau)$:

$$x'^\mu(\tau) \approx x^\mu(\tau+\delta\tau) + \omega^\mu_\nu x^\nu(\tau+\delta\tau) + a^\mu . \tag{2.13}$$

Thus the world-line condition is the requirement that there exists an expression for $\delta\tau$, linear in ω and a, such that we have

$$\{G, x^\mu\}^* \approx \omega^\mu_\nu x^\nu + a^\mu + \left(\frac{\partial' x^\mu}{\partial\tau} + \{x^\mu, \mathcal{H}\}^*\right)\delta\tau . \tag{2.14}$$

This is the general form of the world-line conditions written entirely in terms of the final "physical" DB's for the system. In terms of PB's by means of (2.8) and (2.11) it acquires a more practical form, namely,

$$\{x^\mu, K\}\{G,\chi\} \approx \{x^\mu, K\}\frac{\partial\chi}{\partial\tau}\delta\tau . \tag{2.15}$$

Hence,

$$\delta\tau = \{G,\chi\}\Big/\frac{\partial\chi}{\partial\tau} \tag{2.16}$$

ensures that the world-line condition is obeyed, for any choice of χ.

A more familiar form of the world-line condition is obtained for the specific gauge choice

$$\chi = x^0 - \tau . \tag{2.17}$$

Equation (2.16) yields here

$$\delta\tau = -a^0 - \omega^{0j}x_j . \tag{2.18}$$

As a consequence, that part of condition (2.14) which refers to the Euclidean group is trivially satisfied. The remaining part becomes

$$\{J_{0j}, x^k\}^* = -\delta^k_j\tau - x_j\{x^k, \mathcal{H}\}^* , \tag{2.19}$$

where the Hamiltonian is

$$\mathcal{H} = -p^0 = -(m^2 + \vec{p}^2)^{1/2} . \tag{2.20}$$

These are usually referred to in the literature as *the* world-line conditions, and are satisfied for the free particle. [Note that the DB's in (2.19) are actually the same as ordinary PB's among three-dimensional variables that occur in the traditional statement of the world-line conditions.]

Another interesting gauge choice is the proper-time gauge

$$\chi = p \cdot x - m\tau . \tag{2.21}$$

From Eq. (2.16) we have here

$$\delta\tau = -\frac{a \cdot p}{m} . \tag{2.22}$$

As is to be expected in this gauge the part of condition (2.14) that refers to the homogeneous Lorentz group is trivially satisfied and the nontrivial part refers to the effect of spatial translations:

$$\{x^k, p_j\}^* = \delta^k_j - \frac{p_j}{m}\{x^k, \mathcal{H}\}^* , \tag{2.23}$$

where the Hamiltonian now can be shown to be

$$\mathcal{H} = -m \ln(p^0/m) . \tag{2.24}$$

These conditions are satisfied by the free particle. Notice that x^k and p_k are not canonical coordinates within the DB in (2.23) in distinction to the case for the previous gauge choice in (2.19).

These two examples show that the nontrivial part of the world-line condition depends on the choice of gauge constraint. This is expected, since the Dirac canonical transformation $R^*(\Lambda, a)$, which represents a Poincaré group element in the final form of the theory, is built to preserve the value of τ, and τ has different space-time meanings in different gauges. All this is notwithstanding the fact that in building up a world line we assign spatial position $x^j(\tau)$ at physical (or laboratory) time $x^0(\tau)$, whatever the gauge.

III. THE TWO-PARTICLE SYSTEM

If one tries to construct an action describing two relativistic particles interacting via action-at-a-distance forces and requires invariance under independent reparametrizations of the two world lines, one is compelled to use two independent parameters, one for each world line; and the "equations of motion" are integrodifferential equations which are nonlocal in (physical) time. Consequently, no conventional Lagrangian and Hamiltonian formulation is possible for such a system (see Ref. 2, however). In a local description one can instead try to simulate the presence of the two reparametrization invariances. Had these independent invariances occurred in a single Lagrangian, one would have ended up with two independent first-class constraints. Thus, due to the difficulties with the Lagrangian approach, one can take up Dirac's suggestion[15] and use the constraint formalism within the Hamiltonian formulation as a starting point. This is the essence of the approach of Refs. 5–8.

