# General-relativistic effects in hydrogenic systems

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We examine in detail the behavior of bound systems containing a spin-1/2 fermion (e.g., atomic hydrogen or positronium) in an external gravitational field. Starting with the generally covariant Dirac equation, we derive the effective Hamiltonian for a fermion in a weak gravitational field correct through order $v^2/c^2$, where $v$ is the fermion velocity. The resulting expression is then used to obtain the gravitational Hamiltonian for both hydrogen and positronium including relativistic effects. It is shown that the form of the Hamiltonian for the bound system depends on the choice of center-of-mass and relative coordinates, and several choices of these coordinates are considered. An extensive discussion is given of the relativistic variables used to describe the bound system and their physical significance. The principal focus of this paper is a relativistic gravitational analog of the Stark effect which arises from a set of post-Newtonian terms in the bound-state Hamiltonian. These are shown to mix opposite-parity states in hydrogen, such as $S_{1/2}$ and $P_{1/2}$, and lead to a correlation between the local acceleration of gravity $\vec{g}$ and the photon polarization in electromagnetic decays. We discuss the possibility that a study of this polarization could be used to discriminate among different theories of gravity at the quantum level through its dependence on one of the parametrized post-Newtonian parameters. For a fermion-antifermion system such as positronium, the interaction Hamiltonian can admix states with opposite values of $P$ and $CP$, as we illustrate with several examples. Our results apply specifically to the case of a hydrogenic system supported in a gravitational field by nongravitational forces. The effects of these are not explicitly considered.

## I. INTRODUCTION

The object of this paper is to study the mixing of opposite-parity states which result when a relativistic quantum-mechanical system, such as hydrogen or positronium, is placed in an external gravitational field. This effect, which may be viewed as a relativistic gravitational analog of the Stark effect, is interesting for a number of reasons among which perhaps the most important is the suggestion[1,2] that it may lead to a test of general relativity at the quantum level. The possibility of using the relativistic gravitational Stark (RGS) effect for such a purpose is intriguing because at present our experimental knowledge of the effects of gravity on quantum systems is quite limited. The single most important result comes from the experiment of Colella, Overhauser, and Werner[3] (COW) in which the measured quantum-mechanical phase difference of two neutron beams induced by a gravitational field was shown to agree with the prediction of Newtonian gravity and the Schrödinger equation. However, since all known theories of quantum gravity reduce to the Newtonian result in the nonrelativistic limit their experiment, while important, does not lead to a discrimination among competing theories at the quantum level (e.g., Einstein vs Brans-Dicke theory). To effect such a discrimination it is necessary to look for *relativistic* gravitational effects which depend in some way on the spins and/or momenta of the particles being studied. This would require measuring relativistic effects in the COW experiment, or in other terrestrial experiments,[1] that are approximately 7–10 orders of magnitude below present-day capabilities. The significance of the RGS effect is that it may allow us to obtain the same information from nonterrestrial systems (such as white dwarfs) where the gravitational fields are many orders of magnitude larger than on Earth. We will discuss below exactly how this might come about.

It should be emphasized at the outset that we will restrict our attention exclusively to conventional gravitational interactions which are derivable from Lagrangians which separately conserve all of the discrete space-time symmetries, namely parity ($P$), charge conjugation ($C$), and time reversal ($T$). Apparent violations of parity selection rules then arise, as in the case of the Stark effect, when we ignore parity changes in the external field which compensate those in the quantum system being studied. Our starting point should thus be contrasted with that of other authors[4] who have considered the possibility that the basic gravitational Lagrangian itself violates some discrete space-time symmetry. To make this distinction clearer we consider as an example parity-violating gravitational interactions in a linearized theory where the effective interaction Lagrangian (density) can be written in the form

$$\mathcal{L}_I(x) = \xi T_{\mu\nu}(x) h_{\mu\nu}(x) \equiv \mathcal{L}_I^{(+)}(x) + \mathcal{L}_I^{(-)}(x) . \quad (1.1)$$

Here $h_{\mu\nu}(x)$ denotes the gravitational field, $T_{\mu\nu}(x)$ is an appropriate source, and $\xi$ is a coupling constant. $\mathcal{L}_I^{(\pm)}(x)$ then denote the pieces of $\mathcal{L}_I$ that are, respectively, even and odd under $P$. We can decompose $T_{\mu\nu}$ and $h_{\mu\nu}$ in an analogous way,

$$\begin{aligned} T_{\mu\nu}(x) &= T_{\mu\nu}^{(+)}(x) + T_{\mu\nu}^{(-)}(x) , \\ h_{\mu\nu}(x) &= h_{\mu\nu}^{(+)}(x) + h_{\mu\nu}^{(-)}(x) . \end{aligned} \quad (1.2)$$

Combining Eqs. (1.1) and (1.2) it follows that

$$\begin{aligned} \mathcal{L}_I^{(+)}(x) &= \xi[T_{\mu\nu}^{(+)}(x) h_{\mu\nu}^{(+)}(x) + T_{\mu\nu}^{(-)}(x) h_{\mu\nu}^{(-)}(x)] , \\ \mathcal{L}_I^{(-)}(x) &= \xi[T_{\mu\nu}^{(+)}(x) h_{\mu\nu}^{(-)}(x) + T_{\mu\nu}^{(-)}(x) h_{\mu\nu}^{(+)}(x)] . \end{aligned} \quad (1.3)$$

$\mathcal{L}_I^{(+)}$ is thus analogous to the $(VV+AA)$ term in the weak Lagrangian while $\mathcal{L}_I^{(-)}$ is similar to $(VA+AV)$. It is evident that $\mathcal{L}_I^{(-)}(x)$ can induce $P$-violating (and analogously $C$-, $CP$-, ... violating) transitions, and the consequences of such interactions are considered in Ref. 4. However $\mathcal{L}_I^{(+)}(x)$ can also induce transitions between opposite-parity states which arise through matrix elements of the form

$$\begin{aligned} \langle f | \mathcal{L}_I^{(+)}(x) | i \rangle = \xi [ & \langle f | T_{\mu\nu}^{(+)}(x) | i \rangle h_{\mu\nu}^{(+)}(x) \\ & + \langle f | T_{\mu\nu}^{(-)}(x) | i \rangle h_{\mu\nu}^{(-)}(x) ] , \end{aligned} \quad (1.4)$$

where $i$ and $f$ are initial and final states of some quantum system (such as the $2P_{1/2}$ and $2S_{1/2}$ states in hydrogen) and $h_{\mu\nu}^{(\pm)}(x)$ is an external gravitational field. Since $T_{\mu\nu}^{(-)}(x)$ is odd under $P$ it can connect states $i$ and $f$ which have opposite parity which is the case we wish to study. The analogy of (1.4) to the Stark effect then follows by writing, for example,

$$\langle 2P_{1/2} | e\vec{\mathcal{E}} \cdot \vec{r} | 2S_{1/2} \rangle = e \langle 2P_{1/2} | \vec{r} | 2S_{1/2} \rangle \cdot \vec{\mathcal{E}} , \quad (1.5)$$

where $\vec{\mathcal{E}}$ is an external electric field and $e$ is the electric charge.

In what follows we will focus almost exclusively on opposite-parity transitions in hydrogen and positronium, which in the latter case also lead to $CP$-odd admixtures. To obtain the transition operator (analogous to $e\vec{\mathcal{E}} \cdot \vec{r}$) for the RGS effect we begin in Sec. II by presenting two derivations of the effective Hamiltonian for a fermion in a static gravitational field. In Sec. III the interaction of a bound system of fermions with a gravitational field is discussed. The single-particle Hamiltonians for two fermions ($e$ and $p$ or $e^+$ and $e^-$) are combined with the Hamiltonian describing the interaction of their mutual electromagnetic field with gravity. The resulting expression is then separated into center-of-mass and relative coordinates. Several choices of coordinates are considered in the context of a detailed discussion of relativistic variables. It is shown that the final expression for the interaction Hamiltonian for the composite system depends on which variables are used. Consequently the coefficients of several terms in the RGS operator cannot be specified unambiguously at the present time. However, since the form of the RGS operator is the same irrespective of which coordinates are used, we can write this operator in an appropriately general way and consider its matrix elements in various bound systems. This is done in Secs. IV and V for hydrogen and positronium, respectively. In Sec. VI we present our conclusions, and some additional technical details are given in two Appendices.

It should be emphasized that the RGS effect arises in a system which is being supported against the gravitational field by nongravitational forces. As an example one might consider $He^+$ in an appropriate electromagnetic field. Clearly the effects of these fields must also be taken into account, as we mention briefly in Sec. IV, but a discussion of this problem more generally is beyond the scope of the present paper. For a freely falling atom in a uniform gravitational field the RGS effect presumably vanishes.

## II. EFFECTIVE POTENTIAL FOR A FERMION IN A GRAVITATIONAL FIELD

We present in this section two derivations of the effective potential V for a fermion in a weak gravitational field. Although different assumptions and approximations are made in each of these derivations the final expression for $V$ is the same [see Eq. (2.44) below]. This verifies that $V$ can be specified in an unambiguous way in the weak-field limit. For the sake of definiteness we consider the case of an electron of mass $m$ in the field of the Earth (mass $M_E$) whose spin we neglect. In the first derivation we begin directly with the Dirac equation[5] for the electron in general relativity. In the second derivation we view the Earth as a massive spin-0 (scalar) particle and derive $V$ from the covariant $S$ matrix for the elastic scattering of a spin-$\frac{1}{2}$ and a spin-0 particle via one-graviton exchange. Although each of these approaches is in principle straightforward, care must be taken in specifying the coordinate system in which each result is obtained: In the $S$-matrix approach the calculations are carried out in the center of momentum (CM) of $m$ and $M_E$ which is an inertial coordinate system. In the limit $m/M_E \ll 1$ this system coincides with the rest frame of the Earth, which is the laboratory frame. By contrast the derivation proceeding from the Fock equation describes the electron by a locally flat set of fields called tetrads which are at each space-time point $x$ essentially the square root of the metric tensor $g_{\mu\nu}(x)$. We will see that the results of these two derivations agree and yield for $V$ the expression given in Eq. (2.44) below.

To establish our notation we begin with a discussion of the Dirac equation in Minkowski space,

$$(\gamma^a \partial_a + \kappa)\psi(x) = 0 , \tag{2.1}$$

where $\kappa = mc/\hbar$, $\partial_a \equiv \partial/\partial x^a$, and where the Lorentz index $a$ runs over the values $a = 1,2,3,0$. The $\gamma$'s satisfy the anticommutation relations

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} 1 , \tag{2.2}$$

where 1 is the unit $4 \times 4$ matrix in the space of Dirac matrices, and $\eta^{ab}$ is the metric tensor in Minkowski space ($dx^0 \equiv cdt$),

$$ds^2 = \eta_{ab} dx^a dx^b ,$$

$$\eta_{ab} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix} . \tag{2.3}$$

Lorentz indices $(a, b, \ldots)$ are raised and lowered by $\eta_{ab}$ in the usual way,

$$dx_a = \eta_{ab} dx^b , \text{ etc.,} \tag{2.4}$$

and $\eta^{ab}$ is related to $\eta_{ab}$ via

$$\eta_{ab}\eta^{bc} = \delta^c_a , \tag{2.5}$$

where $\delta^c_a$ is the Kronecker symbol. When the explicit forms of the Dirac matrices $\gamma^a$ are needed we have used the Dirac-Pauli representation,

$$\gamma^k = -i\beta\alpha^k \quad (k = 1,2,3) ,$$

$$\gamma^0 = -i\beta\alpha^0 = -i\beta = -i\gamma^4 , \tag{2.6}$$

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\sigma^k$ are the usual Pauli matrices.

The Dirac equation in an arbitrary (world) coordinate system specified by the metric tensor $g_{\mu\nu}(x)$,

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu , \tag{2.7}$$

is given by

$$[\gamma^\mu(x)\nabla_\mu(x) + \kappa]\psi(x) = 0 . \tag{2.8}$$

Here $\gamma^\mu(x)$ are a set of Dirac matrices satisfying the anticommutation relations

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x) 1 , \tag{2.9}$$

and $\nabla_\mu(x)$ is the covariant derivative of the Dirac field which we discuss below. The form of Eq. (2.8) is dictated by the principle of general covariance which states that the appropriate generalization of a Minkowski-space equation such as (2.1) is obtained by replacing all Lorentz indices by world indices ($\gamma^a \to \gamma^\mu$ etc.) and all ordinary derivatives by covariant derivatives ($\partial_a \to \nabla_\mu$). To exhibit $\gamma^\mu(x)$ and $\nabla_\mu$ explicitly we introduce a set of tetrad (or vierbein) fields $e^a_\mu(x)$ at each space-time point $x$ defined by[5]

$$dx^a = e^a_\mu(x) dx^\mu . \tag{2.10}$$

The tetrad fields relate the world coordinate system, characterized by the index $\mu$, to a locally Minkowskian coordinate system erected at $x$, which is characterized by the index $a$. The fields $e^a_\mu(x)$ are related to $g_{\mu\nu}(x)$ as follows: Combining Eqs. (2.3), (2.7), and (2.10) we have

$$\begin{aligned} ds^2 = \eta_{ab} dx^a dx^b &= \eta_{ab}[e^a_\mu(x)dx^\mu][e^b_\nu(x)dx^\nu] \\ &= [\eta_{ab} e^a_\mu(x) e^b_\nu(x)] dx^\mu dx^\nu \\ &\equiv g_{\mu\nu}(x) dx^\mu dx^\nu . \end{aligned} \tag{2.11}$$

Hence,

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x) = e_{b\mu}(x) e^b_\nu(x) . \tag{2.12}$$

The tetrad fields can thus be viewed as the square root of the metric tensor $g_{\mu\nu}$ in the sense of a matrix equation. Inverting Eq. (2.12) we find

$$\eta_{ab} = g_{\mu\nu}(x) e^\mu_a(x) e^\nu_b(x) , \tag{2.13}$$

where $e^\mu_a(x)$ is the matrix inverse of $e^a_\mu(x)$:

$$e^a_\mu(x) e^\mu_b(x) = \delta^a_b . \tag{2.14}$$

For later purposes we will also need the relation

$$e^\mu_a(x) e_{b\mu}(x) = \eta_{ab} . \tag{2.15}$$

The generalized Dirac matrices $\gamma^\mu(x)$ are given in terms of the tetrad fields by

$$\gamma^\mu(x) = e^\mu_a(x)\gamma^a . \tag{2.16}$$

Using Eqs. (2.2) and (2.12) it follows that

$$\begin{aligned} \{\gamma^\mu(x), \gamma^\nu(x)\} &= e^\mu_a(x) e^\nu_b(x) \{\gamma^a, \gamma^b\} \\ &= 2\eta^{ab} e^\mu_a(x) e^\nu_b(x) 1 = 2g^{\mu\nu}(x) 1 \end{aligned} \tag{2.17}$$

which gives Eq. (2.9).

The covariant derivative $\nabla_\mu$ can also be expressed in terms of the tetrad fields. We define an operator $\Gamma_\mu(x)$ such that

$$\nabla_\mu \psi(x) = [\partial_\mu + \Gamma_\mu(x)]\psi(x) , \tag{2.18}$$

where $\partial_\mu = \partial/\partial x^\mu$. Under a local Lorentz transformation $\Lambda$ which takes $\psi(x)$ into $\psi'(x')$,

$$\psi(x) \to \psi'(x') \equiv D(\Lambda)\psi(x) , \tag{2.19}$$

$\Gamma_\mu(x)$ must transform as[6]

$$\Gamma_\mu(x) \to D(\Lambda)\Gamma_\mu(x)D^{-1}(\Lambda) - [\partial_\mu D(\Lambda)]D^{-1}(\Lambda). \tag{2.20}$$

For the case of infinitesimal rotations or boosts $D(\Lambda)$ is given by

$$D(\Lambda) = 1 + \frac{i}{4}\epsilon_{ab}\sigma^{ab}, \quad \sigma^{ab} = -\frac{i}{2}[\gamma^a, \gamma^b], \tag{2.21}$$

where $\epsilon_{ab}$ are the infinitesimal parameters. Combining Eqs. (2.19)–(2.21) we find after some algebra that $\Gamma_\mu(x)$ is given by

$$\Gamma_\mu(x) = \frac{i}{4}\sigma^{ab}e_a^\nu(x)e_{b\nu;\mu}(x), \tag{2.22}$$

where the covariant derivative $e_{b\nu;\mu}(x)$ is given in terms of the affine connection $\Gamma^\lambda_{\nu\mu}(x)$ by

$$e_{b\nu;\mu}(x) = \partial_\mu e_{b\nu}(x) - \Gamma^\lambda_{\nu\mu}e_{b\lambda}(x). \tag{2.23}$$

The complete Dirac equation can now be written in the form

$$\left\{\gamma^c e_c^\mu(x)\left[\frac{\partial}{\partial x^\mu} + \frac{i}{4}\sigma^{ab}e_a^\nu(x)e_{b\nu;\mu}(x)\right] + \kappa\right\}\psi(x) = 0. \tag{2.24}$$

It thus remains to specify the tetrads $e_c^\mu(x)$ for the case of interest.

