Gravitational radiation by the thermal phonons of a solid

G. Schäfer and H. Dehnen
Fakultät für Physik der Universität Konstanz, D-7750 Konstanz, Postfach 5560, West Germany
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Gravitational radiation by the phonons of a freely vibrating thin elastic cylinder is calculated in the high-frequency limit. For thermal excitation of the phonons the spectral energy distribution is found to be a blackbody one multiplied by the small opalescence factor $(32/105)(MG/c^2R)(c_0/c)^3$. With the help of thermodynamic equilibrium considerations the resonance integral for the absorption cross section of the cylinder is deduced in the high-frequency limit. The application of the result to a neutron star is discussed.

I. INTRODUCTION

In view of the possibility of lattice structures in massive astrophysical objects (neutron stars) the investigation of gravitational radiation by thermal phonons is of exceptional physical interest. Considering this problem more closely one is confronted immediately with some difficulties, which seems to be the reason for the fact that this area has been treated only roughly.

In order to calculate the gravitational radiation power produced by the thermal phonons of a lattice the knowledge of all free vibrations of the solid body is necessary. These are exactly known only for the elastic sphere, but an explicit analytic calculation of the gravitational radiation power in case of thermal excitation of the oscillations has been proved to be very complicated for reasonable temperature ranges. Therefore we restrict ourselves in this paper to a very special model for the solid, namely, to a "thin" elastic cylinder.

Usually the gravitational radiation of a material body is calculated with Einstein's quadrupole radiation formula.[1] However, this equation underlies the condition that the wavelength of the radiation is large compared with the linear dimension of the considered body. This condition is not fulfilled for the radiation of thermal phonons in the most reasonable temperature range. The frequency of the maximum of the radiation power is, as may be expected, of the order

$$\omega_{\max} \simeq kT/\hbar \tag{1.1}$$

with the wavelength $\lambda\!\!\!^{-}_{\max} \simeq \hbar c/kT \simeq T^{-1}$ K cm. Then the usual quadrupole formula is only applicable, if the temperature of the solid fulfils the condition

$$T[\mathrm{K}] \ll 1/L[\mathrm{cm}], \tag{1.2}$$

where L is the linear dimension of the body. For typical linear dimensions, say, 10 cm, the temperature must be much lower than 10^{-1} K for the applicability of Einstein's quadrupole formula. This means that for the large range of temperatures $T \gtrsim 10^{-1}$ K the wavelength of the radiation is comparable to or small compared with the linear dimension of the solid and that the gravitational radiation of the thermal phonons must be calculated with a modified method. Whereas the low-temperature range can be treated with Einstein's formula without difficulties, for instance for the elastic sphere (see the article of Wagoner *et al.*[2]), this is not the case for the most interesting high-temperature range, which shall be investigated in this paper.

The modification for calculating the gravitational radiation field, necessary now, has been given already by Halpern *et al.*[3] and Press[4] and shall be repeated briefly in such a form which is used in the following only. Here we set $c=G=1$ (G gravitational constant) and choose the metric signature $(-,+,+,+)$.

Starting from the linearized Einstein equations of gravitation[5]

$$\hat{h}^{\mu\nu|\alpha}{}_{|\alpha} = -16\pi T^{\mu\nu} \tag{1.3a}$$

with

$$\begin{aligned} &\hat{h}^{\mu\alpha}{}_{|\alpha}=0, \quad \hat{h}^{\mu\nu}=h^{\mu\nu}-\tfrac{1}{2}\eta^{\mu\nu}h, \quad h=h^{\mu\nu}\eta_{\mu\nu}, \\ &h^{\mu\nu}=g^{\mu\nu}-\eta^{\mu\nu}, \quad \eta^{\mu\nu}=\mathrm{diag}(-1,+1,+1,+1) \end{aligned} \tag{1.3b}$$

($g^{\mu\nu}$ is the metric tensor, $\eta^{\mu\nu}$ is the Minkowski metric), one finds that the far-field solution can be set in the form

$$\hat{h}^{jk}(x^\alpha) = \frac{2}{\partial}\frac{d^2}{dt^2}\int (T^{00}-2T^{0i}n_i + T^{lm}n_l n_m)_{\mathrm{ret}}\, x'^j x'^k d^3x' , \tag{1.4}$$

whereas the remaining components of $\hat{h}^{\mu\nu}$ follow from the de Donder condition (1.3b). Herein $\partial^2 = \vec{x}\cdot\vec{x}$ and $\vec{n}=\vec{x}/\partial$, where $\vec{x}$ is the space vector of the field point. The word "retarded" means evaluation of the components of the energy-momentum-stress tensor $T^{\mu\nu}$ and their time derivatives at the time $t'=t-|\vec{x}-\vec{x}'|$ for each $\vec{x}'$ before integration over the three-dimensional space. Besides the usual quadrupole contribution, Eq.

(1.4) contains all higher multipole radiation terms.

The radial gravitational radiation flux density is given by (see, e.g., Misner *et al.*,[6] p. 992)

$$T^{0r}_{(\mathrm{GW})} = \frac{1}{32\pi} \langle h^{\mathrm{TT}}_{jk|0} h^{\mathrm{TT}jk}{}_{|0} \rangle \tag{1.5}$$

and the total radiation power, identical with the energy loss of the material system, takes the form

$$P \equiv -\frac{dE}{dt} = \hat{d}^2 \int T^{0r}_{(\mathrm{GW})} d\Omega \,. \tag{1.6}$$

The connection between the transverse-traceless (TT)-gauged wave field h^{TT}_{jk} and the solution (1.4) is established by the projection

$$h^{\mathrm{TT}}_{jk} = P_{jl}\hat{h}^{lm}P_{mk} - \tfrac{1}{2}P_{jk}P_{lm}\hat{h}^{lm} \tag{1.7}$$

with the projection operator

$$P_{lm} = \eta_{lm} - n_l n_m \tag{1.7$'$}$$

($n^l = x^l/\hat{d}$ is the unit radial vector).