Considering the two-particle system we define a sixteen-dimensional phase space Γ with canonical coordinates x^μ_α, $p_{\mu\alpha}$, $\alpha = 1, 2$, the only nonvanishing PB's being

$$\{x^\mu_\alpha, p_{\nu\beta}\} = \delta^\mu_\nu \delta_{\alpha\beta} . \tag{3.1}$$

x^μ_1 and x^μ_2 represent the space-time position vectors of particles 1 and 2. A canonical realization of the Poincaré group via canonical transformations $R(\Lambda, a)$ is given by the action

$$x'^\mu_\alpha = \Lambda^\mu{}_\nu x^\nu_\alpha + a^\mu , \quad p'^\mu_\alpha = \Lambda^\mu{}_\nu p^\nu_\alpha . \tag{3.2}$$

The generators for this realization are sums of the momenta and angular momenta of the two particles

$$J_{\mu\nu} = \sum_{\alpha=1}^{2} (x_{\mu\alpha}p_{\nu\alpha} - x_{\nu\alpha}p_{\mu\alpha}) , \quad P_\mu = \sum_{\alpha=1}^{2} p_{\mu\alpha} \tag{3.3}$$

and through their PB's they satisfy the Poincaré algebra.

We must now choose two independent first-class constraints. In the literature the following proposal is made[5] (see also Refs. 6 and 7):

$$K_1 = P^2 + q^2 - 2(m_1{}^2 + m_2{}^2) - 8mV\left(r^2 - \frac{(P\cdot r)^2}{P^2}\right) \approx 0 ,$$
$$K_2 = P \cdot q - m_1{}^2 + m_2{}^2 \approx 0 . \tag{3.4}$$

Here, relative and "center-of-mass" variables have been used according to

$$q^\mu = p^\mu_1 - p^\mu_2 , \quad r^\mu = \tfrac{1}{2}(x^\mu_1 - x^\mu_2) ,$$
$$X^\mu = \tfrac{1}{2}(x^\mu_1 + x^\mu_2) , \quad m = \frac{m_1 m_2}{m_1 + m_2} . \tag{3.5}$$

The motivation for this particular form is that V shall correspond to the usual classical potential in the nonrelativistic limit. K_1 and K_2 are manifestly invariant under the canonical transformations $R(\Lambda, a)$, so they have vanishing PB's with $J_{\mu\nu}$ and

P_μ. Moreover, they are first class since

$$\{K_1, K_2\} = 0 . \qquad (3.6)$$

The region in Γ where both K_1 and K_2 vanish is a fourteen-dimensional constraint hypersurface to be denoted by Σ. (We deal in the sequel with just one connected branch of Σ.) Clearly, the canonical transformations $R(\Lambda, a)$ map Σ onto itself. Moreover, the first-class property of K_1 and K_2 means that the canonical transformations generated by each of these functions also map Σ onto itself. Denote by ξ the collection of coordinates for Γ. If to some point ξ in Σ we apply all the canonical transformations generated by K_1 and K_2, we get a two-dimensional region in Σ; this is the orbit of ξ under this two-parameter Abelian set of canonical transformations. It is clear that Σ is the union of such two-dimensional orbits, so the orbits form a twelve-parameter family. We shall refer to orbits as sheets S. A way to build up these sheets is to solve the generalized Hamiltonian equations of motion (the derivatives on the left-hand sides are partial):

$$\begin{aligned} \frac{d\xi}{d\tau} &\approx v_1\{\xi, K_1\} + v_2\{\xi, K_2\}, \\ \frac{d\xi}{d\sigma} &\approx u_1\{\xi, K_1\} + u_2\{\xi, K_2\}, \end{aligned} \qquad (3.7)$$

where v_1, v_2 and u_1, u_2 may be chosen at will provided $v_1u_2 - v_2u_1 \neq 0$. The solutions of these equations are sheets and if one point of such a sheet lies in Σ the entire sheet lies in Σ. The canonical transformations $R(\Lambda, a)$ can now be seen not only to map Σ onto itself, but to carry each sheet S onto another sheet S'.