Since we are concerned with the interaction of an electron in a static spherically symmetric gravitational field, the appropriate tetrads $e_c^\mu(x)$ are those corresponding to the Schwarzschild solution,[7]

*standard coordinates*:

$$ds^2 = \left(1 - \frac{2GM_E}{rc^2}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 - \left(1 - \frac{2GM_E}{rc^2}\right)c^2dt^2, \tag{2.25}$$

*isotropic coordinates*:

$$ds^2 = \left(1 + \frac{GM_E}{2\rho c^2}\right)^4(dx^2 + dy^2 + dz^2) - \left(\frac{1 - \frac{GM_E}{2\rho c^2}}{1 + \frac{GM_E}{2\rho c^2}}\right)^2 c^2dt^2,$$
$$\rho = (x^2 + y^2 + z^2)^{1/2}, \tag{2.26}$$
$$r = \rho\left(1 + \frac{GM_E}{2\rho c^2}\right)^2.$$

Here $G = 6.672 \times 10^{-8}$ cm$^3$ g$^{-1}$ sec$^{-2}$ is the Newtonian gravitational constant and $M_E$ is the mass of the Earth. For our purposes it is more convenient to use the expression for the Schwarzschild solution in terms of isotropic coordinates, because the tetrads corresponding to these coordinates can be read off by inspection. In the weak-field case,

$$\Phi \equiv \frac{GM_E}{\rho c^2} \ll 1, \tag{2.27}$$

we have[8]

$$g_{\mu\nu}(x) = \delta_{\mu\nu}(1 + 2\Phi) - 2\delta_{\mu 0}\delta_{\nu 0}, \tag{2.28a}$$

$$g^{\mu\nu}(x) = \delta_{\mu\nu}(1 - 2\Phi) - 2\delta_{\mu 0}\delta_{\nu 0}, \tag{2.28b}$$

$$e_a^\mu(x) = \delta_{\mu a}(1 - \Phi) + 2\delta_{\mu 0}\delta_{a0}\Phi, \tag{2.28c}$$

$$e_{b\nu}(x) = \delta_{b\nu}(1 + \Phi) - 2\delta_{b0}\delta_{\nu 0}. \tag{2.28d}$$

If we further define the vector $g_\mu$,

$$g_\mu(x) = c^2\frac{\partial\Phi}{\partial x^\mu} = (\vec{g}, 0), \tag{2.29}$$

then

$$\Gamma^\lambda_{\mu\nu}(x) = \frac{1}{c^2}[\eta_{\nu\lambda}g_\mu(x) + \eta_{\mu\lambda}g_\nu(x) - \delta_{\mu\nu}\eta_{\sigma\lambda}g_\sigma(x)], \tag{2.30a}$$

$$e^\nu_{b;\mu}(x) = \frac{1}{c^2}[\eta_{\mu\nu}g_b(x) - \delta_{\mu b}\eta_{\sigma\nu}g_\sigma(x)]. \tag{2.30b}$$

Using Eqs. (2.28)–(2.30) it follows that the matrix $\Gamma_\mu = (\vec{\Gamma}, \Gamma_0)$ in Eq. (2.22) is given by

$$\Gamma = -\Gamma^\dagger = \frac{-i}{2c^2}\vec{\Sigma} \times \vec{g}, \tag{2.31a}$$

$$\vec{\Gamma}_0 = \vec{\Gamma}_0^\dagger = \frac{-1}{2c^2}\vec{\alpha} \cdot \vec{g}, \tag{2.31b}$$

where $\vec{\Sigma}$ is given by

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \tag{2.32}$$

Combining Eqs. (2.24) and (2.28)–(2.32) the Dirac equation can now be written as

$$\left\{\gamma^j(1 - \Phi)\left[\partial_j - \frac{i}{2c^2}(\vec{\Sigma} \times \vec{g})_j\right] + \gamma^0(1 + \Phi)\left(\partial_0 - \frac{1}{2c^2}\vec{\alpha} \cdot \vec{g}\right) + \kappa\right\}\psi(x) = 0. \tag{2.33}$$

Multiplying Eq. (2.33) on the left by $\hbar c(1 - \Phi)\beta$, and dropping terms of order $\Phi^2$, $\Phi\vec{g}$, etc., we obtain finally

$$i\hbar\frac{\partial\psi(x)}{\partial t} = \left[-i\hbar c(1 - 2\Phi)\vec{\alpha} \cdot \vec{\partial} - \frac{i\hbar}{2c}\vec{\alpha} \cdot \vec{g} + \beta mc^2(1 - \Phi)\right]\psi(x) \equiv H\psi(x). \tag{2.34}$$

The Hamiltonian in Eq. (2.34) is Hermitian when the requisite spatial integrations are carried out using the correct measure, i.e.,

$$\langle H\rangle = \int d^3x(\hat{g})^{1/2}\phi^\dagger(x)H\psi(x) = \langle H^\dagger\rangle , \qquad \hat{g} = \det(g_{ij}) \cong 1 + 6\Phi . \tag{2.35}$$

However, it is more convenient to absorb $(\hat{g})^{1/2}$ into the wave functions $\psi$ and $\phi^\dagger$ in order that $H$ appear as a Hermitian Hamiltonian when integrated with respect to the Euclidean coordinates $\int d^3x$. This can be achieved[9] by introducing a new wave function $\tilde{\psi}(x)$,

$$\tilde{\psi}(x) = (1 + \tfrac{3}{2}\Phi)\psi(x) \equiv \theta\psi(x) , \tag{2.36}$$

and a corresponding Hamiltonian $\tilde{H} = \theta H \theta^{-1}$, in terms of which the Dirac equation in Eq. (2.34) becomes[10]

$$\tilde{H}\tilde{\psi}(x) = i\hbar \frac{\partial}{\partial t}\tilde{\psi}(x) , \tag{2.37a}$$

$$\begin{aligned}\tilde{H} &= -i\hbar c(1-2\Phi)\vec{\alpha}\cdot\vec{\partial} + \frac{i\hbar}{c}\vec{\alpha}\cdot\vec{g} + \beta mc^2(1-\Phi) \\ &= -i\hbar c\vec{\alpha}\cdot\vec{\partial} + i\hbar c(\Phi\vec{\alpha}\cdot\vec{\partial} + \vec{\alpha}\cdot\vec{\partial}\Phi) + \beta mc^2(1-\Phi) .\end{aligned} \tag{2.37b}$$

$\tilde{H}$ is now manifestly Hermitian in the usual sense. We will henceforth drop the tildes in Eqs. (2.37) and denote $\tilde{\psi}$ by $\psi$ and $\tilde{H}$ by $H$.[11]

The Hamiltonian of Eq. (2.37) couples the upper and lower components of a relativistic Dirac wave function through the "odd" operator $\vec{\alpha}$. For most applications however, we wish to use nonrelativistic wave functions and treat the gravitational interaction as an ordinary perturbation in nonrelativistic quantum mechanics. Moreover, the Hamiltonian of Eq. (2.37) cannot be used even with relativistic wave functions without running into consistency problems as we discuss below. To express this interaction as an effective nonrelativistic operator we carry out a Foldy-Wouthuysen expansion which decouples the upper and lower Dirac components. Following Bjorken and Drell[12] we write the Hamiltonian of Eq. (2.37) in the form $(\mu \equiv mc^2)$

$$H = \mathcal{O} + \mathcal{E} + \beta\mu , \tag{2.38}$$

where the "odd" and "even" operators $\mathcal{O}$ and $\mathcal{E}$ are given by

$$\mathcal{O} = c\vec{\alpha}\cdot\left[(1-2\Phi)\vec{p} + \frac{i\hbar}{c^2}\vec{g}\right] , \tag{2.39a}$$

$$\mathcal{E} = -\beta\mu\Phi . \tag{2.39b}$$

We note that

$$\{\beta, \mathcal{O}\} = [\beta, \mathcal{E}] = 0 , \tag{2.40}$$

just as in the electromagnetic case considered in Ref. 12. To carry out the Foldy-Wouthuysen expansion of $H$ (in powers of $1/\mu$) we rewrite Eqs. (2.37) in the form

$$H'\Psi' = i\hbar\frac{\partial}{\partial t}\Psi' , \qquad \Psi' = e^{iS}\Psi, \quad H' = e^{iS}He^{-iS} , \tag{2.41}$$

where we have introduced an operator $S$ which is to be chosen in such a way that $H'$ contains no odd operators to leading order in $1/\mu$, which is $O(1)$. The same procedure is then repeated twice more, $H \to H' \to H'' \to H'''$, which is as far as we need expand the original Hamiltonian for present purposes. To this order the transformed Hamiltonian $H'''$ is given by

$$H''' = \beta\mu + \mathcal{E} + \frac{1}{2\mu}\beta\mathcal{O}^2 - \frac{1}{8\mu^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] + O\left(\frac{1}{\mu^3}\right) , \tag{2.42}$$

and hence through $O(1/\mu^2)$ contains only even operators which do not connect upper and lower wave-function components. In deriving the above result we assume that we are considering a system for which the kinetic energy of the electron and its gravitational potential energy are small compared to $\mu$. In addition we neglect all terms which contain more than one power of the gravitational interaction. Making these approximations we find

$$[\mathcal{O},\mathcal{E}] \cong -\frac{i\hbar\mu}{c}\beta\vec{\alpha}\cdot\vec{g} + 2\mu c\Phi\beta\vec{\alpha}\cdot\vec{p} , \tag{2.43a}$$

$$-\frac{1}{8\mu^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] \cong \beta\left(-\frac{i\hbar}{2\mu}\vec{g}\cdot\vec{p} + \frac{c^2}{2\mu}\Phi\vec{p}^2 - \frac{\hbar}{4\mu}\vec{g}\cdot\vec{\sigma}\times\vec{p}\right), \tag{2.43b}$$

$$\frac{1}{2\mu}\beta\mathcal{O}^2 \cong \beta\left(\frac{c^2\vec{p}^2}{2\mu} + \frac{2i\hbar}{\mu}\vec{g}\cdot\vec{p} - \frac{2c^2}{\mu}\Phi\vec{p}^2 + \frac{\hbar}{\mu}\vec{g}\cdot\vec{\sigma}\times\vec{p}\right) . \tag{2.43c}$$

We have exhibited the separate contributions to $H'''$ in Eqs. (2.43) to demonstrate the interesting point that terms having the same structure arise in what appear to be two different orders in the Foldy-Wouthuysen expansion, namely $1/\mu$ and $1/\mu^2$, something which does not occur to lowest order for the electromagnetic interaction. Combining the various terms in Eqs. (2.43) we find

$$\begin{aligned}H''' &= \beta\left[\mu + \frac{c^2\vec{p}^2}{2\mu} - \mu\Phi + \frac{3}{2\mu}\left(i\hbar\vec{g}\cdot\vec{p} - c^2\Phi\vec{p}^2 + \frac{\hbar}{2}\vec{g}\cdot\vec{\sigma}\times\vec{p}\right)\right] \\ &\equiv \beta\left(\mu + \frac{c^2\vec{p}^2}{2\mu} + V\right) .\end{aligned} \tag{2.44}$$

The first two terms in Eq. (2.44) give the usual expression for the relativistic energy of the electron correct to order $v^2/c^2$, where $v$ is the electron's velocity. The remaining terms give the Newtonian contribution with momentum-dependent corrections which may be viewed as representing the effects of the relativistic increase of the electron's mass with velocity.

Having completed our discussion of the single-particle Foldy-Wouthuysen transformation we return to the consistency problem raised earlier. It might be thought that the correct way to treat gravitational effects in hydrogen is to simply evaluate matrix elements of the Hamiltonian in Eqs. (2.37) between four-component relativistic wave functions. It turns out that this is not the case for reasons having to do with the separation of the center of mass (c.m.) and relative motion of the two-body system. We note that the Newtonian term $-\beta mc^2\Phi$ in Eqs. (2.37) will ultimately determine the motion of the c.m. whereas the remaining gravitational terms will induce the opposite-parity transitions of interest. To isolate the latter terms we would like to drop the Newtonian contribution, but this leads to a problem: The term $-\beta mc^2\Phi = \mathcal{E}$ which would thereby be eliminated also contributes to the parity-mixing transition operator through the term proportional to $[\mathcal{O},[\mathcal{O},\mathcal{E}]]$. Thus to distinguish between these two contributions from $\mathcal{E}$ it is necessary to begin with the Foldy-Wouthuysen form of the Hamiltonian in Eq. (2.42) rather than with the original Hamiltonian in Eqs. (2.37).

Equation (2.44) gives the basic single-particle interaction which we will use in Sec. III to derive the expression for the RGS Hamiltonian appropriate to hydrogen or positronium. To verify that Eq. (2.44) is indeed correct we will rederive it below from the $S$ matrix for the one-graviton-exchange contribution to the scattering of a spin-$\frac{1}{2}$ and a spin-0 particle. Consider the process $m(\vec{p}) + M_E(\vec{q}) \to m(\vec{p}\,') + M_E(\vec{q}\,')$ shown in Fig. 1, where $m$ and $M_E$ denote the electron and scalar particle, respectively. We first derive the $S$ matrix in an arbitrary frame, and then specialize to the CM defined by $\vec{p}+\vec{q}=0$. Following Barker, Gupta, and

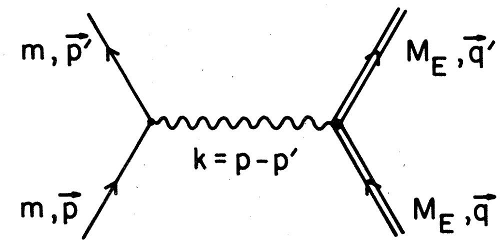


FIG. 1. One-graviton-exchange contribution to the scattering of a spin-$\frac{1}{2}$ particle with mass $m$ and a scalar particle with mass $M_E$. The wavy line denotes the graviton.

Haracz[13] (BGH) the couplings of the electron ($e$) and scalar ($s$) to the gravitational field $h_{\mu\nu}(x)$ are given in the linear approximation by (setting $\hbar = c = 1$)

$$\mathcal{H}^{(e)}(x) = \tfrac{1}{8}\hat{\kappa} h_{\mu\nu}(x)[\bar{\psi}(x)\gamma_\mu \partial_\nu \psi(x) + \bar{\psi}(x)\gamma_\nu \partial_\mu \psi(x) - (\partial_\nu \bar{\psi}(x))\gamma_\mu \psi(x) - (\partial_\mu \bar{\psi}(x))\gamma_\nu \psi(x)]\,,$$

$$\mathcal{H}^{(s)}(x) = \tfrac{1}{2}\hat{\kappa} h_{\mu\nu}(x)[\partial_\mu \phi(x)\partial_\nu \phi(x) - \tfrac{1}{2}\delta_{\mu\nu}(\partial_\rho \phi(x))^2 - \tfrac{1}{2}\delta_{\mu\nu} M_E{}^2 \phi^2(x)]\,, \qquad (2.45)$$

where $\psi(x)$ and $\phi(x)$ are the field operators for the electron and scalar, respectively. The coupling constant $\hat{\kappa}$ is related to the Newtonian gravitational constant $G$ by

$$G = \frac{\hat{\kappa}^2}{16\pi}\,. \qquad (2.46)$$

The expressions in square brackets in Eq. (2.45) are the energy-momentum tensors $T^{(e)}_{\mu\nu}(x)$ and $T^{(s)}_{\mu\nu}(x)$ for the electron and the scalar. The graviton propagator is given by

$$\langle 0 | T[h_{\mu\nu}(x) h_{\lambda\rho}(x')] | 0 \rangle = -i(\delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\mu\nu}\delta_{\lambda\rho}) D_F(x - x')\,, \qquad (2.47\text{a})$$

$$D_F(x - x') = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-x')} \frac{1}{k^2 - i\epsilon}\,. \qquad (2.47\text{b})$$

The $S$-matrix element for the transition $i \to f$ is defined by

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta^4(p + q - p' - q') \left(\frac{m^2}{p_0 p_0' V^2}\right)^{1/2} \left(\frac{1}{4 q_0 q_0' V^2}\right)^{1/2} \mathfrak{M}_{fi}\,, \qquad (2.48\text{a})$$

$$\begin{aligned} -i\mathfrak{M}_{fi} &= \left(-\frac{i\hat{\kappa}}{2}\right)\left(\frac{\hat{\kappa}}{8}\right) \bar{u}(p')[\gamma_\mu(p_\nu + p_\nu') + \gamma_\nu(p_\mu + p_\mu')] u(p) [q_\lambda q_\rho' + q_\rho q_\lambda' - \delta_{\lambda\rho}(q\cdot q' + M_E{}^2)] \\ &\quad \times \left(\frac{-i}{k^2}\right)(\delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\mu\nu}\delta_{\lambda\rho}) \\ &= -\frac{\hat{\kappa}^2}{4k^2}\, \bar{u}(p')[2imM_E{}^2 + \gamma\cdot q(p + p')\cdot(q + q')] u(p)\,. \end{aligned} \qquad (2.48\text{b})$$