All the above steps are direct generalizations of what was done in Sec. II. The eight-dimensional phase space is replaced by a sixteen-dimensional one; the seven-dimensional constraint hypersurface is replaced by a fourteen-dimensional one, since the single constraint K is replaced by the first-class pair K_1, K_2. The lines L of the last section become the sheets S.

Owing to the form of K in (3.4) the total momentum P_μ is constant on each sheet S. The way P^2 enters the argument of V requires $P^2 \neq 0$ and from a physical point of view we demand $P^2 > 0$.

Consider now a general solution of the equations of motion (3.7). Since the sheet S is two dimensional it requires two independent variables to serve as coordinates over S; and it is intuitively clear from the forms of K_1 and K_2 that x_1^0 and x_2^0 could be used for this purpose. Each point of S then determines two world points in a space-time picture. However, except in the noninteracting situation, we expect that *each* of x_1^j and x_2^j will depend on *both* of x_1^0 and x_2^0 as we wander over S. Hence, it seems as if x_1^j and x_2^j in general will trace out world sheets instead of world lines. Therefore, some conditions *must* be adopted in order to get (a unique pair of) world lines if all points of S are used in the space-time reconstruction. Such conditions will then be part of a complete theory. We shall now analyze the situation explicitly.

A set of gauge conditions has the general form

$$\chi_\alpha(x_1, x_2, p_1, p_2, \tau, \sigma) \approx 0 , \quad \alpha = 1, 2 \qquad (3.8)$$

with explicit dependence on both the parameters τ and σ. For each value of τ and σ these two gauge constraints pick out one point on S, and as τ and σ vary this point traces out the sheet S completely. The conditions for (3.8) to be gauge constraints is

$$\mathrm{Det}\{\chi_\alpha, K_\beta\} \neq 0 , \qquad (3.9)$$

i.e., the χ's and K's must form a second-class set.[15,16]

Before imposing the gauge constraints (3.8) the equations of motion are determined by (3.7) which contains arbitrary functions v_α and u_α. Imposition of (3.8) now determines these functions by the conditions

$$\frac{d\chi_\alpha}{d\tau} = 0 , \quad \frac{d\chi_\alpha}{d\sigma} = 0 . \qquad (3.10)$$

We find

$$v_\alpha = -a_{\alpha\beta}\frac{\partial\chi_\beta}{\partial\tau}, \quad u_\alpha = -a_{\alpha\beta}\frac{\partial\chi_\beta}{\partial\sigma}, \qquad (3.11)$$

where $a_{\alpha\beta}$ is the inverse to the matrix appearing in (3.9), i.e.,

$$a_{\alpha\beta}\{\chi_\beta, K_\gamma\} = \delta_{\alpha\gamma} . \qquad (3.12)$$

The general theory now assures us that suitable Hamiltonians $\mathcal{H}_\tau$ and $\mathcal{H}_\sigma$ can always be found such that the equations of motion for any function f take the forms

$$\begin{aligned} \frac{df}{d\tau} &\approx \frac{\partial f}{\partial\tau} - \{f, K_\alpha\}a_{\alpha\beta}\frac{\partial\chi_\beta}{\partial\tau} \approx \frac{\partial' f}{\partial\tau} + \{f, \mathcal{H}_\tau\}^* , \\ \frac{df}{d\sigma} &\approx \frac{\partial f}{\partial\sigma} - \{f, K_\alpha\}a_{\alpha\beta}\frac{\partial\chi_\beta}{\partial\sigma} \approx \frac{\partial' f}{\partial\sigma} + \{f, \mathcal{H}_\sigma\}^* , \end{aligned} \qquad (3.13)$$

where $\{\, , \,\}^*$ is the DB relative to the χ's and K's given by

$$\begin{aligned} \{f, g\}^* = \{f, g\} &- a_{\alpha\beta}(\{f, K_\alpha\}\{\chi_\beta, g\} - \{f, \chi_\beta\}\{K_\alpha, g\}) \\ &- \{f, K_\alpha\}a_{\alpha\beta}\{\chi_\beta, \chi_{\beta'}\}a_{\alpha'\beta'}\{K_{\alpha'}, g\} . \end{aligned} \qquad (3.14)$$