Up to this point the calculation of $S_{fi}$ is completely covariant since no specification has been made of any frame. We now go to the CM by setting

$$\vec{p}+\vec{q}=\vec{p}\,'+\vec{q}\,'=0\,. \tag{2.49}$$

In the limit $m/M_E \ll 1$, which we will ultimately consider, the electron can be viewed as scattering from a fixed target in such a way that its energy is conserved but its three-momentum is not. Under these conditions we can write

$$k^2=(p'-p)^2\cong\vec{k}^2\,,$$

and (2.50)

$$\vec{p}\cdot\vec{k}\cong-\tfrac{1}{2}\vec{k}^2\,.$$

Using Eqs. (2.50) $\mathfrak{M}$ becomes

$$\mathfrak{M}=\frac{\hat{\kappa}^2}{2\vec{k}^2}\,\{\bar{u}(p')u(p)[m(M_E{}^2+2\vec{p}^2-\tfrac{1}{2}\vec{k}^2+2p_0q_0)] -\bar{u}(p')\gamma_4u(p)[(p_0+q_0)(2\vec{p}^2-\tfrac{1}{2}\vec{k}^2+2p_0q_0)]\}\,. \tag{2.51}$$

To convert $\mathfrak{M}$ into an effective gravitational potential for the electron, we reexpress the Dirac spinors $u(p)$, $\bar{u}(p')$ in terms of Pauli spinors $\chi$, $\chi^\dagger$ by writing

$$\bar{u}(p')u(p)=N\chi_f^\dagger[\alpha\beta-\tfrac{1}{4}(\vec{s}^2-\vec{k}^2) -i\vec{\sigma}\cdot\vec{k}\times\vec{p}\,]\chi_i\,, \tag{2.52a}$$

$$\bar{u}(p')\gamma_4u(p)=N\chi_f^\dagger[\alpha\beta+\tfrac{1}{4}(\vec{s}^2-\vec{k}^2) +i\vec{\sigma}\cdot\vec{k}\times\vec{p}\,]\chi_i\,, \tag{2.52b}$$

$$N=(4m^2\alpha\beta)^{-1/2}\,, \tag{2.52c}$$

$$\alpha=(p_0+m)\,,\quad \beta=(p_0'+m)\,,\qquad \vec{s}=\vec{p}+\vec{p}\,'\,,\quad \vec{k}=\vec{p}\,'-\vec{p}\,. \tag{2.52d}$$

Combining Eqs. (2.51) and (2.52) we see that $\mathfrak{M}$ can be written in the form

$$\mathfrak{M}=\chi_f^\dagger[F_1(\vec{p},\vec{k})+i\vec{\sigma}\cdot\vec{k}\times\vec{p}F_2(\vec{p},\vec{k})]\chi_i\,, \tag{2.53}$$

where the leading contributions to $F_1$ and $F_2$ are given by

$$F_1\cong-4\pi GM_Em\left[\frac{1}{\vec{k}^2}+\frac{1}{M_Em}\left(4+\frac{3M_E}{2m}+\frac{3m}{2M_E}\right)\vec{p}\cdot\frac{1}{\vec{k}^2}\vec{p}\right] +4\pi G\left(1+\frac{3}{8}\frac{M_E}{m}\right)\,, \tag{2.54a}$$

$$F_2\cong-\frac{4\pi G}{\vec{k}^2}\left(1+\frac{3M_E}{4m}\right)\,. \tag{2.54b}$$

We have written the $\vec{p}$-dependent term in Eq. (2.54a) in such a way as to lead to a manifestly Hermitian coordinate-space potential. The momentum-space matrix element in Eq. (2.53) gives rise to an effective potential $V$ in coordinate space whose matrix elements between free-particle wave functions reproduce $\mathfrak{M}$. Thus, taking the Fourier transform of the expression in brackets in Eqs. (2.53) and (2.54), we find

$$V=-\frac{GM_Em}{r}+\frac{1}{M_Emc^2}\left(4+\frac{3M_E}{2m}+\frac{3m}{2M_E}\right)\vec{p}\cdot\left(-\frac{GM_Em}{r}\right)\vec{p}+G\left(1+\frac{3M_E}{4m}\right)\frac{\hbar\vec{\sigma}\cdot\vec{r}\times\vec{p}}{c^2r^3}\,, \tag{2.55}$$

where we have reinstated the factors of $\hbar$ and $c$ in an obvious way, and have dropped the contribution proportional to $\delta^3(\vec{r})$ arising from the last term in $F_1$. Noting that $M_E/m\gg1$, the second term in Eq. (2.55) gives

$$-\tfrac{3}{2}\vec{p}\cdot\left(\frac{GM_E}{mc^2r}\right)\vec{p}=\frac{3}{2mc^2}(-c^2\Phi\vec{p}^2+i\hbar\vec{g}\cdot\vec{p})\,. \tag{2.56}$$

We can also rewrite the third term,

$$\frac{3}{4}\frac{GM_E}{mc^2r^3}\hbar\vec{\sigma}\cdot\vec{r}\times\vec{p}=\frac{3}{4}\frac{1}{mc^2}\frac{1}{r}\frac{d}{dr}(-c^2\Phi(r))\hbar\vec{\sigma}\cdot\vec{r}\times\vec{p}\,, \tag{2.57}$$

by using the identity

$$\frac{1}{r}\frac{d\Phi(r)}{dr}\vec{r}=\vec{\partial}\Phi(r)=\frac{1}{c^2}\vec{g}(r)\,, \tag{2.58}$$

which is valid for a spherically symmetric $\Phi(r)$.

Hence combining Eqs. (2.55)–(2.58) we find ($\mu=mc^2$)

$$V=-\mu\Phi+\frac{3}{2\mu}\left(-c^2\Phi\vec{p}^2+i\hbar\vec{g}\cdot\vec{p}+\frac{\hbar}{2}\vec{g}\cdot\vec{\sigma}\times\vec{p}\right)\,, \tag{2.59}$$

in complete agreement with Eq. (2.44). We note in passing that the spin term in Eqs. (2.44) and (2.59) can be written as

$$\frac{3\hbar}{4\mu}\vec{g}\cdot\vec{\sigma}\times\vec{p}=\frac{3}{2}\frac{GM_E}{mc^2r^3}\vec{S}\cdot\vec{L}\,, \tag{2.60}$$

where use has been made of Eq. (2.58), and $\vec{S}=\hbar\vec{\sigma}/2$. In the form of Eq. (2.60) this term is seen to be the usual spin-orbit interaction which at the macroscopic level would give rise to the geodetic precession[14] of an orbiting gyroscope. (Since we are neglecting the Earth's spin there is no analog of the Lense-Thirring precession which arises from a spin-spin interaction.)

The potential of Eq. (2.44) follows from the metric of Eq. (2.26) which is the form appropriate to the general theory of relativity. Although the experimental support for general relativity at the *macroscopic* level is quite compelling[15,16] there is, by contrast, little (if any) information which bears on its validity at the *quantum* level. Since this is one of the questions which we would like to explore by a study of the RGS effect we consider next the generalization of Eq. (2.44) for an arbitrary metric theory. It is convenient to describe metric theories of gravity by use of the parametrized post-Newtonian (PPN) formalism[17] in which the metric of Eq. (2.26) is replaced by[18]

$$\begin{aligned} -g_{00} &= 1 - 2\Phi + 2\beta'\Phi^2 \,, \\ g_{0i} &= 0 \,, \quad i = 1,2,3 \\ g_{ij} &= (1 + 2\gamma'\Phi)\delta_{ij} \equiv f\delta_{ij} \,. \end{aligned} \tag{2.61}$$

$\gamma'$ and $\beta'$ are parameters which distinguish among different metric theories of gravity with general relativity corresponding to $\gamma' = \beta' = 1$. To lowest order in $\Phi$, $\beta'$ makes no contribution but $\gamma'$ leads to a generalization of the expression for $\tilde{H}$ in Eq. (2.37b):

$$\begin{aligned} \tilde{H} \to & -i\hbar c\vec{\alpha}\cdot\vec{\partial} + \tfrac{1}{2}(1+\gamma')i\hbar c(\Phi\vec{\alpha}\cdot\vec{\partial} + \vec{\alpha}\cdot\vec{\partial}\Phi) \\ & + \beta mc^2(1-\Phi) \,. \end{aligned} \tag{2.62}$$

This leads in turn to a modification of the single-particle interaction in Eq. (2.44):

$$\begin{aligned} H''' \to \beta\bigg[ & \mu + \frac{c^2\vec{p}^{\,2}}{2\mu} - \mu\Phi \\ & + \frac{1}{\mu}(\gamma' + \tfrac{1}{2})\left( i\hbar\vec{g}\cdot\vec{p} - c^2\Phi\vec{p}^{\,2} + \frac{\hbar}{2}\vec{g}\cdot\vec{\sigma}\times\vec{p}\right)\bigg] \,. \end{aligned} \tag{2.63}$$

The derivation of Eqs. (2.62) and (2.63) is discussed in Appendix A.

## III. INTERACTION OF A COMPOSITE SYSTEM WITH A GRAVITATIONAL FIELD

We turn in this section to a derivation of the effective two-body transition operator for the RGS effect in hydrogen and for an analogous effect in positronium. We begin with Eq. (2.63) and write for hydrogen

$$\begin{aligned} H^{(0)}(e-p) = & H^{(\mathrm{em})}(e-p) + H(e) + H(p) \\ & + H^{(\mathrm{ge})}(e-p) \,. \end{aligned} \tag{3.1}$$

Here $H^{(\mathrm{em})}(e-p)$ denotes the electromagnetic interaction, and $H(e)$ and $H(p)$ are the appropriate single-particle operators from Eq. (2.63) expressed in terms of the electron and proton variables, respectively. (Since we are dealing with low-energy and low-momentum-transfer processes, we will ignore form-factor contributions and treat the proton as an elementary Dirac particle for our purposes.) $H^{(\mathrm{ge})}(e-p)$ represents the coupling of the gravitational field to the electromagnetic field of the combined $e-p$ system. From the discussion of Appendix B this is given by

$$\begin{aligned} H^{(\mathrm{ge})} = \frac{Ze^2}{r}\bigg[ & (\gamma'+1)\Phi_E + (\gamma'+1)\frac{\vec{g}\cdot\vec{R}}{c^2} \\ & + (\gamma'+1)\frac{(m_p - m_e)}{2Mc^2}\vec{g}\cdot\vec{r}\bigg] \,, \end{aligned} \tag{3.2}$$

where $r = |\vec{r}|$ is the electron-proton separation as measured in isotropic coordinates. For the definitions of other quantities see Appendix B. The contribution from $H^{(\mathrm{ge})}$ was not included in Refs. 1 and 2.

We now wish to combine $H(e)$ and $H(p)$ and express their sum in terms of center-of-mass and relative coordinates. In the expressions for $H(e)$ and $H(p)$ in Eq. (2.63) the contributions of interest to us are those proportional to $(\gamma' + \frac{1}{2})$. Since these terms are relativistic "corrections" to the Newtonian contribution $-\mu\Phi$, it is necessary to exercise some care in defining the c.m. coordinates for the composite system in order that various inconsistencies be avoided. In this section we begin by briefly reviewing the existing literature on this problem with a view towards applying the appropriate results to the case of a hydrogen atom in a gravitational field. Our discussion closely follows that given by Krajcik and Foldy[19] (KF), who also reference much of the earlier literature.

Before proceeding with the discussion of relativistic variables, it is important to distinguish the present treatment of gravitational effects in hydrogen from that given recently by several other authors.[20] In our work we explicitly view the hydrogen atom as a *two-body* system with the electron, proton, and their mutual electromagnetic field all interacting with an external gravitational field. By contrast, the authors of Ref. 20 view the electron as moving in a fixed Coulomb field and hence treat hydrogen as a fixed *one-body* system interacting with the gravitational field. To the extent that effects due to the recoil of the proton, or the motion of the c.m., can in fact be neglected such an approximation is presumably legitimate. However, these effects represent relativistic corrections to the lowest-order results which can in principle be of the same order as the (relativistic) post-Newtonian effects that we wish to study. This is particularly true for fermion-antifermion systems such as positronium where the mass of the system is comparable to that of any of its constituents. We note, for example, that Will's[20] analysis of gravitational red-shift measurements in hydrogen, as tests of nonmetric theories of gravity, depends on retaining the terms proportional to $\beta'$ and $\gamma'$ in $-g_{00}$ and $g_{ij}$, which correspond to his functions $T$ and $H$, respectively. Although it may well turn out that viewing hydrogen as a one-body system for these purposes is correct in the end, this point has heretofore not been properly addressed for gravitational effects. Furthermore, as the subsequent discussion will indicate, it is a nontrivial question which cannot be completely answered at the present time. In this section we study three sets of variables describing composite systems: (a) The nonrelativistic variables used in Refs. 1 and 2, (b) the relativistic c.m. variables discussed by KF (Ref. 19), and (c) a set of relativistic center-of-energy variables.[21] We will show explicitly that each set has certain advantages and disadvantages relative to the others, and that all three lead to different results in the present case. The most extensive discussion of the problem is due to KF (Ref. 19), whose treatment we consider first.

The realization that inconsistencies could arise if nonrelativistic variables were used to describe the relativistic interactions of a composite system emerged from the work of Barton and Dombey.[22] These authors showed that the Drell-Hearn-Gerasimov (DHG) sum rule for the absorption of photons by nucleons apparently fails when applied to a *composite* system such as $^3$H or $^3$He. It was subsequently shown by Brodsky and Primack[23] (BP) that this problem can be traced to the fact that the interaction of a loosely bound composite system with an external electromagnetic field is not simply given by the sum of the separate interactions of the constituents with the field, where each interaction is evaluated in the Foldy-Wouthuysen (FW) approximation.[24] BP, and also Osborn,[25] exhibited the terms needed in order to correct the naive FW result so as to achieve agreement with the DHG sum rule. More recent work[19,26] has focused on the view that the correct Hamiltonian for a composite system can be obtained to $O(v^2/c^2)$ from the sum of the individual interactions if one uses an appropriate set of relativistic variables in place of the usual nonrelativistic variables.

To understand how these relativistic variables are constructed[19,27] we begin by first considering the case of a *nonrelativistic* composite system with $N$ constituents. Each constituent $j$ ($j=1,\ldots,N$) is described by the variables $\vec{r}_j$, $\vec{p}_j$, and $\vec{s}_j$ which give its coordinate, momentum, and spin relative to some chosen coordinate system. Given the $3N$ vectors $(\vec{r}_j,\vec{p}_j,\vec{s}_j)$ one can define the c.m. $\vec{R}$ of the composite system, and its total momentum $\vec{P}$, by writing

$$\vec{R}=\frac{1}{M}\sum_j m_j\vec{r}_j\,, \tag{3.3a}$$

$$\vec{P}=\sum_j \vec{p}_j\,, \tag{3.3b}$$

where $M=\sum_j m_j$ gives the total mass of the composite system in terms of the constituent masses $m_j$. Finally, a new set of variables $\vec{\rho}_j$, $\vec{\pi}_j$, and $\vec{\sigma}_j$ are introduced via the relations

$$\vec{r}_j=\vec{\rho}_j+\vec{R}\,,\quad \vec{p}_j=\vec{\pi}_j+m_j\frac{\vec{P}}{M}\,,\quad \vec{s}_j=\vec{\sigma}_j\,. \tag{3.4}$$

Since $\vec{\rho}_j$ and $\vec{\pi}_j$ satisfy the constraints

$$\sum_j m_j\vec{\rho}_j=0\,,\quad \sum_j \vec{\pi}_j=0\,, \tag{3.5}$$

it follows that $(\vec{R},\vec{P},\vec{\rho}_j,\vec{\pi}_j,\vec{\sigma}_j)$ comprises an alternate set of $3N$ vectors in terms of which the composite system can be described. Evidently $\vec{\rho}_j$ specifies the location of constituent $j$ relative to the c.m. $\vec{R}$, with a similar interpretation for $\vec{\pi}_j$. If we define in addition the total internal angular momentum $\vec{S}$ of the composite system,

$$\vec{S}=\sum_j(\vec{\sigma}_j+\vec{\rho}_j\times\vec{\pi}_j)\,, \tag{3.6}$$

then the generators $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$ of the Galilean group assume the "single-particle" form ($U$ is the internal interaction)

$$\vec{\mathcal{P}}=\vec{P}\,, \tag{3.7a}$$

$$\vec{\mathcal{J}}=\vec{R}\times\vec{P}+\vec{S}\,, \tag{3.7b}$$

$$\vec{\mathcal{K}}=M\vec{R}-t\vec{P}\,, \tag{3.7c}$$

$$\mathcal{H}=h+\frac{\vec{P}^2}{2M}\,, \tag{3.7d}$$

$$h=Mc^2+\sum_j\frac{\vec{\pi}_j^{\,2}}{2m_j}+U\,. \tag{3.7e}$$

Here $h$ is a rotationally and translationally invariant function of the c.m. variables which corresponds to the internal Hamiltonian. The operators $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$ represent respectively the generators for space translations, rotations, boosts, and time translations. It follows[19] from Eqs. (3.7) that the c.m. variables have the property that they describe the composite system as if it were in fact a single particle with coordinate $\vec{R}$, momentum $\vec{P}$, and spin $\vec{S}$, independent of the details of its internal structure. Since this description corresponds to our expectation for the behavior of the c.m. motion of the composite system, the variables $\vec{R}$, $\vec{P}$, and $\vec{S}$ are the appropriate ones to use in the Galilean case.

We turn next to the *relativistic* case where we seek a set of relations analogous to Eqs. (3.3) and (3.4) which allow the generators of the (relativistic) Poincaré group to be expressed in a single-particle form analogous to Eqs. (3.7). The commutation relations of the Poincaré group, expressed in terms of $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$ are given in Eqs. (2.2) of KF. These can be expressed in the usual form

$$[M_{\alpha\beta},M_{\nu\lambda}]=\delta_{\alpha\lambda}M_{\beta\nu}+\delta_{\beta\nu}M_{\alpha\lambda}-\delta_{\alpha\nu}M_{\beta\lambda}-\delta_{\beta\lambda}M_{\alpha\nu}\ , \tag{3.8a}$$

$$[M_{\alpha\beta},p_\lambda]=\delta_{\beta\lambda}p_\alpha-\delta_{\alpha\lambda}p_\beta,\quad \alpha,\beta,\nu,\lambda=1,\ldots,4 \tag{3.8b}$$

if we identify

$$\mathcal{J}_a=-\frac{i}{2}\epsilon_{abc}M_{bc}\ , \tag{3.9a}$$

$$\mathcal{K}_a=-\frac{1}{c}M_{a4}\ , \tag{3.9b}$$

$$\mathcal{P}_a=p_a\ ,\ \mathcal{H}=-icp_4\ ,\quad a,b,c=1,2,3\ . \tag{3.9c}$$

The key observation of KF is that there exists a Hermitian operator $\phi=\phi(\vec{R},\vec{P},\vec{\rho}_j,\vec{\pi}_j,\vec{\sigma}_j)$ which has the property that, if Eq. (3.4) is replaced by

$$\vec{r}_j=e^{i\phi}(\vec{\rho}_j+\vec{R})e^{-i\phi}\ , \tag{3.10a}$$

$$\vec{p}_j=e^{i\phi}\left(\vec{\pi}_j+\frac{m_j}{M}\vec{P}\right)e^{-i\phi}\ , \tag{3.10b}$$

$$\vec{s}_j=e^{i\phi}\vec{\sigma}_je^{-i\phi}\ , \tag{3.10c}$$

then the Poincaré generators $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$ for an *isolated* system can be cast into a single-particle form in terms of the newly defined variables $\vec{r}_j$, $\vec{p}_j$, and $\vec{s}_j$. For example, the correct relativistic expression for $\vec{\mathcal{P}}$ is obtained by replacing Eqs. (3.7a) and (3.3b) by

$$\vec{\mathcal{P}}=\sum_j e^{i\phi}\left(\vec{\pi}_j+\frac{m_j}{M}\vec{P}\right)e^{-i\phi}\ . \tag{3.11}$$

As discussed by KF, the operator $\phi$ defines possible relationships between the sets of variables $\{\vec{r}_j,\vec{p}_j,\vec{s}_j\}$ and $\{\vec{R},\vec{P},\vec{\rho}_j,\vec{\pi}_j,\vec{\sigma}_j\}$. Thus $\vec{p}_j$ and $\vec{\pi}_j$ retain their interpretation as the momentum and internal momentum of constituent $j$, but their relationship is altered by $\phi$. KF discuss the construction of the operator $\phi$ which, when expanded in powers of $1/c$, allows Eq. (3.10a) to be written in the form

$$\vec{r}_j=\vec{\rho}_j+\vec{R}+\frac{1}{c^2}\vec{N}_j\ , \tag{3.12}$$

where the explicit form of $\vec{N}_j$ is given in Eq. (2.27) of KF. The expressions for $\vec{p}_j$ and $\vec{s}_j$ have a similar form. If we drop the arbitrary function $\Pi^{(1)}$ appearing in Eqs. (2.26) and (2.27) of KF, then the remaining contributions to $\vec{N}_j$ (and to the analogous terms in $\vec{p}_j$ and $\vec{s}_j$) contain an explicit factor of $1/M$. This means that in the limit $M\to\infty$ the relativistic and nonrelativistic variables coincide, which is intuitively reasonable: As noted by Brodsky and Primack[23] the corrections to the naive FW result arise from the fact that the bound-state wave function is different in the c.m. and laboratory frames. Clearly such effects would be expected to vanish as $M\to\infty$ since then the composite system could be taken to be permanently at rest. It follows that if one neglects effects of $O(m_e/m_p)$ then it would be sufficient to use nonrelativistic variables, particularly in our problem where all energies are small compared to $m_p$. This assumption was used in Refs. 1 and 2. We shall see, however, that contrary to intuition the relativistic contributions of KF do *not* vanish as $M\to\infty$ in the gravitational case. This is an important point to which we will return shortly.

For our purposes an important question is the extent to which the results of KF are applicable to the case of a composite system in an arbitrary external field, particularly a gravitational field. This question has been considered recently by Sebastian[28] but is not completely understood at present. It is not hard to see that an external gravitational field poses new problems: In addition to demanding that the relativistic variables lead to a single-particle form for $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$, we also require that the motion of the c.m. correspond to a uniform acceleration $|\vec{g}|$ *which is independent of the internal composition of the body*. The latter condition follows, of course, from the Eötvös-Dicke-Braginsky (EDB) experiments[29] which establish the composition independence of the acceleration of different bodies to a precision of at least $10^{-11}$. From the outset it has not been shown that these two conditions can be implemented in a mutually compatible way. For example, the composition independence of $|\vec{g}|$ for systems interacting via electromagnetic interactions has been demonstrated theoretically for a class of gravitational theories by Lightman and Lee[30] and by Haugan and Will.[31] However, the center-of-energy variables that these authors employ are not those of KF, and are not constrained by the requirement that the generators of the Lorentz group assume a single-particle form. By contrast the relativistic variables of KF have not been shown to lead to a composition independence of $|\vec{g}|$.

Returning to Eqs. (2.27) of KF, $\beta\vec{W}^{(1)}$ for the $e-p$ system ($1\equiv e$, $2\equiv p$) is given by[32]

$$\beta \vec{W}^{(1)} = \frac{e_1 e_2}{2} \frac{(\vec{\rho}_1 + \vec{\rho}_2)}{|\vec{\rho}_1 - \vec{\rho}_2|} . \tag{3.13}$$

Since $\beta \vec{W}^{(1)}$ is a contribution to the relativistic correction $\vec{N}_j/c^2$ in Eq. (3.12) it suffices to use the *nonrelativistic* relations in Eqs. (3.3)–(3.5) to express $\vec{W}^{(1)}$ in terms of the internal variables $\vec{r} \equiv \vec{\rho}_1 - \vec{\rho}_2$ and $\vec{k} = \vec{\pi}_1 = -\vec{\pi}_2$. From Eq. (3.5) we have

$$m_1 \vec{\rho}_1 + m_2 \vec{\rho}_2 = 0 \tag{3.14}$$

and hence, using Eq. (3.4),

$$\vec{\rho}_1 = \frac{m_2}{M} \vec{r} , \quad \vec{\rho}_2 = \frac{-m_1}{M} \vec{r} . \tag{3.15}$$

Equations (3.15), and the analogous results for $\vec{\pi}_{1,2}$,

$$\vec{\pi}_1 = -\vec{\pi}_2 = \frac{1}{M} (m_2 \vec{p}_1 - m_1 \vec{p}_2) , \tag{3.16}$$

allow $\vec{\rho}_{1,2}$ and $\vec{\pi}_{1,2}$ to be eliminated in favor of the internal coordinates $\vec{r}$ and $\vec{k}$. From Eqs. (3.13) and (3.15) we then have, with $e_1 = -e_2 = e$,

$$\beta \vec{W}^{(1)} = \frac{(m_1 - m_2)}{2M} \frac{e^2 \vec{r}}{r} \tag{3.17}$$

and also,

$$\beta \int_0^{\vec{P}} d\vec{P} \cdot \vec{W}^{(1)} = \frac{(m_1 - m_2)}{2M} \frac{e^2}{r} \vec{r} \cdot \vec{P} . \tag{3.18}$$

Equations (3.17) and (3.18), when combined with Eqs. (2.27) of KF, give the complete expression for $\vec{N}_j$ in Eq. (3.12). At this stage we again use the fact that $\vec{N}_j/c^2$ is a relativistic correction to argue that $\vec{N}_j$ may be dropped in any term in Eq. (3.1) which is itself a relativistic correction to the leading contribution. In practice this means that $\vec{N}_j$ contributes only to the kinetic energy and Newtonian gravitational potential energy terms in Eq. (3.1). For example, the term proportional to $\vec{W}^{(1)}$ modifies the Newtonian contribution in $H(e) + H(p)$ as follows:

$$-m_1 \vec{g} \cdot \vec{r}_1 - m_2 \vec{g} \cdot \vec{r}_2 \to M \vec{g} \cdot \vec{R} + \frac{e^2 (m_1 - m_2)}{2Mc^2} \vec{g} \cdot \vec{r} , \tag{3.19}$$

where the first term on the right-hand side of (3.19) is the usual nonrelativistic result. In all the remaining terms the nonrelativistic relations in Eq. (3.4) are sufficient. The complete expression for the Hamiltonian $H(e-p)$ describing a hydrogen atom in a gravitational field is then given by

$$H(e-p) = H^{(0)}(e-p) + \Delta H(e-p) , \tag{3.20}$$

where

$$\begin{aligned} H^{(0)}(e-p) = & -\frac{Ze^2}{|\vec{r}|} + Mc^2 + \frac{\vec{P}^2}{2M} + \frac{\vec{k}^2}{2\mu_R} - M(c^2 \Phi_E + \vec{g} \cdot \vec{R}) - (\gamma' + \tfrac{1}{2}) \Phi_E \left( \frac{\vec{P}^2}{M} + \frac{\vec{k}^2}{\mu_R} \right) \\ & - (\gamma' + \tfrac{1}{2}) \frac{1}{c^2} \left[ \vec{g} \cdot \vec{R} \frac{\vec{P}^2}{M} + \frac{1}{M} (\vec{g} \cdot \vec{r} \vec{P} \cdot \vec{k} + \vec{P} \cdot \vec{r} \vec{g} \cdot \vec{k}) + \frac{1}{\mu_R} \vec{g} \cdot \vec{R} \vec{k}^2 + \left( \frac{1}{m_e} - \frac{1}{m_p} \right) \vec{g} \cdot \vec{r} \vec{k}^2 \right] \\ & + (\gamma' + \tfrac{1}{2}) \frac{i\hbar}{c^2} \left[ \left( \frac{1}{m_e} - \frac{1}{m_p} \right) \vec{g} \cdot \vec{k} + \frac{2}{M} \vec{g} \cdot \vec{P} \right] + (\gamma' + \tfrac{1}{2}) \frac{1}{c^2} \left[ \vec{g} \cdot \left( \frac{\vec{s}_e}{m_e} - \frac{\vec{s}_p}{m_p} \right) \times \vec{k} + \vec{g} \cdot \frac{\vec{S}}{M} \times \vec{P} \right] \\ & + (\gamma' + 1) \frac{Ze^2}{|\vec{r}|} \left[ \Phi_E + \frac{\vec{g} \cdot \vec{R}}{c^2} + \frac{(m_p - m_e)}{2Mc^2} \vec{g} \cdot \vec{r} \right] , \end{aligned} \tag{3.21a}$$

$$\begin{aligned} \Delta H(e-p) = & \frac{1}{2c^2} \left( \frac{1}{m_e} - \frac{1}{m_p} \right) (\vec{g} \cdot \vec{r} \vec{k}^2 - i\hbar \vec{g} \cdot \vec{k}) - \frac{1}{2c^2} \vec{g} \cdot \left( \frac{\vec{s}_e}{m_e} - \frac{\vec{s}_p}{m_p} \right) \times \vec{k} \\ & + \frac{1}{2Mc^2} (-i\hbar \vec{g} \cdot \vec{P} + \vec{g} \cdot \vec{r} \vec{P} \cdot \vec{k} + \vec{P} \cdot \vec{r} \vec{g} \cdot \vec{k}) + \frac{Ze^2}{2Mc^2} (m_e - m_p) \frac{\vec{g} \cdot \vec{r}}{|\vec{r}|} , \end{aligned} \tag{3.21b}$$

$$M = m_e + m_p , \quad \mu_R = \frac{m_e m_p}{m_e + m_p} , \tag{3.21c}$$

$$\vec{S} = \vec{s}_e + \vec{s}_p + \vec{r} \times \vec{k} . \tag{3.21d}$$

In Eqs. (3.21) $\Phi_E$ and $\vec{R}$ are defined by the expansion

$$c^2 \Phi(r) = \frac{GM_E}{r} \simeq \frac{GM_E}{R_E} + \vec{g} \cdot \vec{R} = c^2 \Phi_E + \vec{g} \cdot \vec{R} , \tag{3.22}$$

and thus $\vec{R}$ is the location of the c.m. measured from the surface of the Earth. $\vec{P}$ and $\vec{k}$ denote the operators

$$\vec{P} = -i\hbar \partial / \partial \vec{R} , \quad \vec{k} = -i\hbar \partial / \partial \vec{r} \tag{3.23}$$

and $\vec{s}_{e,p}$ are the spins of the electron and proton. The first and last terms in (3.21a) give $H^{(em)}(e-p)$ and $H^{(ge)}$, respectively, while the remaining terms arise from $H(e)+H(p)$. Turning to Eq. (3.21b) we note that $\Delta H(e-p)$ does not vanish in the limit when $m_p$ (and hence $M$) becomes very large, in contrast to what is seen in the electromagnetic case. Hence even when the atom is sufficiently massive that its recoil could be neglected, there remains a contribution to the gravitational interaction from the relativistic correction. Since this appears to run counter to the intuitive expectations discussed earlier, it is not clear at present whether the relativistic coordinates of KF require additional modifications to describe gravitational interactions. We observe, however, that $\Delta H(e-p)$ does not depend on the PPN parameter $\gamma'$. This follows from the fact that, as noted above, only the Newtonian gravitational contribution and the kinetic energy are modified by the use of relativistic coordinates, and these contributions are independent of $\gamma'$. From this observation we conclude more generally that, irrespective of the form of the relativistic correction, $\Delta H(e-p)$ must be independent of $\gamma'$ and hence can never completely cancel $H^{(0)}(e-p)$. This means that the operator for the RGS effect will in general be nonzero except, perhaps, for some special choice of $\gamma'$. Returning to Eqs. (3.21) we can combine the expressions for $H^{(0)}(e-p)$ and $\Delta H(e-p)$ to give

$$\begin{aligned} H(e-p) &= H^{(0)}(e-p)+\Delta H(e-p) \\ &= -\frac{Ze^2}{|\vec{r}|}+Mc^2-M(c^2\Phi_E+\vec{g}\cdot\vec{R})+\left[1-(2\gamma'+1)\Phi_E-(2\gamma'+1)\frac{\vec{g}\cdot\vec{R}}{c^2}\right]\left(\frac{\vec{P}^2}{2M}+\frac{\vec{k}^2}{2\mu_R}\right) \\ &\quad+(\gamma'+\tfrac{1}{2})\frac{i\hbar}{Mc^2}\vec{g}\cdot\vec{P}+\frac{\gamma'}{c^2}\left(\frac{1}{m_e}-\frac{1}{m_p}\right)(-\vec{g}\cdot\vec{r}\,\vec{k}^2+i\hbar\vec{g}\cdot\vec{k})+\frac{\gamma'}{c^2}\vec{g}\cdot\left(\frac{\vec{S}_e}{m_e}-\frac{\vec{S}_p}{m_p}\right)\times\vec{k}+\frac{(\gamma'+\frac{1}{2})}{Mc^2}\vec{g}\cdot\vec{S}\times\vec{P} \\ &\quad-\frac{\gamma'}{Mc^2}(\vec{g}\cdot\vec{r}\,\vec{P}\cdot\vec{k}+\vec{P}\cdot\vec{r}\,\vec{g}\cdot\vec{k}-i\hbar\vec{g}\cdot\vec{P})+\frac{\gamma' Ze^2(m_p-m_e)}{2Mc^2}\frac{\vec{g}\cdot\vec{r}}{|\vec{r}|}+(\gamma'+1)\frac{Ze^2}{|\vec{r}|}\left(\Phi_E+\frac{\vec{g}\cdot\vec{R}}{c^2}\right). \end{aligned} \tag{3.24}$$

We have written Eq. (3.24) in such a way that each set of terms is separately Hermitian.