These final DB equations of motion describe the evolution with respect to τ and σ determined by (3.8). The condition that these equations determine world lines and not world sheets is

$$\frac{dx_1^\mu}{d\tau} \propto \frac{dx_1^\mu}{d\sigma}, \qquad \frac{dx_2^\mu}{d\tau} \propto \frac{dx_2^\mu}{d\sigma}. \tag{3.15}$$

Using Eqs. (3.13) one finds that

$$\{x_\gamma^\mu, K_\alpha\}a_{\alpha\beta}\frac{\partial \chi_\beta}{\partial \tau} = c_\gamma\{x_\gamma^\mu, K_\alpha\}a_{\alpha\beta}\frac{\partial \chi_\beta}{\partial \sigma}, \quad \gamma = 1, 2 \tag{3.16}$$

which implies

$$\frac{\partial \chi_\alpha}{\partial \tau} \propto \frac{\partial \chi_\alpha}{\partial \sigma}, \tag{3.17}$$

unless the matrices $\{x_1^\mu, K_\alpha\}$, $\{x_2^\mu, K_\alpha\}$ considered as matrices in μ and α all have rank less than 2. Thus (3.17) is not implied when either of the following conditions is satisfied:

(1) $V = \text{const}$, no interaction,
(2) $P \cdot r = 0$, (3.18)
(3) other, stronger restrictions on phase space.

Any one of the last two conditions will make the particles lose their Newtonian degrees of freedom if it is not implied by the imposed gauge conditions. This feature is unsatisfactory and leaves only the two possibilities no interactions or Eq. (3.17).

Equation (3.17) implies, on the other hand, that the gauge constraint (3.8) must have the forms

$$\chi_\alpha(x_1, x_2, p_1, p_2, \tau') \approx 0, \quad \alpha = 1, 2 \tag{3.19}$$

where τ' is a function of τ and σ. Calling τ' just τ from now on, the equations of motion will only be with respect to τ and explicitly follow from

$$\frac{df}{d\tau} \approx \frac{\partial f}{\partial \tau} - \{f, K_\alpha\}a_{\alpha\beta}\frac{\partial \chi_\beta}{\partial \tau} \approx \frac{\partial' f}{\partial \tau} + \{f, \mathcal{H}\}^*. \tag{3.20}$$

We have thus lost one gauge degree of freedom. A "state of motion" is only defined by a curve instead of a sheet.

Effectively, only one of the gauge constraints (3.19) has an explicit τ dependence. The other constraint determines then a curve C on the sheet S and the choice of such a condition can no longer be viewed as a gauge choice giving different descriptions of "the same theory" since in general it leads to different pairs of world lines. Rather, such a choice is a part of a complete theory.

In order to set up the conditions for the objective reality of the world lines we must switch from the realization of the Poincaré group given by the ordinary canonical transformations $R(\Lambda, a)$ to one in terms of Dirac canonical transformations $R^*(\Lambda, a)$. The generators $J_{\mu\nu}$, P_μ of $R(\Lambda, a)$ can again be used for this purpose since via their DB's they do provide a realization of the Poincaré algebra (because their PB's with K_α vanish). If $R(\Lambda, a)$ maps a sheet S onto the sheet S', so will $R^*(\Lambda, a)$; but on individual points of S the effects of $R(\Lambda, a)$ and $R^*(\Lambda, a)$ may not be the same. In particular, if C and C' are the curves on S and S', respectively, determined by the gauge conditions (3.19), $R^*(\Lambda, a)$ will map C onto C' (preserving the τ value), while $R(\Lambda, a)$ in general will not do so. Once the gauge choice has been made, we are only interested in the curve C on each sheet S and not in the rest of S, so a change of inertial frame $O \to O'$ is hereafter to be represented by $R^*(\Lambda, a)$.

The world-line conditions are now easy to set up. Let the point τ on the curve C on a sheet S lead to the pair of world points $x_1^\mu(\tau), x_2^\mu(\tau)$ in an inertial frame O. Let O' be related to O by an infinitesimal transformation (Λ, a). Then the expressions

$$x_\alpha^\mu(\tau) + \{G, x_\alpha^\mu(\tau)\}^*, \quad \alpha = 1, 2 \qquad G = \tfrac{1}{2}\omega^{\lambda\nu}J_{\lambda\nu} - a^\lambda P_\lambda \tag{3.21}$$

are the space-time coordinates, in O', of the two points on the world lines to which O' assigns the common parameter value τ. The world lines are objectively real if we can find quantities $\delta_1\tau, \delta_2\tau$ such that the above expressions are the (geometrical) Lorentz transforms of $x_1^\mu(\tau + \delta_1\tau)$ and $x_2^\mu(\tau + \delta_2\tau)$, respectively. [So $x_\alpha^\mu(\tau + \delta_\alpha\tau)$ are the space-time coordinates in O of the point on world line α with parameter value $\tau + \delta_\alpha\tau$: two world points to which O' assigns equal parameter values may possess different parameter values in O.] Thus the world-line condition is the requirement that there exist expressions $\delta_1\tau$, $\delta_2\tau$ linear in $\omega^{\lambda\nu}$ and a^λ such that

$$x_\alpha^\mu(\tau) + \{G, x_\alpha^\mu(\tau)\}^* = x_\alpha^\mu(\tau + \delta_\alpha\tau) + \omega_\nu^\mu x_\alpha^\nu(\tau + \delta_\alpha\tau) + a^\mu. \tag{3.22}$$

Retaining only first-order terms and using (3.20) we get

$$\{G, x_\alpha^\mu\}^* = \omega_\nu^\mu x_\alpha^\nu + a^\mu + \left(\frac{\partial' x_\alpha^\mu}{\partial \tau} + \{x_\alpha^\mu, \mathcal{H}\}^*\right)\delta_\alpha\tau, \quad \alpha = 1, 2 \tag{3.23}$$

which are the world-line conditions for a two-particle system in the present formalism, stated in terms of the final physical DB's. By means of Eqs. (3.14) and (3.20) and using the fact that the PB's $\{G, K_\beta\}$ vanish, these conditions become in terms of PB's

$$\{x_\alpha^\mu, K_\beta\}a_{\beta\gamma}\{G, \chi_\gamma\} \approx \{x_\alpha^\mu, K_\beta\}a_{\beta\gamma}\frac{\partial \chi_\gamma}{\partial \tau}\delta_\alpha\tau, \quad \alpha = 1, 2. \tag{3.24}$$

If none of the restrictions in (3.18) are satisfied, these conditions reduce to

$$\{G, \chi_\gamma\} \approx \frac{\partial \chi_\gamma}{\partial \tau} \delta_\alpha \tau, \quad \alpha = 1, 2 \tag{3.25}$$

which imply $\delta_1 \tau = \delta_2 \tau = \delta\tau$ and *the gauge condition which has no explicit τ dependence must be manifestly Poincaré invariant.*

The consequences for some natural gauge choices are easily worked out. Consider first the gauge choice when the physical times x_1^0, x_2^0 are both set equal to τ:

$$\chi_1 = r^0, \quad \chi_2 = X^0 - \tau. \tag{3.26}$$

Since χ_1 is not Poincaré invariant this choice is only consistent with the world-line conditions if one of the restrictions in (3.18) is satisfied and the only choice consistent with Newtonian degrees of freedom is to have no interaction between the particles. We have therefore recovered the result of the no-interaction theorem in the present model. Notice that the space components of the positions may be chosen as canonical coordinates. Likewise, for gauge constraints of the more general type

$$\chi_1 = \eta \cdot r, \quad \chi_2 = \eta \cdot X - \tau, \tag{3.27}$$

where η is a constant numerical timelike or lightlike vector, the world-line conditions can be satisfied only if the interaction vanishes since χ_1 is not Poincaré invariant.