We turn next to the equations of motion for the coordinate, momentum, and spin variables in both the single-particle case and for the composite system. If we denote any of these operators by $\Omega$ then the Heisenberg equations of motion are

$$\frac{d\Omega}{dt}=\frac{1}{i\hbar}[\Omega,H]+\frac{\partial\Omega}{\partial t}, \tag{3.25}$$

where in our case $\partial\Omega/\partial t=0$. Using Eq. (2.63) we find for a single particle in the Earth's field

$$\frac{d\vec{r}}{dt}=\left[1-(2\gamma'+1)\left(\Phi_E+\frac{\vec{g}\cdot\vec{r}}{c^2}\right)\right]\frac{\vec{p}}{m}+\frac{(\gamma'+\frac{1}{2})}{mc^2}(i\hbar\vec{g}-\vec{s}\times\vec{g}), \tag{3.26a}$$

$$\frac{d\vec{p}}{dt}=m\vec{g}\left[1+(\gamma'+\tfrac{1}{2})\frac{\vec{p}^2}{m^2c^2}\right], \tag{3.26b}$$

$$\frac{d\vec{s}}{dt}=\frac{(\gamma'+\frac{1}{2})}{mc^2}\vec{s}\times(\vec{g}\times\vec{p}). \tag{3.26c}$$

Using $H(e-p)$ in Eq. (3.24) the corresponding equations for the motion of the two-particle system become

$$\frac{d\vec{R}}{dt}=\left[1-(2\gamma'+1)\left(\Phi_E+\frac{\vec{g}\cdot\vec{R}}{c^2}\right)\right]\frac{\vec{P}}{M}+(2\gamma'+\tfrac{1}{2})\frac{i\hbar}{Mc^2}\vec{g}-\frac{(\gamma'+\frac{1}{2})}{Mc^2}\vec{S}\times\vec{g}, \tag{3.27a}$$

$$\frac{d\vec{P}}{dt}=M\vec{g}\left[1+(\gamma'+\tfrac{1}{2})\frac{\vec{P}^2}{M^2c^2}+(\gamma'+\tfrac{1}{2})\frac{\vec{k}^2}{\mu_R Mc^2}\right]+(\gamma'+1)\frac{Ze^2}{rc^2}\vec{g}, \tag{3.27b}$$

$$\begin{aligned}\frac{d\vec{S}}{dt}&=\frac{(\gamma'+\frac{1}{2})}{Mc^2}\vec{S}\times(\vec{g}\times\vec{P})+\frac{\gamma'}{c^2}\vec{g}\times\left[\left(\frac{\vec{S}_e}{m_e}-\frac{\vec{S}_p}{m_p}\right)\times\vec{k}\right]\\&\quad+\frac{\gamma'}{c^2}\left(\frac{1}{m_e}-\frac{1}{m_p}\right)(\vec{r}\times\vec{g}\,\vec{k}^2+i\hbar\vec{g}\times\vec{k})\\&\quad+\frac{\gamma'}{Mc^2}(\vec{r}\times\vec{g}\,\vec{P}\cdot\vec{k}-\vec{r}\cdot\vec{g}\,\vec{P}\times\vec{k}\\&\quad+\vec{r}\times\vec{P}\,\vec{g}\cdot\vec{k}-\vec{r}\cdot\vec{P}\,\vec{g}\times\vec{k}),\end{aligned} \tag{3.27c}$$

$$\vec{S}=\vec{S}_e+\vec{S}_p+\vec{r}\times\vec{k}. \tag{3.27d}$$

We see from Eqs. (3.26) and (3.27) that the composite system behaves as a single particle only in the special case $\gamma'=0$, provided that we replace $m$ in Eq. (3.26) by the internal Hamiltonian $h$,

$$h=Mc^2+\frac{\vec{k}^2}{2\mu_R}-\frac{Ze^2}{|\vec{r}|}. \tag{3.28}$$

This can be understood by noting that $\gamma'=0$ corresponds to Einstein's 1911 theory in which the gravitational potential of a particle with energy $E$ would be given by

$$V(r)=\frac{-GM_E(E/c^2)}{r}, \tag{3.29}$$

in contrast to the more general result for $\gamma'\neq 0$,

$$V(r)=\frac{-GM_E(E/c^2)}{r}(1+\gamma'\beta^2). \tag{3.30}$$

Thus for $\gamma'=0$ a particle behaves gravitationally as if its mass were simply replaced by $E/c^2 \cong m+\vec{p}^2/2mc^2$, which is what Eqs. (3.26) indicate. For a composite system (ignoring the field energy) the energy would be the sum of the energies of the individual constituents and thus the gravitational energy for $\gamma'=0$ would necessarily assume a single-particle form. It is important to stress, however, that when the coupling of the gravitational field to the internal electromagnetic field is included, Eqs. (3.27) do not assume the single-particle form even when $\gamma'=0$. This is similar to what happens for the electromagnetic case where the coupling of the field to internal exchange currents, for example, prevents the equations of motion of a composite system from assuming the single-particle form.

We turn next to consider another set of relativistic variables which we call center-of-energy (CE) variables. Let $h_j$ be the single-particle Hamiltonian for constituent $j$. We then define the CE variables $\vec{R}$, $\vec{P}$ and the internal variables $\vec{r}$, $\vec{k}$ as follows:

$$\vec{R}=\tfrac{1}{2}\left(\frac{h_1\vec{r}_1+h_2\vec{r}_2}{h_1+h_2}+\text{H.c.}\right),\quad \vec{P}=\vec{p}_1+\vec{p}_2, \tag{3.31a}$$

$$\vec{r}=\vec{r}_1-\vec{r}_2,\quad \vec{k}=\tfrac{1}{2}\left(\frac{h_2\vec{p}_1-h_1\vec{p}_2}{h_1+h_2}+\text{H.c.}\right). \tag{3.31b}$$

The *classical* analogs of Eqs. (3.31), which evidently reduce to the usual expressions in the nonrelativistic limit, have been used by the authors of Refs. 30 and 31 to discuss the composition independence of $\vec{g}$. Here we wish to interpret those equations as *quantum-mechanical* relations, which is the reason why we use the average of each expression and its Hermitian conjugate (H.c.). Operators appearing in the denominator are to be understood through a Taylor series expansion.

To apply Eqs. (3.31) we begin with the expression for the single-particle Hamiltonian $h_i$, neglecting temporarily electromagnetic effects:

$$h_i=m_ic^2+\frac{\vec{p}_i^{\,2}}{2m_i}-m_i(c^2\Phi_E+\vec{g}\cdot\vec{r}_i)+\frac{(\gamma'+\frac{1}{2})}{m_ic^2}[i\hbar\vec{g}\cdot\vec{p}_i-(c^2\Phi_E+\vec{g}\cdot\vec{r}_i)\vec{p}_i^{\,2}-\vec{s}_i\cdot\vec{g}\times\vec{p}_i]. \tag{3.32}$$

To proceed we combine the single-particle expressions for the electron and proton [using either (2.63) or (3.32)] and replace the constituent variables by the CE variables using the inverses of Eqs. (3.31):

$$\vec{r}_1=\vec{R}+\tfrac{1}{2}\left(\frac{h_2\vec{r}}{h_1+h_2}+\text{H.c.}\right),\quad \vec{r}_2=\vec{R}-\tfrac{1}{2}\left(\frac{h_1\vec{r}}{h_1+h_2}+\text{H.c.}\right), \tag{3.33a}$$

$$\vec{p}_1=\vec{k}+\tfrac{1}{2}\left(\frac{h_1\vec{p}}{h_1+h_2}+\text{H.c.}\right),\quad \vec{p}_2=-\vec{k}+\tfrac{1}{2}\left(\frac{h_2\vec{p}}{h_1+h_2}+\text{H.c.}\right). \tag{3.33b}$$

We note that when Eqs. (3.33) are used, the constituent variables are replaced by expressions involving the $h_i$ which depend in turn on the constituent variables themselves through Eq. (3.32). Thus the CE variables must be introduced by an iterative procedure, each application of which removes the dependence on the constituent variables to one higher order in $1/c^2$. For present purposes a single iteration is sufficient and, as in the case of the KF relativistic variables, only the kinetic energy and Newtonian gravitational terms need relativistic corrections. We find

$$H(1)+H(2)=Mc^2-Mc^2\Phi_E+\tilde{N}+[1-(2\gamma'+1)\Phi_E]\tilde{K}-\frac{(\gamma'+\frac{1}{2})}{c^2}\left[\vec{g}\cdot\vec{R}\frac{\vec{P}^2}{M}+\frac{2}{M}\vec{g}\cdot\vec{r}\vec{P}\cdot\vec{k}+\frac{1}{\mu_R}\vec{g}\cdot\vec{R}\vec{k}^2+\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\vec{g}\cdot\vec{r}\vec{k}^2\right]$$
$$+(\gamma'+\tfrac{1}{2})\frac{i\hbar}{c^2}\left[\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\vec{g}\cdot\vec{k}+\frac{2}{M}\vec{g}\cdot\vec{P}\right]+\frac{(\gamma'+\frac{1}{2})}{c^2}\left[\vec{g}\cdot\left(\frac{\vec{s}_1}{m_1}-\frac{\vec{s}_2}{m_2}\right)\times\vec{k}+\vec{g}\cdot(\vec{s}_1+\vec{s}_2)\times\vec{P}\right], \tag{3.34a}$$

$$\tilde{N}=-\vec{g}\cdot(m_1\vec{r}_1+m_2\vec{r}_2), \tag{3.34b}$$

$$\tilde{K}=\frac{\vec{p}_1^{\,2}}{2m_1}+\frac{\vec{p}_2^{\,2}}{2m_2}. \tag{3.34c}$$

Since $\tilde{N}$ is already $O(G)$ we can neglect the gravitational terms in $h_i$ and hence, to the required order in $1/c^2$,

$$\tilde{N}=-M\vec{g}\cdot\vec{R}+\frac{1}{2c^2}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)(\vec{g}\cdot\vec{r}\vec{k}^2-i\hbar\vec{g}\cdot\vec{k})+\frac{1}{Mc^2}\left(\vec{g}\cdot\vec{r}\vec{P}\cdot\vec{k}-\frac{i\hbar}{2}\vec{g}\cdot\vec{P}\right). \tag{3.35}$$

By contrast, in $\tilde{K}$ we can neglect the kinetic energy contributions to the $h_i$, but we retain those due to Newtonian gravity. Using the relations

$$\frac{m_2h_1 - m_1h_2}{m_1m_2(h_1+h_2)} \cong \frac{-\vec{g}\cdot\vec{r}}{Mc^2}, \tag{3.36a}$$

$$\frac{h_1}{h_1+h_2} = \frac{m_1}{M}\left(1 - \frac{m_2}{M}\frac{\vec{g}\cdot\vec{r}}{c^2}\right), \tag{3.36b}$$

$$\frac{h_2}{h_1+h_2} = \frac{m_2}{M}\left(1 + \frac{m_1}{M}\frac{\vec{g}\cdot\vec{r}}{c^2}\right), \tag{3.36c}$$

$$[\vec{P},\vec{r}]=0, \tag{3.36d}$$

we find

$$\tilde{K} = \frac{\vec{k}^2}{2\mu_R} + \frac{\vec{P}^2}{2M} - \frac{1}{Mc^2}\vec{g}\cdot\vec{r}\vec{P}\cdot\vec{k} + \frac{i\hbar}{2Mc^2}\vec{g}\cdot\vec{P}. \tag{3.37}$$

We now return to Eqs. (3.34) and include the effects of electromagnetism which arise from two sources, $H^{(ge)}$ in Eq. (3.2), and the electromagnetic contributions to $h_1$ and $h_2$ in Eqs. (3.33). From Eq. (B16) we see that the electromagnetic contribution to either $h_1$ or $h_2$ is just $H^{(ge)}/2$ and, since this correction affects only the Newtonian contribution $\tilde{N}$, the corresponding correction $\delta\tilde{N}$ to $\tilde{N}$ is just

$$\delta\tilde{N} = -\frac{\vec{g}\cdot\vec{r}}{2Mc^2}[m_1\phi_2(\vec{x}_1) - m_2\phi_1(\vec{x}_2)]$$

$$= \frac{Ze^2}{2Mc^2}(m_1 - m_2)\frac{\vec{g}\cdot\vec{r}}{r}. \tag{3.38}$$

We note that $\delta\tilde{N}$ is the same expression that arises in Eq. (3.17) from the $\beta\vec{W}^{(1)}$ contribution to the KF variables. Collecting the previous results together and introducing as before

$$\vec{S} = \vec{s}_1 + \vec{s}_2 + \vec{r}\times\vec{k}, \tag{3.39}$$

we arrive at an expression for $H(1-2) = H(e-p)$ which is similar to that given in Eq. (3.24) for the KF variables, but yet not identical to it. In Table I we compare the coefficients of the four sets of terms which have different coefficients depending on whether one uses nonrelativistic (NR), Krajcik-Foldy (KF), or center-of-energy (CE) variables. The remaining terms in (3.24) are identical in all three sets of variables.

We conclude this section with a brief summary of the advantages and disadvantages of the three sets of center-of-mass variables that we have been considering. The NR variables have the obvious advantage of simplicity and are an intuitively reasonable choice when recoil effects (of order $m_e/m_p$) can be neglected. If this approximation is made *before* the variables are inserted into the two-body RGS Hamiltonian in (3.24), then the KF, CE, and NR results coincide.

TABLE I. Comparison of relativistic variables. NR denotes the usual nonrelativistic variables for a two-body system where the center-of-mass coordinate $\vec{R}$ and the relative coordinate $\vec{r}$ are given by $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1+m_2)$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$. CE denotes the relativistic center-of-energy variables obtained by replacing the masses by the corresponding relativistic energies in the NR variables as discussed in the text. KF denotes the relativistic variables of Krajcik and Foldy, Ref. 19. The entries in the table give the coefficients of the corresponding terms in Eq. (3.40) for each choice of variables.

| | Term | NR | CE | KF |
|---|---|---|---|---|
| $C_K$ | $\frac{(m_p - m_e)}{m_e m_p c^2}(-\vec{g}\cdot\vec{r}\vec{k}^2 + i\hbar\vec{g}\cdot\vec{k})$ | $\gamma'+\frac{1}{2}$ | $\gamma'$ | $\gamma'$ |
| $C_S$ | $\frac{1}{c^2}\vec{g}\cdot\left(\frac{\vec{s}_e}{m_e} - \frac{\vec{s}_p}{m_p}\right)\times\vec{k}$ | $\gamma'+\frac{1}{2}$ | $\gamma'+\frac{1}{2}$ | $\gamma'$ |
| $C_E$ | $\frac{Ze^2}{c^2}\frac{(m_p - m_e)}{M}\frac{\vec{g}\cdot\vec{r}}{r}$ | $\frac{\gamma'+1}{2}$ | $\frac{\gamma'}{2}$ | $\frac{\gamma'}{2}$ |
| $C_X$ | $\frac{-1}{Mc^2}(\vec{g}\cdot\vec{r}\vec{P}\cdot\vec{k} + \vec{P}\cdot\vec{r}\vec{g}\cdot\vec{k} - i\hbar\vec{g}\cdot\vec{P})$ | $\gamma'+\frac{1}{2}$ | $\gamma'+\frac{1}{2}$ | $\gamma'$ |

(For nongravitational interactions it makes no difference when the limit $m_p\to\infty$ is taken.) The KF variables have the advantage of being relativistic (in contrast to NR), and constructed so as to maintain the single-particle form for the Poincaré generators of the composite system. However, in the presence of an external gravitational field they no longer lead to a single-particle form. Additionally they have not yet been shown to lead to the composition independence of $|\vec{g}|$ for composite systems, in the framework of a rigorous quantum-mechanical calculation. It is also not clear what significance to attach to the fact that the KF variables do not lead to the naive nonrelativistic result when $m_p\to\infty$. The CE variables, which are also relativistic, have the advantage that they have been explicitly shown to lead to the composition independence of $|\vec{g}|$, at least in the context of a semiclassical calculation. They do not coincide with the KF variables and hence presumably do not lead to a one-particle form for the Poincaré generators.