However, when we choose to describe the particles in terms of the time in the rest frame of the system, i.e., when the gauge constraints are

$$\chi_1 = P \cdot r, \quad \chi_2 = P \cdot X - M\tau, \quad M = (P^2)^{1/2}, \tag{3.28}$$

the world-line conditions can be satisfied for any "potential" V. Here $\chi_1 \approx 0$ fixes a curve C on each sheet S in a Poincaré invariant way [cf. (3.18)], and the space components of the positions are no longer canonical coordinates and hence the no-interaction theorem does not apply.

In conclusion, the world-line conditions are not always violated in the presence of interaction, but the situation depends on the gauge chosen and this gauge choice is then part of the specific theory (cf. the statements in Ref. 18).

IV. THE RELATIVISTIC OSCILLATOR MODEL

In this section we shall explicitly solve a simple model to illustrate the results of the previous section. This is the relativistic oscillator model with equal masses and is characterized by the pair of constraints

$$K_1 = P^2 + q^2 - 4m^2 + \beta^2\left(r^2 - \frac{(P\cdot r)^2}{P^2}\right), \qquad K_2 = P \cdot q. \tag{4.1}$$

Denote the set of four-vectors r, X, q, P by ξ. A sheet S in Σ, the hypersurface for which K_1 and K_2 vanish, is obtained when the canonical transformations generated by K_1 and K_2 are applied to a phase-space point $\xi^{(0)}$ lying in Σ. We may assume that ξ depends on two parameters τ and σ such that

$$\frac{d\xi}{d\tau} \approx \{\xi, \tfrac{1}{2}K_1\}, \quad \frac{d\xi}{d\sigma} \approx \{\xi, K_2\}, \qquad \xi|_{\tau=\sigma=0} \equiv \xi^{(0)}. \tag{4.2}$$

This is just a particular choice of (3.7). The solution for the total momentum P is immediate: it is independent of both parameters. (P is the generator of translations and is gauge invariant.) It is then convenient to split each of the other vectors r, X, q into components parallel and perpendicular to P ($P^2 > 0$):

$$r_\parallel^\mu = \frac{P\cdot r}{P^2} P^\mu, \quad r_\perp^\mu = r^\mu - r_\parallel^\mu, \text{ etc.} \tag{4.3}$$

The solution to (4.2) may then be written as

$$\begin{aligned} r(\tau,\sigma) &= r_\parallel^{(0)} + P\sigma + r_\perp^{(0)}\cos\beta\tau + q_\perp^{(0)}\frac{\sin\beta\tau}{\beta}, \\ X(\tau,\sigma) &= X^{(0)} - \frac{P\cdot r^{(0)}}{P^2} q_\perp^{(0)} + P\tau \\ &\quad + \left(\sigma + \frac{P\cdot r^{(0)}}{P^2}\right)(q_\perp^{(0)}\cos\beta\tau - \beta r_\perp^{(0)}\sin\beta\tau), \\ q(\tau,\sigma) &= q_\perp^{(0)}\cos\beta\tau - \beta r_\perp^{(0)}\sin\beta\tau. \end{aligned} \tag{4.4}$$

One may check that K_1 and K_2 vanish at any point $\xi(\tau,\sigma)$, and that $J_{\mu\nu}$ are independent of τ and σ. The set of points $\xi(\tau,\sigma)$ in phase space gives the sheet S containing $\xi^{(0)}$. Now different gauge choices correspond to different parametrizations of the sheet S. Let $\xi'(\tau',\sigma')$ be another parametrization of S. ξ' is then a solution to (3.7) and $\xi'(\tau',\sigma') = \xi[f(\tau',\sigma'), g(\tau',\sigma')]$ provided $v_1 = \frac{1}{2}\partial f/\partial\tau'$, $v_2 = \partial g/\partial\tau'$, $u_1 = \frac{1}{2}\partial f/\partial\sigma'$, and $u_2 = \partial g/\partial\sigma'$. Thus (4.4) is implicitly the solutions of the equations of motion for any gauge choice.