As we see from Table I, the effect of the different choices of center-of-mass variables is simply to change the $\gamma'$-dependent coefficients of the four terms given in the table. For purposes of evaluating matrix elements of this operator in the next two sections we will replace these coefficients with four constants $C_K(\gamma')$, $C_S(\gamma')$, $C_E(\gamma')$, and $C_X(\gamma')$ so that

$$H(e-p) = -\frac{Ze^2}{|\vec{r}|} + Mc^2 - M(c^2\Phi_E + \vec{g}\cdot\vec{R}) + \left[1 - (2\gamma'+1)\Phi_E - (2\gamma'+1)\frac{\vec{g}\cdot\vec{R}}{c^2}\right]\left(\frac{\vec{P}^2}{2M} + \frac{\vec{k}^2}{2\mu_R}\right)$$
$$+(\gamma'+\tfrac{1}{2})\frac{i\hbar}{Mc^2}\vec{g}\cdot\vec{P} + \frac{C_K}{c^2}\left(\frac{1}{m_e} - \frac{1}{m_p}\right)(-\vec{g}\cdot\vec{r}\vec{k}^2 + i\hbar\vec{g}\cdot\vec{k})$$
$$+\frac{C_S}{c^2}\vec{g}\cdot\left(\frac{\vec{S}_e}{m_e} - \frac{\vec{S}_p}{m_p}\right)\times\vec{k} + \frac{(\gamma'+\frac{1}{2})}{Mc^2}\vec{g}\cdot\vec{S}\times\vec{P} - \frac{C_X}{Mc^2}(\vec{g}\cdot\vec{r}\vec{P}\cdot\vec{k} + \vec{P}\cdot\vec{r}\vec{g}\cdot\vec{k} - i\hbar\vec{g}\cdot\vec{P})$$
$$+\frac{C_E Ze^2(m_p - m_e)}{Mc^2}\frac{\vec{g}\cdot\vec{r}}{|\vec{r}|} + (\gamma'+1)\frac{Ze^2}{|\vec{r}|}\left(\Phi_E + \frac{\vec{g}\cdot\vec{R}}{c^2}\right). \quad (3.40)$$

For later application to fermion-antifermion systems such as positronium, we simply set $m_e = m_p = m$ which then leads to the expression given in Eq. (5.2). The normalization of the $C$'s has been chosen so that they are given by the entries of Table I for each choice of center-of-mass variables.

## IV. THE RELATIVISTIC GRAVITATIONAL STARK EFFECT IN HYDROGEN

We consider in this section the internal transitions induced by the gravitational Hamiltonian in Eq. (3.40). We define the internal transition operator as the $\vec{P}=0$ contribution in Eq. (3.40) which is

$$H(e-p)|_{\vec{P}\to 0} = -\frac{Ze^2}{r} + \frac{\vec{k}^2}{2\mu_R} + \frac{C_K}{c^2}\left(\frac{1}{m_e} - \frac{1}{m_p}\right)(-\vec{g}\cdot\vec{r}\vec{k}^2 + i\hbar\vec{g}\cdot\vec{k}) + \frac{C_S}{c^2}\vec{g}\cdot\left(\frac{\vec{S}_e}{m_e} - \frac{\vec{S}_p}{m_p}\right)\times\vec{k} + C_E Ze^2\frac{(m_p - m_e)}{Mc^2}\frac{\vec{g}\cdot\vec{r}}{r}. \quad (4.1)$$

Note that $\vec{R}$ is constant when $\vec{P}=0$ and hence terms such as $Mc^2(\Phi_E + \vec{g}\cdot\vec{R})$ are trivial additive constants and can also be dropped. The relativistic gravitational Stark (RGS) effect arises from the terms proportional to $C_K$ and $C_E$ which (like the usual Stark interaction) can mix states of opposite parity. The term proportional to $C_S$, which acts in some ways as a gravitational analog of the Zeeman interaction, can also mix states of opposite parity. For the sake of brevity we will lump the $C_K$, $C_E$, and $C_S$ terms together as the RGS effect. If we consider, for example, the metastable $2S_{1/2}$ state in hydrogen, the terms proportional to $C_i$ $(i=K,S,E)$ can admix into this state $nP_{1/2}$ components, where $n \geq 2$. When this state emits or absorbs electromagnetic radiation, the interference between the resulting $EL$ and $ML$ multipoles leads to a characteristic polarization of the radiation from which the $C_i$, and hence $\gamma'$, can in principle be inferred. In the remainder of this section we elaborate on the RGS effect in greater detail.

Consider to start with the term proportional to $C_K$. Since $m_p \gg m_e$ we can set $1/m_p \simeq 0$. Using Eq. (3.23) to write $\vec{k} = -i\hbar\partial/\partial\vec{r} \equiv -i\hbar\vec{\nabla}$ this term can be written in the form

$$\frac{C_K}{m_e c^2}(-\vec{g}\cdot\vec{r}\vec{k}^2 + i\hbar\vec{g}\cdot\vec{k}) = C_K\frac{g\hbar^2}{m_e c^2}\hat{g}\cdot(\vec{r}\nabla^2 + \vec{\nabla}). \quad (4.2)$$

If we multiply and divide the coefficient on the right-hand side of Eq. (4.2) by $e^2$, where $e$ is the electric charge, then we can rewrite it in the form

$$C_K\frac{g\hbar^2}{m_e c^2} = C_K\left(\frac{g\hbar}{c}\right)\left(\frac{e^2}{\hbar c}\right)\left(\frac{\hbar^2}{m_e e^2}\right) = C_K\eta\alpha a_0, \quad (4.3)$$

where $\alpha$ is the fine structure constant and $a_0$ is the Bohr radius. We have introduced the constant $\eta$,

$$\eta = \frac{g\hbar}{c}, \quad (4.4)$$

which sets the energy scale for the RGS effect, and which clearly expresses in a compact form the fact that such transitions represent relativistic quantum gravitational effects. Table II gives the values of $\eta$ for various systems of interest. The data for the white dwarfs, entries 3–5, are

TABLE II. Values of $\eta = g\hbar/c$ for various systems. The data for entries 3–5 are taken from Ref. 33.

| System | $\eta$ (eV) | $\eta/\eta_\oplus$ |
|---|---|---|
| 1. Earth | $2.2\times10^{-23}$ | 1 |
| 2. Sun | $6.0\times10^{-22}$ | 28 |
| 3. 40 Eridani $B$ $M=0.372M_\odot$ $R=0.0152R_\odot$ | $9.7\times10^{-19}$ | $4.5\times10^{4}$ |
| 4. Sirius $B$ | $1.1\times10^{-17}$ | $5.3\times10^{5}$ |
| 5. $WD$ 2359-43 | $4.9\times10^{-17}$ | $2.3\times10^{6}$ |
| 6. Neutron star $M=0.4M_\odot$ $\rho=1\times10^{15}$ g cm$^{-3}$ | $3.5\times10^{-12}$ | $1.6\times10^{11}$ |

taken from Ref. 33. We see from this table one of the advantages of the RGS effect from the point of view of testing relativistic theories of gravity at the quantum level: Since we can study the spectrum of hydrogen from distant sources, we can realize a large amplification in $\eta$, of order $10^6$ in white dwarfs and $\sim 10^{11}$ for the hydrogen believed to be accreted around some neutron stars. Combining Eqs. (4.2) and (4.3), and noting that matrix elements of $\vec{\nabla}$ in hydrogen are of order $1/a_0$, we see that the amplitude for admixing in opposite-parity components will be proportional to the factor $Z\alpha\eta$, where $Z$ is the nuclear charge. The terms proportional to $C_S$ and $C_E$ in Eq. (4.1) can be treated in a similar fashion and, on dimensional grounds, their matrix elements must also be proportional to $Z\alpha\eta$.

Suppose we consider, for sake of illustration, transitions from the metastable $2S_{1/2}$ state, which is the most suitable candidate for terrestrial experiments. Admixtures of $nP_{1/2,3/2}$ for $n \geq 2$ can be induced into this state, not only by the gravitational Hamiltonian in Eq. (4.1) but also by the electromagnetic and weak interactions as well. The latter case has been considered in detail by Dunford, Lewis, and Williams.[34] The expressions for the resulting wave functions are extremely complicated, and are considered in detail in Ref. 21. For present purposes we will thus exhibit only a few illustrative contributions, of which the most important is that due to the $2P_{1/2}$ state, which is separated from $2S_{1/2}$ by the Lamb shift $L = 4.38 \times 10^{-6}$ eV. Combining Eqs. (4.1)–(4.3) we can write the gravitational matrix element in the form (for arbitrary principal quantum number $n$)

$$\begin{aligned}\langle nP_{1/2}m'_J|\eta\alpha a_0\Big[C_K\vec{g}\cdot(\vec{r}\nabla^2+\vec{\nabla})-\frac{iC_S}{2}\hat{g}\cdot\vec{\sigma}_e\times\vec{\nabla}\Big] \\ +C_E\frac{Ze^2}{c^2}\,\vec{g}\cdot\hat{r}\,|nS_{1/2}m_J\rangle \\ =(2C_E-C_K)Z\alpha\eta\frac{(n^2-1)^{1/2}}{6n} \\ \times\langle\chi^{m'_J}_{1/2}|\vec{\sigma}_e\cdot\hat{g}|\chi^{m_J}_{1/2}\rangle\,,\end{aligned} \tag{4.5}$$

where the $\chi$'s are Pauli spinors with $m_J = \pm\frac{1}{2}$. Note that there is no contribution from $C_S$ in Eq. (4.5). This is a consequence of the fact that the radial integral vanishes for matrix elements of the spin term when taken between Coulomb wave functions with the same principal quantum number $n$. Hence the spin term can admix into the $2S_{1/2}$ state only $nP_{1/2,3/2}$ with $n \geq 3$. For sake of comparison the analog of Eq. (4.5) for the Stark mixing induced by an external electric field $\vec{\mathcal{E}}$ is

$$\langle nP_{1/2}m'_J|e\vec{\mathcal{E}}\cdot\vec{r}|nS_{1/2}m_J\rangle = \frac{e\mathcal{E}a_0}{2}\frac{n}{Z}(n^2-1)^{1/2}\langle\chi^{m'_J}_{1/2}|\vec{\sigma}_e\cdot\hat{\mathcal{E}}|\chi^{m_J}_{1/2}\rangle\,. \tag{4.6}$$

To pursue the present example we consider the polarization of the radiation in the transition $2S_{1/2}\to 1S_{1/2}+\gamma$ due to the combined presence of an electric and a gravitational field. From Eqs. (4.5) and (4.6) the combined transition matrix element is just $\langle\chi'_{1/2}|\vec{\sigma}_e\cdot\vec{\mathfrak{F}}|\chi_{1/2}\rangle$, where

$$\vec{\mathfrak{F}} = \hat{g}\left[(2C_E-C_K)Z\alpha\eta\frac{\sqrt{3}}{12}\right]+\hat{\mathcal{E}}\left[e\mathcal{E}a_0\frac{\sqrt{3}}{Z}\right]. \tag{4.7}$$

The $2P_{1/2}$ contribution to the amplitude for $2S_{1/2}\to 1S_{1/2}+\gamma$ in the presence of $\vec{g}$ and $\vec{\mathcal{E}}$ is then proportional to

$$\begin{aligned}\frac{1}{L}\sum_{m'_J}\langle 1S_{1/2}+\gamma|\vec{\sigma}_e\cdot\vec{\epsilon}^*|2P_{1/2}(m'_J=\pm\tfrac{1}{2})\rangle\langle 2P_{1/2}(m'_J=\pm\tfrac{1}{2})|\vec{\sigma}_e\cdot\vec{\mathfrak{F}}|2S_{1/2}\rangle \\ =\frac{1}{L}\langle\chi^{m'_J}_{1/2}(1S)|\vec{\epsilon}^*\cdot\vec{\mathfrak{F}}+i\vec{\sigma}_e\cdot\vec{\epsilon}^*\times\vec{\mathfrak{F}}|\chi^{m_J}_{1/2}(2S)\rangle\,.\end{aligned} \tag{4.8}$$

Here $\vec{\sigma}_e\cdot\vec{\epsilon}^*$ is, up to an overall constant, the usual $E1$ operator, and $\vec{\epsilon}^* = \vec{\epsilon}^*(\vec{k})$ is the polarization vector for the emitted photon with momentum $\vec{k}$. In practice one is interested in the transition rate $W_{21}$ for $2S_{1/2}\to 1S_{1/2}+\gamma$ for the situation in which the initial electron is polarized in some direction $\hat{n}$ due, say, to the ambient magnetic fields which are known to be present in stars. In this case

$$\begin{aligned}W_{21} &\propto \mathrm{Tr}[M\tfrac{1}{2}(1+\vec{\sigma}_e\cdot\hat{n})M^\dagger]\,, \\ M &= \vec{\epsilon}^*\cdot\vec{\mathfrak{F}}+i\vec{\sigma}_e\cdot\vec{\epsilon}^*\times\vec{\mathfrak{F}}\,.\end{aligned} \tag{4.9}$$

Evaluating the trace in Eqs. (4.9) we find

$$\mathrm{Tr}(\cdots)=\vec{\epsilon}\cdot\vec{\epsilon}^*|\vec{\mathfrak{F}}|^2+i\vec{\epsilon}\times\vec{\epsilon}^*\cdot[\vec{\mathfrak{F}}(\hat{n}\cdot\vec{\mathfrak{F}})-\vec{\mathfrak{F}}\times(\hat{n}\times\vec{\mathfrak{F}})]\,, \tag{4.10}$$

where we have used the fact that $\vec{\mathfrak{F}}$ is a real vector to simplify the expression leading to Eq. (4.10). The term proportional to $i\vec{\epsilon}\times\vec{\epsilon}^*$ leads to a circular polarization $P_\gamma$ of the emitted radiation, the magnitude of which is determined by the expression in square brackets in Eq. (4.10). Combining Eqs. (4.10) and (4.7) we see that $P_\gamma$ will depend on $\vec{g}$ through correlations such as $\vec{k}\cdot\vec{g}\times(\hat{n}\times\vec{\mathcal{E}})$. These correlations give rise to a characteristic variation of $P_\gamma$ over the surface of a star, as shown in Fig. 2, and can in principle be used to

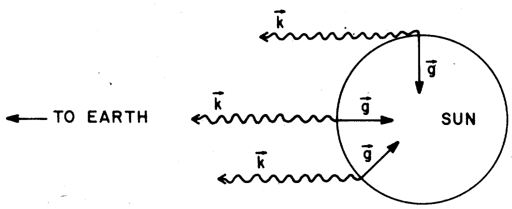


FIG. 2. Variation of $P_\gamma$ across the surface of the Sun. Note that since only the forward hemisphere is visible there are no cancellations from contributions emanating from the rear hemisphere. See text for further details.

distinguish between the gravitational and nongravitational contributions to $P_\gamma$. The RGS and Stark contributions to $P_\gamma$ can also be distinguished in at least two other ways: (i) The gravitational contribution to $P_\gamma$ is proportional to $\eta$ and hence to $|\vec{g}|$. We can thus compare $|P_\gamma|$ in various white dwarfs for which $\eta/\eta\oplus$ varies between $\approx 10^4$-$10^6$ as we see from Table II. (ii) We see from Eqs. (4.5) and (4.6) that the RGS and Stark matrix elements have a different dependence on $Z/n$:

$$\frac{\langle \mathrm{RGS}\rangle}{\langle \mathrm{Stark}\rangle} \propto \frac{Z^2}{n^2}. \tag{4.11}$$

Thus if we compare hydrogenic systems such as H and $He^+$ in the same white dwarfs, then the variation of $|P_\gamma|$ with $Z$ can be used to distinguish between the RGS and Stark contributions. The same end can also be achieved by comparing transitions from states with different values of $n$ in hydrogen. It must be emphasized, however, that any actual attempt to measure the gravitational contribution to $P_\gamma$ will be extremely difficult due to the much larger contributions from nongravitational sources.[35] However, our discussion applies as well to terrestrial experiments where the nongravitational contributions can be more readily controlled.

## V. GRAVITY-INDUCED EFFECTS IN FERMION-ANTIFERMION SYSTEMS

We consider in this section some gravitational effects in fermion-antifermion systems, where a change in parity induced by an external gravitational field leads to change in the eigenvalue of $CP$. For any fermion-antifermion $(f\text{-}\bar{f})$ system the eigenvalues of charge conjugation $(C)$ and parity $(P)$ are given by

$$C=(-1)^{L+S},\quad P=(-1)^{L+1},\quad CP=(-1)^{S+1}, \tag{5.1}$$

where $L$ and $S$ denote the orbital and spin angular momentum of $f-\bar{f}$. (Here $f-\bar{f}$ is treated in the nonrelativistic limit where $L$ and $S$ are good quantum numbers.) Since the gravitational field $h_{\mu\nu}$ in Eq. (2.45) is even under $C$ it follows that for $f-\bar{f}$ any change in $L$ induced by the gravitational field must be compensated for by a corresponding change in $S$. From Eqs. (5.1) it then follows that a gravitational field can mix states with the same total angular momentum but opposite eigenvalues of $CP$.

If we return to the two-body Hamiltonian in Eq. (3.40) and set $m_e=m_p=m$ we find

$$\begin{aligned} H(f-\bar{f}) = {} & 2mc^2 - 2m(c^2\Phi_E+\vec{g}\cdot\vec{R}) + \frac{(\gamma'+\frac{1}{2})}{2mc^2}\,\vec{g}\cdot\vec{S}\times\vec{P} + \left[1-(2\gamma'+1)\left(\Phi_E+\frac{\vec{g}\cdot\vec{R}}{c^2}\right)\right]\left(\frac{\vec{P}^2}{4m}+\frac{\vec{k}^2}{m}\right) \\ & + \frac{(\gamma'+\frac{1}{2})}{2mc^2}\, i\hbar\vec{g}\cdot\vec{P} - \frac{e^2}{r}\left[1-(\gamma'+1)\left(\Phi_E+\frac{\vec{g}\cdot\vec{R}}{c^2}\right)\right] + \frac{C_S}{mc^2}\,\vec{g}\cdot(\vec{s}_1-\vec{s}_2)\times\vec{k} \\ & - \frac{C_X}{2mc^2}(\vec{g}\cdot\vec{r}\,\vec{P}\cdot\vec{k}+\vec{P}\cdot\vec{r}\,\vec{g}\cdot\vec{k}-i\hbar\,\vec{g}\cdot\vec{P}). \end{aligned} \tag{5.2}$$

We note the following features of $H(f-\bar{f})$: (a) All terms of the form $\vec{g}\cdot\vec{r}$ and $\vec{g}\cdot\vec{r}\nabla^2$ are absent. These terms are spin independent and, from the preceding discussion, would have led to $C$-violating transitions. (b) There is a spin-dependent term $V^\sigma(f-\bar{f})$ which survives in Eq. (5.2) which can be expressed in the form

$$V^\sigma(f-\bar{f}) = -C_S\,\frac{i\hbar\eta}{2mc}\,\hat{g}\cdot(\vec{\sigma}_1-\vec{\sigma}_2)\times\frac{\partial}{\partial\vec{r}}. \tag{5.3}$$

Here $1=f$, $2=\bar{f}$, $\vec{r}=\vec{r}_1-\vec{r}_2$, and $m$ is the mass of $f$ or $\bar{f}$. Since $V^\sigma$ changes both $L$ and $S$ it can change the eigenvalue of $CP$ without at the same time changing the total angular momentum $J$ of

$\eta/\eta$

the system. We note, however, that since $h_{\mu\nu}$ is a spin-2 field, $H(f-\bar{f})$ should also lead to transitions with $|\Delta\vec{J}|=2$ or 1. That this is indeed the case is shown in Table III where we exhibit selected matrix elements of $V^\sigma$. (When taken between pure Coulomb wave functions, radial matrix elements of $V^\sigma$ vanish when the principal quantum numbers $n$ and $n'$ for the initial and final states are the same.) We further note that $\langle {}^3P_0|V^\sigma|{}^1S_0\rangle=0$ on the basis of angular momentum conservation.

It is interesting to contrast the properties of $\langle V^\sigma\rangle$ in positronium with those of the Zeeman and Stark interactions. The Zeeman Hamiltonian for

TABLE III. Selected matrix elements of $V^\sigma$ in positronium. The spatial wave functions for the state $|n;{}^{2S+1}L_J\rangle$ are those obtained using only the Coulomb interaction.

| $\Delta J$ | Matrix element in units of $Z\alpha\eta C_S$ |
|---|---|
| 0 | $\langle 1;{}^3S_1(S_z=\pm1)\|V^\sigma\|2;{}^1P_1(L_z=\pm1)\rangle = \mp\frac{32\sqrt{2}}{81}$ |
| | $\langle 4;{}^3F_2(L_z=\pm1)\|V^\sigma\|3;{}^1D_2(L_z=\pm2)\rangle = \frac{\pm\sqrt{2}\,2^{13}\,3^3}{7^8}$ |
| 1 | $\langle 2;{}^3P_1(L_z=\pm1)\|V^\sigma\|1;{}^1S_0\rangle = \pm\frac{32\sqrt{2}}{81}$ |
| 2 | $\langle 3;{}^1D_2(J_z=0)\|V^\sigma\|2;{}^3P_0(S_z=\pm1)\rangle = \mp\frac{1}{\sqrt{3}}\,\frac{4\,608}{15\,625}$ |

positronium, $H^Z$, is given by[36]

$$H^Z = \frac{e\hbar}{2mc}(\vec{\sigma}_1 - \vec{\sigma}_2)\cdot\vec{B}\,. \tag{5.4}$$

Without loss of generality we can take $\vec{B}$ along the $z$ axis in which case

$$\begin{aligned}\langle n';{}^3S_1(S_z=0)\,|H^Z|\,n;{}^1S_0\rangle &= \frac{e\hbar}{mc}B\delta_{n,n'}\,,\\ \langle n';{}^3S_1(S_z=\pm1)\,|H^Z|\,n;{}^1S_0\rangle &= 0\,.\end{aligned} \tag{5.5}$$

We see that $H^Z$ has nonvanishing matrix elements between ${}^1S_0$ ($CP=-1$) and ${}^3S_1$ ($CP=+1$) as could have been anticipated from the discussion following Eqs. (5.1). However, ${}^3S_1$ and ${}^1S_0$ have different values of $J$, as would any two states connected by the purely spin-dependent interaction $H^Z$. It follows that $H^Z$ differs from $V^\sigma$ in that the Zeeman interaction cannot coherently admix states of opposite $CP$ whereas $V^\sigma$ can. We turn next to the Stark interaction $H^S$,

$$H^S = e\vec{\mathscr{E}}\cdot(\vec{r}_1 - \vec{r}_2) = e\vec{\mathscr{E}}\cdot\vec{r}\,. \tag{5.6}$$

Since $H^S$ is spin independent it follows from Eqs. (5.1) that the Stark interaction cannot connect states of opposite $CP$, in contrast to $V^\sigma$ which can. We emphasize that although $V^\sigma$ differs from both $H^Z$ and $H^S$, the effects of $V^\sigma$ can be simulated by higher-order electromagnetic processes. We also note in passing that both $V^\sigma$ and $H^Z$ are even under time reversal even though they connect states of opposite $CP$. This in no way conflicts with the CPT theorem, since the changes in $CP$ induced in positronium are offset by corresponding changes in the external gravitational or magnetic fields.

The preceding considerations apply *mutatis mutandis* to matrix elements of $V^\sigma$ between hadrons of well-defined $C$ (e.g., $\rho^0, \omega, \phi, \psi/J, \Upsilon, \ldots$) if we view these as quark-antiquark bound states. Our

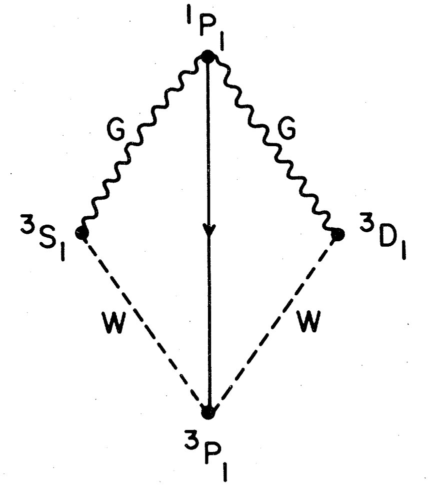


FIG. 3. Gravity-induced transitions in fermion-antifermion systems. The $J=1$ states are designated by the usual spectroscopic notation ${}^{2S+1}L_J$, where $L$, $S$, and $J$ are the orbital, spin, and total angular momentum. The wavy line indicates gravity-induced transitions between states with the same eigenvalue of $C$ but opposite eigenvalues of $P$ and $CP$. The dashed line indicates transitions that are induced by the $C$-odd, $P$-odd, and $CP$-even part of the weak Hamiltonian. The solid line denotes a $C$-odd, $CP$-odd transition that can be induced by a combination of the weak and gravitational interactions.

discussion indicates that these states can have gravity-induced admixtures of opposite-$CP$ components, although the magnitudes of these admixtures are expected to be quite small.

As noted earlier, $V^\sigma$ leads to transitions between states with the same $C$ but opposite $P$, and hence opposite $CP$. It is interesting to note that, when combined with the $C$-odd $P$-odd part of the weak Hamiltonian $H_w$, $V^\sigma$ can lead to transitions between states of opposite $C$ and $CP$. This is illustrated schematically in Fig. 3 for a $J=1\ f-\bar{f}$ system, where $G$ and $W$ denote transitions induced by the gravitational and weak interactions, respectively. We see that these interactions lead to a combined second-order transition which admixes the ${}^1P_1$ and ${}^3P_1$ states which differ in both $C$ and $CP$.

## VI. CONCLUSIONS

The present study of hydrogen in a gravitational field is part of an ongoing program to devise experimental tests of general relativity (GR) at the quantum level.[1] Motivated by the results of Ref. 2, we anticipated that an analysis of the relativistic gravitational Stark effect would lead to a quantum test of GR which had certain advantages over those previously considered.[1] These include the suggestion of testing GR in regions with relatively strong gravitational fields by measuring the polarization of light in white dwarfs,[2] and the possibility of adapting some existing high-precision atomic physics techniques to such tests.[2]

In addition to supplying the details of the calculations leading to the results of Refs. 1 and 2, we have included in this paper a detailed discussion of two points that we had not previously considered: The first is the effect on the RGS Hamiltonian of the coupling of the gravitational field to the internal electromagnetic field in hydrogen (or positronium), and the second is the effect of using relativistic (rather than nonrelativistic) center-of-mass variables. Although the coupling $H^{(ge)}$ of the gravitational and electromagnetic fields gives rise to a transition operator which is nominally higher order than those considered in Refs. 1 and 2, it turns out that *matrix elements* of this operator are of the same order as those previously considered.[1,2] The expression for $H^{(ge)}$ is unambiguously calculable and is given in Eq. (3.2).

The point to which we have devoted the greatest consideration in this paper is the need for a proper set of center-of-mass variables, as discussed in Sec. III. We have focused on three sets of variables, (a) nonrelativistic (NR), (b) the relativistic variables of Krajcik and Foldy (KF), and (c) center-of-energy (CE) variables. Each has certain advantages and disadvantages relative to the others, and each leads to a different expression for the RGS operator as shown in Table I. The motivation for using NR variables in Refs. 1 and 2 was the observation that in the (reasonable) approximation $m_e/m_p \to 0$ (or $m_p \to \infty$) the KF and NR variables coincided. The difficulty is that if this limit is taken *after* the KF variables are inserted into the RGS operator $H^{(0)}(e-p)$ in Eq. (3.1), the results of using the KF and NR variables are no longer the same. The very fact that the final result depends on how the $m_p \to \infty$ limit is taken points up some of the problems that we confront.

Based on our work and on discussions with experts in this field, it is our opinion that the problem of relativistic coordinates in the presence of post-Newtonian gravitational interactions needs further study. We also feel that this problem may influence previous work[20] on post-Newtonian effects in hydrogen, as well as the analysis of the EDB experiments in the framework of various relativistic theories of gravity.[30,31] Because of these uncertainties we cannot draw any final conclusions about the magnitude of the RGS effect in hydrogen, and hence we will content ourselves at this stage with the following summary of possibilities. Our focus will be on the RGS admixtures into the metastable $2S_{1/2}$ state in hydrogen which is the most likely candidate for a terrestrial experiment.

(a) The most important RGS contribution arises from the nearby $2P_{1/2}$ state. Using either KF or CE variables this vanishes to leading order, i.e., to $O(Z\alpha\eta)$. Using NR variables there is a surviving contribution to this order, but it is independent of $\gamma'$.

(b) For reasons discussed in Sec. III, the choice of center-of-mass variables should not affect the form of the post-Newtonian contributions proportional to $\gamma'$. Since from (a) above the $2P_{1/2}$ admixture is independent of $\gamma'$, irrespective of which coordinates are used, it follows that this contribution cannot be used to discriminate among different gravity theories at the quantum level.

(c) To order $Z\alpha\eta$ one can next consider admixtures into $2S_{1/2}$ from $nP_{1/2}$ where $n \geq 3$. For these contributions the radial integrals arising from the spin-dependent term proportional to $\vec{g}\cdot\vec{\sigma}\times\vec{k}$ no longer vanish, as they do for transitions between states with the same value of $n$. These contributions, along with those proportional to $\vec{g}\cdot\vec{\nabla}$ and $\vec{g}\cdot\vec{r}\nabla^2$ will thus contribute a term of order $Z\alpha\eta$ which depends on $\gamma'$. However, this term will be suppressed relative to the NR contribution in (a) by factors of order $L/\Delta E \cong 10^{-6}$, where $L = 4.38\times 10^{-6}$ eV is the Lamb shift separation and $\Delta E \cong 1$ eV is a characteristic spacing between levels of different $n$. The sum of all contributions to $O(Z\alpha\eta)$ will thus be nonzero in general and explicitly dependent on $\gamma'$.

(d) A contribution comparable to that in (c) can arise from $2P_{1/2}$ if we retain terms of order $(Z\alpha)^3\eta$. Since the evaluation of such terms is extremely tedious (partly because of the need to construct appropriate Bethe-Salpeter wave functions) there is little point in carrying them out at this stage, pending clarification of the center-of-mass problem. Since these contributions are expected to be roughly comparable in magnitude to those in (c), and also $\gamma'$ dependent, the sum of all RGS contributions will almost certainly be nonzero, although smaller than anticipated in Refs. 1 and 2.

Faced with the combined problems of discriminating among different relativistic theories of gravity, and among different choices of center-of-mass variables, it might be asked how we can hope to sort out one problem from another. One possibility is to consider the RGS effect in a centrifugal field, where the Dirac equation can be written down unambiguously,[37] and hence where the only uncertainty is due to the choice of center-of-mass variables. Centrifugal fields of order $10^5$ g have already been used in gravity experiments,[38] and the advance of technology might make it feasible to carry out experiments looking for a polarization of radiation induced by such a field.

*Note Added*: For the case $\vec{P}=0$, the RGS Hamiltonian in Eq. (4.1) has nonzero matrix elements even for an atom in a uniform field. This can be understood by noting that for an atom *at rest* in a gravitational field the RGS effect arises from a difference in the gravitational potential over the dimensions of the atom. If we consider, for example, the term proportional to $-\vec{g}\cdot\vec{r}\vec{k}^2$ in Eq. (4.1), then the energy difference $\Delta E$ arising from the variation of this term over a distance $a_0$ is ($v/c \sim \alpha$)

$$\Delta E = m_e g a_0 \left(\frac{v}{c}\right)^2 = m_e g\left(\frac{\hbar}{m_e c\alpha}\right)\alpha^2 = \left(\frac{g\hbar}{c}\right)\alpha = \eta\alpha, \tag{6.1}$$

which gives Eqs. (4.3)–(4.5). This is analogous to the situation for the COW effect in Ref. 3 in which the interference effect arises from the difference in gravitational potential over the vertical dimension of a crystal which is at rest in a uniform field. Moreover, it is easy to show that the respective energy scales in the RGS and COW effects are just in the ratio of $a_0$ to the vertical dimension of the crystal, which is typically of order 2 cm. By contrast, for an atom in *free fall* in a uniform field, we presume on the basis of the equivalence principle that no effect would exist. However, this remains to be formally demonstrated in the context of the preceding discussion of relativistic coordinates. We have not considered this problem in any greater detail both because our knowledge of relativistic center-of-mass coordinates is inadequate at present, and also because of the complexities of decoupling the c.m. and internal interactions. There are, however, cases of practical interest, such as $He^+$ in a white dwarf, where the atom would not be in free fall due to its coupling to ambient electromagnetic fields. In such a circumstance the $He^+$ would not be at rest with respect to the gravitational field and hence $P_\gamma$, although nonzero, would not be given by the simple results of Sec. IV. A detailed calculation of $P_\gamma$ under these conditions is, however, beyond the scope of this paper. We note in the same spirit that the results for positronium do not suggest any immediate application, since positronium is presumably in free fall in terrestrial experiments.