We shall consider the gauge constraints (3.28) which are compatible with the world-line conditions. We impose this gauge choice on the solution (4.4), i.e.,

$$P\cdot r(\tau,\sigma) = 0, \quad P\cdot X(\tau,\sigma) = tM, \tag{4.5}$$

which then determine τ and σ. We find

$$\tau = \frac{1}{P^2}(Mt - PX^{(0)}), \quad \sigma = -\frac{P\cdot r^{(0)}}{P^2}. \tag{4.6}$$

Setting this back into (4.4) we obtain the solution of the equations of motion in the gauge (4.5):

$$r(t) = r_\perp^{(0)} \cos\beta\left(\frac{t}{M} - \frac{P\cdot X^{(0)}}{M^2}\right) + \frac{1}{\beta} q_\perp^{(0)} \sin\beta\left(\frac{t}{M} - \frac{P\cdot X^{(0)}}{M^2}\right),$$

$$X(t) = X_\perp^{(0)} - \frac{P\cdot r^{(0)}}{M^2} q_\perp^{(0)} + t\frac{P}{M}, \tag{4.7}$$

$$q(t) = q_\perp^{(0)} \cos\beta\left(\frac{t}{M} - \frac{P\cdot X^{(0)}}{M^2}\right) - \beta r_\perp^{(0)} \sin\beta\left(\frac{t}{M} - \frac{P\cdot X^{(0)}}{M^2}\right),$$

where t is the time of the rest frame of the system as implied by (3.28). In this form, the point $\xi^{(0)}$ may not lie on the curve $\xi(t)$. If we agree to choose $\xi^{(0)} = \xi(0)$, we have the additional conditions

$$P\cdot X^{(0)} \approx 0, \quad P\cdot r^{(0)} \approx 0, \tag{4.8}$$

so the curve $\xi(t)$ is now described in the simpler form

$$r(t) = r_\perp^{(0)} \cos\beta\frac{t}{M} + \frac{1}{\beta} q_\perp^{(0)} \sin\beta\frac{t}{M},$$

$$X(t) = X_\perp^{(0)} + \frac{t}{M} P, \tag{4.9}$$

$$q(t) = q_\perp^{(0)} \cos\beta\frac{t}{M} - \beta r_\perp^{(0)} \sin\beta\frac{t}{M}.$$

The world lines for particles 1 and 2 are then given by

$$x_{1,2}(t) = X_\perp^{(0)} + \frac{t}{M}P \pm \left[r_\perp^{(0)} \cos\beta\frac{t}{M} + \frac{1}{\beta} q_\perp^{(0)} \sin\beta\frac{t}{M}\right]. \tag{4.10}$$

This is a uniform motion in the direction of P superimposed on circular motion in the hyperplane perpendicular to P.

If one repeats the same procedure for the unsatisfactory gauge choice (3.26), one will obtain solutions which are different from (4.10).

The generators $J_{\mu\nu}, P_\mu$ of Poincaré transformations are "constants of motion." Each of $J_{\mu\nu}$ and P_μ is the sum of a contribution from particle number 1 at the point $x_1(t)$ on its world line and a contribution from particle 2 at the point $x_2(t)$ on its world line, and these points are not simultaneous in the frame in which the world lines are plotted. Still it is such sums that are conserved and act as generators of the physical transformations $R^*(\Lambda, a)$ in the present formalism. Having obtained an acceptable system of world lines we may of course examine them at points which are simultaneous in the physical sense. For a given state of motion, we can find parameter values t_1 and t_2 for which we have

$$x_1^0(t_1) = x_2^0(t_2) = t^0. \tag{4.11}$$

The important thing to observe is that if we sum up the individual particle momenta at simultaneous points on the world lines we get

$$\tilde{P}(t^0) \equiv p_1(t_1) + p_2(t_2) = P + \tfrac{1}{2}[q(t_1) - q(t_2)], \tag{4.12}$$

i.e., the conserved total four-momentum P is expressible in terms of quantities at a common physical time t^0 in this way:

$$P = \tilde{P}(t^0) + \tfrac{1}{2}[q(t_2) - q(t_1)]. \tag{4.13}$$

A similar expression obtains for the total conserved angular momentum $J_{\mu\nu}$. The last term in (4.13) represents an interaction momentum which we normally would interpret to be carried by a force field. Such expressions are typical of action-at-a-distance theories, as was pointed out by van Dam and Wigner.[19] It is this property that inhibits a satisfactory Hamiltonian formulation of relativistic many-body systems in terms of dynamical variables "at one time."

V. CONCLUDING REMARKS

In the present paper we have examined recent proposals of relativistic particle action-at-a-distance theories for two-particle systems that are local in an evolution parameter. We have analyzed their structure within Dirac's formalism for constrained systems not necessarily based on a Lagrangian. We have shown that the solutions of the equations of motion in general are world sheets instead of world lines. Taking the point of view that the theory must give rise to world lines, we have derived a set of conditions which generalize the world-line conditions usually given in the literature. In order to satisfy these conditions in the presence of interaction we had to restrict the class of solutions $x_\alpha(\tau, \sigma)$ by fixing the parameter σ in a Poincaré-invariant way. This we could do, e.g., by means of the constraint

$$P\cdot r \approx 0, \tag{5.1}$$

which by the way leads to a theory which has a simple Lagrangian, namely,

$$L(\tau) = \frac{1}{2}\sum_{\alpha=1}^{2}\left(\frac{\dot{x}_\alpha^2}{V_\alpha} + m_\alpha^2 V_\alpha\right) - m(V_1 + V_2)V(r^2), \tag{5.2}$$

where V_1 and V_2 are two einbein variables [cf. (2.1)]. [When the equations of motion of V_α are set back into (5.2) the Lagrangian acquires the form proposed in Ref. 8.] Since the form of the constraints K_α cannot be fundamentally different from (3.4) for a physically understandable model, we think our results are rather general. Of course, there could exist completely different ways to construct world lines by, e.g., averaging over one of the evolution parameters which would yield nonlocal theories.

In the theory (5.2) the gauge choice

$$P\cdot X\approx \tau M\,,\quad M=(P^2)^{1/2} \tag{5.3}$$

has a natural interpretation because the evolution parameter τ is then the time in the rest frame of the two-body system. We have tried to interpret this theory by expressing the conserved Poincaré generators in terms of dynamical variables at a common physical time; and in doing so we have found that the conserved total four-momentum contains a piece in addition to the sums of the individual particle momenta. Hence in a sense the idea of a force field has not been completely eliminated; its remnants appear if we analyze the particle theory in terms of its components at equal times.

It is interesting to compare the above interpretation with the properties of a quantum version of a two-body system of essentially the type we have considered in the present paper. In Refs. 11 and 12 the Green's function and a generalized scattering theory were constructed. The Green's function is nonlocal and this nonlocality is built up by coherentlike states of zero-mass particles. In Ref. 12 these states were considered to be built up by gluons—the force field of strong interactions according to quantum chromodynamics. This interpretation agrees well with the result of the present paper since this nonlocality may be identified with the remnants of the force field above. Furthermore, it is easily seen that the Green's function is indeed local in the rest frame $P_j=0$.

Note added: After the completion of this work we learned about a paper by Molotkov and Todorov[20] in which the frame dependence of world lines is discussed.

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