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## APPENDIX A: THE DIRAC EQUATION IN THE PPN FORMALISM

We present in this appendix the intermediate steps leading to the derivation of the effective single-particle Hamiltonian given in Eq. (2.63) in the PPN formalism. As noted earlier the parameter $\beta'$ in Eqs. (2.61) makes no contribution since we are working to lowest order in $\Phi$ everywhere. The generalizations of Eqs. (2.28) and (2.30) are

$$g_{\mu\nu}(x) = \delta_{\mu\nu}(1+2\gamma'\Phi) - 2\delta_{\mu 0}\delta_{\nu 0}[1-(1-\gamma')\Phi], \tag{A1a}$$

$$g^{\mu\nu}(x) = \delta_{\mu\nu}(1-2\gamma'\Phi) - 2\delta_{\mu 0}\delta_{\nu 0}[1+(1-\gamma')\Phi], \tag{A1b}$$

$$e_a^\mu(x) = \delta_{\mu a}(1-\gamma'\Phi) + \delta_{\mu 0}\delta_{a0}(1+\gamma')\Phi, \tag{A1c}$$

$$e_{b\nu}(x) = \delta_{b\nu}(1+\gamma'\Phi) - \delta_{b0}\delta_{\nu 0}[2-(1-\gamma')\Phi], \tag{A1d}$$

$$e^\nu_{b;\mu}(x) = \frac{\gamma'}{c^2}[\eta_{\mu\nu}g_b(x) - \delta_{\mu b}\eta_{\sigma\nu}g_\sigma(x)] - \frac{(1-\gamma')}{c^2}[\delta_{\mu 0}\delta_{\nu 0}g_b(x) + \delta_{\mu 0}\delta_{b0}\eta_{\sigma\nu}g_\sigma(x)]. \tag{A1e}$$

$\Gamma_0$ in Eq. (2.31b) is left unchanged, but $\vec{\Gamma}$ gets multiplied by $\gamma'$. The Hamiltonian in Eq. (2.34) now reads

$$H = -i\hbar c[1-(1+\gamma')\Phi]\vec{\alpha}\cdot\vec{\partial} - \frac{i\hbar}{2c}(2\gamma'-1)\vec{\alpha}\cdot\vec{g} + \beta mc^2(1-\Phi), \tag{A2}$$

and Eq. (2.36) generalizes to

$$\tilde{\psi}(x) = (1+\tfrac{3}{2}\gamma'\Phi)\psi(x), \tag{A3}$$

which leads to the Hamiltonian of Eq. (2.62). Carrying out the Foldy-Wouthuysen transformation we note that $[\mathcal{O},\mathcal{E}]$ and $[\mathcal{O},[\mathcal{O},\mathcal{E}]]$ in Eqs. (2.43) do not depend on $\gamma'$ (to lowest order in the gravitational interaction) and hence are left unchanged. However, each gravitational term in Eq. (2.43c) picks up a factor $\frac{1}{2}(1+\gamma')$. Combining these results with Eq. (2.42) then leads to Eq. (2.63).

## APPENDIX B: MAXWELL'S EQUATIONS IN A GRAVITATIONAL FIELD

As noted in Sec. III the Hamiltonian describing a hydrogen atom in a gravitational field contains a term $H^{(ge)}(e-p)$ which represents the coupling of the gravitational field to the electromagnetic field of the $e-p$ system. In this appendix we derive $H^{(ge)}$ as given in Eq. (3.2).

In the absence of gravity Maxwell's equations can be written in the form[39] (here $c=1$)

$$\partial_\alpha F^{\alpha\beta}(x) = -J^\beta(x), \tag{B1a}$$

$$\partial_\alpha F_{\beta\gamma}(x) + \partial_\beta F_{\gamma\alpha}(x) + \partial_\gamma F_{\alpha\beta}(x) = 0, \tag{B1b}$$

where $F^{\alpha\beta}$ is the electromagnetic field-strength tensor and $J^\beta = (\vec{J}, \rho)$ is the current four-vector. Using the principle of general covariance, the gravity-modified Maxwell (GMM) equations are obtained by replacing the Minkowski indices $\alpha, \beta, \gamma$ by world indices $\mu, \nu, \lambda$ and converting the ordinary derivatives to covariant derivatives. Equations (B1) thus become

$$F^{\mu\nu}{}_{;\mu}(x) = -J^\nu(x), \tag{B2a}$$

$$F_{\mu\nu;\lambda}(x) + F_{\nu\lambda;\mu}(x) + F_{\lambda\mu;\nu}(x) = 0. \tag{B2b}$$

Because the $F$'s are antisymmetric, the covariant derivatives in Eq. (B2b) can be replaced by ordinary derivatives and hence the homogeneous Maxwell equations assume their usual forms,

$$\vec{\nabla} \times \vec{E} + \partial \vec{B}/\partial t = 0, \tag{B3a}$$

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{B3b}$$

Equation (B2a) can be written as

$$\frac{\partial}{\partial x^\mu} (\sqrt{g} F^{\mu\nu}) = -\sqrt{g} J^\nu, \tag{B4}$$

where $g = |\det g_{\mu\nu}|$. If we write out $J^\nu$ explicitly for a system of charged particles then, for a static spherically symmetric (SSS) gravitational field, Eq. (B4) can be cast into the form

$$\vec{\nabla} \cdot (\epsilon \vec{E}) = \rho, \tag{B5a}$$

$$\vec{\nabla} \times (\vec{B}/\mu) = \vec{J} + \partial(\epsilon \vec{E})/\partial t, \tag{B5b}$$

$$\epsilon = \mu = (-f/g_{00})^{1/2}, \tag{B5c}$$

$$ds^2 \equiv f(r)(d\vec{x})^2 + g_{00}(r)(dx^0)^2. \tag{B5d}$$

We see that the effect of a gravitational field on Maxwell's equations is to impart to space an effective dielectric constant $\epsilon$ and a permeability $\mu$ which are specific functions of the components of the metric tensor. It follows that a region of space can also be thought of as having an effective index of refraction $n$ given by

$$n = (\epsilon\mu)^{1/2} = (-f/g_{00})^{1/2}. \tag{B6}$$

Equation (B6) can be verified by noticing that since $ds^2 = 0$ for photons we can divide Eq. (B5d) by $f(r)$ to give

$$0 = ds^2 = (d\vec{x})^2 - [-g_{00}(r)/f(r)](dx^0)^2. \tag{B7}$$

In the form of Eq. (B7) light is viewed as propagating in a Minkowskian space-time, but with a local index of refraction given by Eq. (B6). A ray of light propagating in the vicinity of a gravitating mass would thus be deflected from its original path. Equation (B6) has been used recently[40] to calculate the second-order contribution to the gravitational deflection of light by the Sun.

From Eq. (B6) we also deduce that the Coulomb interaction between the electron and proton is modified as it would be in any other dielectric medium:

$$\begin{aligned} &-\frac{e^2}{|\vec{x}|} \to \frac{1}{\epsilon(\vec{x})}\left(\frac{-e^2}{|\vec{x}|}\right) = eA_0(\vec{x}), \\ &|\vec{x}| = |\vec{r}_e - \vec{r}_p|. \end{aligned} \tag{B8}$$

Since the effects we are interested in studying arise from a variation of the gravitational potential over the dimensions of an atom, care must be taken to retain the appropriate terms in the expansion of $A_0(\vec{x})$. Let $\vec{x}_i$ and $\vec{x}$ be a source point (for a charge $q_i$) and a field point, respectively, relative to an arbitrary origin. If $\epsilon(\vec{x})$ were a constant, $\epsilon_0$, then the potential $\phi_i^{(0)}(\vec{x})$ due to the charge $q_i$ would be given from Eq. (B5a) by

$$\phi_i^{(0)}(\vec{x}) = \frac{q_i}{\epsilon_0 |\vec{x} - \vec{x}_i|}. \tag{B9}$$

To account for the variation of $\epsilon$ with $\vec{x}$ we follow Refs. 30 and 31 and write

$$\epsilon(\vec{x}) = \epsilon_0 + \epsilon_0' \vec{g} \cdot \vec{x}, \tag{B10a}$$

$$\phi_i(\vec{x}) = \phi_i^{(0)}(\vec{x}) + \phi_i^{(1)}(\vec{x}), \tag{B10b}$$

where $\epsilon_0$ and $\epsilon_0'$ are constants. Combining Eqs. (B10) with (B5a) the differential equation for $\phi_i^{(1)}(x)$ is

$$\nabla^2 \phi_i^{(1)}(\vec{x}) = \frac{\epsilon_0'}{\epsilon_0^2} \rho_i(\vec{x}) \vec{g} \cdot \vec{x} - \frac{\epsilon_0'}{\epsilon_0} \vec{g} \cdot \vec{\nabla} \phi_i^{(0)}(\vec{x}). \tag{B11}$$

The solution for $\phi_i^{(1)}(\vec{x})$ is then given by

$$\phi_i^{(1)}(\vec{x}) = \frac{-\epsilon_0'}{\epsilon_0} \vec{g} \cdot \vec{x} \phi_i^{(0)}(\vec{x}) + \frac{\epsilon_0'}{2\epsilon_0^2} q_i \vec{g} \cdot \vec{\nabla} |\vec{x} - \vec{x}_i|. \tag{B12}$$

If we now specialize to the case of an atom at the surface of the Earth then from Eqs. (B5c) and (2.61),

$$\begin{aligned} &\epsilon(\vec{x}) = 1 + (\gamma' + 1)(\Phi_E + \vec{g} \cdot \vec{x}/c^2), \\ &\Phi_E = \frac{GM_E}{R_E c^2}, \end{aligned} \tag{B13}$$

and hence,

$$\epsilon_0 = 1 + (\gamma' + 1)\Phi_E, \quad \epsilon_0' = (\gamma' + 1)/c^2. \tag{B14}$$

Combining the previous results we can then write

$$\begin{aligned} \phi_i(\vec{x}) = &\frac{q_i}{|\vec{x} - \vec{x}_i|}[1 - (\gamma' + 1)\Phi_E] \\ &- \frac{(\gamma' + 1)}{c^2} q_i \frac{\vec{g} \cdot \vec{x}}{|\vec{x} - \vec{x}_i|} \\ &+ \frac{(\gamma' + 1)}{2c^2} q_i \frac{\vec{g} \cdot (\vec{x} - \vec{x}_i)}{|\vec{x} - \vec{x}_i|}. \end{aligned} \tag{B15}$$

Equation (B15) gives the potential at a point $\vec{x}$ due to the presence of a source charge $q_i$ at $\vec{x}_i$. To calculate $H^{(ge)}$ we use the contributions from either the electron or proton in the other's field:

$$\begin{aligned} H^{(em)} + H^{(ge)} &= \tfrac{1}{2}[q_e\,\phi_p(\vec{x}_e) + q_p\,\phi_e(\vec{x}_p)] \\ &= q_e\,\phi_p(\vec{x}_e) \\ &= q_p\phi_e(\vec{x}_p) \\ &= \frac{-Ze^2}{|\vec{x}_e - \vec{x}_p|}\left[1 - (\gamma'+1)\Phi_E - \frac{(\gamma'+1)\vec{g}\cdot(\vec{x}_e+\vec{x}_p)}{2c^2}\right]. \end{aligned} \tag{B16}$$

Since the gravitational terms in Eq. (B16) are already $O(1/c^2)$ we can use nonrelativistic variables and write

$$\vec{x}_e = \vec{R} + \frac{m_p}{M}\vec{r}, \quad \vec{x}_p = \vec{R} - \frac{m_e}{M}\vec{r}. \tag{B17}$$

Although additional contributions to the leading Coulomb term would arise from the use of relativistic c.m. variables, as discussed in Sec. III, these are nongravitational in nature and will be dropped, as we have done elsewhere. Hence, finally,

$$H^{(ge)} = \frac{Ze^2}{r}\left[(\gamma'+1)\Phi_E + (\gamma'+1)\frac{\vec{g}\cdot\vec{R}}{c^2} + \frac{(\gamma'+1)(m_p - m_e)}{2Mc^2}\vec{g}\cdot\vec{r}\right]. \tag{B18}$$

Note that $H^{(ge)}$ is symmetric in the coordinates of the electron and proton. Equation (B18) agrees with the results of Refs. 30 and 31 as well as with those of Barker and O'Connell.[41]

We observe that, on dimensional grounds, the matrix elements of $H^{(ge)}$ in a hydrogenic system of charge $Ze$ will in general be of order

$$\langle H^{(ge)}\rangle \sim Z\alpha\eta, \tag{B19}$$

where $\eta = g\hbar/c$. Thus $\langle H^{(ge)}\rangle$ is of the same order as $\langle H(e) + H(p)\rangle$ in Eq. (3.1).[42] We next show that there are no other terms of this order which arise from the covariant Dirac equation in (2.8). Returning to Eqs. (2.38) and (2.39), we add to $H$ the electromagnetic interaction terms and carry out the FW expansion leading to $H'''$ in Eq. (2.44). The electromagnetic terms generate an additional contribution $\Delta H'''$ to Eq. (2.44) given by ($\beta = \gamma^4$)

$$\begin{aligned} \Delta H''' = &-eA_0 + \frac{\beta}{2\mu}(e^2\vec{A}^2 + ie\hbar c\vec{\partial}\cdot\vec{A} - 2ec\,\vec{A}\cdot\vec{p} - e\hbar c\vec{\sigma}\cdot\vec{B}) - \frac{e\hbar c^2}{8\mu^2}(i\hbar\vec{\sigma}\cdot\vec{\partial}\times\vec{E} + \hbar\vec{\partial}\cdot\vec{E} + 2\vec{\sigma}\cdot\vec{E}\times\vec{p}) \\ &+ \beta(2\gamma'+1)\frac{e\hbar}{\mu c}\left(\tfrac{1}{4}\vec{\sigma}\cdot\vec{g}\times\vec{A} - \frac{i}{2}\vec{g}\cdot\vec{A} + \frac{c^2}{2}\Phi\vec{\sigma}\cdot\vec{B} + \frac{c^2}{\hbar}\Phi\vec{A}\cdot\vec{p} - \frac{ic^2}{2}\Phi\vec{\partial}\cdot\vec{A}\right) \\ &+ \frac{(\gamma'+1)}{2}\frac{e\hbar^2}{\mu^2}\left(\tfrac{1}{4}\vec{g}\cdot\vec{E} + \frac{i}{2}\vec{\sigma}\cdot\vec{g}\times\vec{E} + \frac{c^2}{\hbar}\Phi\vec{\sigma}\cdot\vec{E}\times\vec{p} + \tfrac{1}{2}c^2\Phi\vec{\partial}\cdot\vec{E} + \frac{i}{2}c^2\Phi\vec{\sigma}\cdot\vec{\partial}\times\vec{E}\right). \end{aligned} \tag{B20}$$

The terms appearing in the first and second parentheses are the standard results[12] for the FW expansion of the electromagnetic interaction. The expressions in the third and fourth parentheses are new and evidently represent an additional coupling of the electromagnetic and gravitational fields. For an electron in the Coulomb field of the proton (or vice versa) we can set $\vec{A} = 0$, and hence the remaining terms are of order

$$\frac{e\hbar^2}{\mu^2}\vec{g}\cdot\langle\vec{E}\rangle \sim (Z\alpha)^3\eta, \tag{B21}$$

and hence are negligible. We thus conclude that the leading contributions to $\langle H^{(0)}(e-p)\rangle$ come from the terms given in Eq. (3.1).

If it were not for the problems which arise from the need to use relativistic coordinates, the Hamiltonian $H^{(0)}(e-p)$ would be the complete gravitational contribution. However, the relativistic coordinates introduce an additional contribution $\Delta H(e-p)$ given in Eqs. (3.21) so that the complete expression for $H(e-p) = H^{(0)}(e-p) + \Delta H(e-p)$ is given by Eq. (3.24).

For a system of $k$ particles described by a *nonmetric* theory of gravity, the functions $f(r)$ and $g_{00}(r)$ are replaced by two functions $H(r)$ and $T(r)$, which are defined by the Lagrangian[30]

$$L = \sum_k \int dt[-m_{0k}c^2(T - H\vec{\beta}_k^{\,2})^{1/2} + e_k A_\mu \beta_k^\mu], \tag{B22}$$

where $\beta^\mu = v^\mu/c$. Equation (B22) also describes metric theories if we identify

$$T = -g_{00}(r), \quad H = f(r). \tag{B23}$$

For the more general theory defined by Eq. (B22), Maxwell's equations still hold in the form given in Eqs. (B3a), (B3b), (B5a), and (B5b) but $\epsilon$ and $\mu$ are no longer necessarily given by Eq. (B5c). In such a formalism (the so-called "$TH\epsilon\mu$ formalism"[30]) the effects of gravity on electromagnetic systems are thus characterized by the four arbitrary functions $T$, $H$, $\epsilon$, and $\mu$. The discussion[30,31] of the composition independence of $|\vec{g}|$ referred to in Sec. III is based on this formalism.

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$$\frac{d}{d\rho}=\frac{d}{dr}+O\left((\text{gravity})^2\right)$$

which follows from Eqs. (2.25) and (2.26).

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