The application of this formalism (with exception of the projection procedure) to the gravitational radiation of lattice vibrations has been performed already by Halpern *et al.*[3] and Sacchetti *et al.*[7] Both papers consider two-phonon processes and use propagating phonons with Born-von Kármán cyclic boundary conditions. In contrast to this we choose in the following one-phonon processes and take such eigenvibrations of the solid which satisfy the condition that the total surface of the crystal is *free* of tractions (free oscillations, bound phonons). We find that the results differ essentially (on the acoustic mode level) with respect to the directional characteristic of the radiation and its temperature and frequency dependence in case of thermal excitation of the phonons, although in the paper of Halpern *et el.* the shape of the solid is just the same as used by us. On the other hand, Halpern *et al.* and Sacchetti *et al.* use a discrete lattice whereas we shall take a continuous matter distribution; but the deviations of the results do not come from this difference because they remain in the continuum limit.

Because gravitation is a long-range nonscreenable interaction, it is to be expected that the gravitational radiation effects depend essentially on the used eigenvibrations. It is known that even the optical absorption spectrum of an ionic lattice changes if the cyclic boundary condition is replaced by the condition of free surfaces,[8] which effect is more pronounced the smaller the ratio between volume and surface of the crystal.[9] From this point of view the differences in the results mentioned above are understandable.

On the other hand with respect to the conservation law

$$T_\mu{}^\nu{}_{|\nu} = 0, \tag{1.8}$$

which is necessarily connected with the field equations (1.3a) and (1.3b) and their solution (1.4), only the *free* eigenvibrations of the solid are allowed for calculation of gravitational effects. Therefore it seems to us that the results of Halpern *et al.* and Sacchetti *et al.* are doubtful.

Concerning an application of our results to real physical objects we are limited by the very special shape of our solid body. In spite of this we give a rough estimation of the thermal gravitational luminosity of a "young" neutron star, according to which the gravitational energy loss is comparable to the electromagnetic one and should be important for its cooling.

II. THE MODEL OF THE SOLID

As a solid we consider a circular cylinder with free surfaces consisting of homogeneous isotropic elastic material characterized by the constant shear modulus $\mu (\mu > 0)$ and the constant bulk modulus $K = \lambda + \frac{2}{3}\mu (\lambda \geq 0)$. The conditions $\mu > 0$, $\lambda \geq 0$ mean that Poisson's ratio becomes nonnegative. The general energy-momentum-stress tensor of matter without heat flow reads (see, e.g., Ashby *et al*,[10])

$$T^{\mu\nu} = \rho v^\mu v^\nu + p(g^{\mu\nu} + v^\mu v^\nu) - \sigma^{\mu\nu} \tag{2.1}$$

(ρ density, p pressure, $\sigma^{\mu\nu}$ shear-stress tensor, and v^μ four-velocity of the substratum) with the requirements

$$\sigma_{\mu\nu}v^\nu = 0, \quad v^\mu v_\mu = -1, \quad \sigma^\mu{}_\mu = 0 \tag{2.2}$$

and the boundary conditions in case of a body with free surfaces

$$p\eta_\mu - \sigma_{\mu\nu}\eta^\nu = 0, \tag{2.3}$$

wherein η^ν ($\eta^\nu\eta_\nu = +1$) is the normal of the surface. Restriction to our homogeneous and isotropic material means

$$\sigma_{km} = 2\mu(e_{km} - \tfrac{1}{3}\delta_{km}\, e^l_l), \quad \sigma_{0\mu} = 0 \tag{2.4a}$$

with

$$e_{kl} = \tfrac{1}{2}(u_{k|l} + u_{l|k}), \tag{2.4b}$$

wherein u_k is the displacement of matter from the equilibrium position connected with v^μ according to

$$v^i = \dot{u}^i \quad \left(\dot{u}^i \equiv \frac{du^i}{dt}\right). \tag{2.5}$$

Hereby we have restricted ourselves to the nonrelativistic limit ($g^{\mu\nu} = \eta^{\mu\nu}, \dot{u}^i\dot{u}_i \ll 1$).

With the perturbation ansatz

$$\rho=\rho_{(0)}+\rho_{(1)},\quad p=p_{(0)}+p_{(1)},\qquad \rho_{(0)}\gg|\rho_{(1)}|\gg|p_{(1)}| \tag{2.6}$$

($\rho_{(0)}$ and $p_{(0)}$ are the undisturbed density and pressure, constant inside the body) one finds from the conservation relation (1.8) for the first-order perturbations of ρ and p linear in u^i the representation

$$\rho_{(1)}=-(\rho_{(0)}u^i)_{|i}, \tag{2.7}$$

$$p_{(1)}=-u^i p_{(0)|i}-Ku^i{}_{|i} \tag{2.8}$$

and for u^i the equation of motion linearized in u^i:

$$\rho_{(0)}\frac{\partial^2}{\partial t^2}u^i=(\lambda u^l{}_{|l})^{|i}+[\mu(u^{i|l}+u^{l|i})]_{|l}. \tag{2.9}$$

In the case of our circular cylinder we have in cylindrical coordinates

$$\rho_{(0)}(r,\varphi,z)=\rho_0\,\theta(R-r)\,\theta(L-z)\,\theta(z),\quad p_{(0)}\equiv 0 \tag{2.10}$$

(ρ_0 rest mass density), whereby the axis of the material cylinder (L length, R radius) lies on the z axis. In view of the "thin"cylinder (defined later) we restrict ourselves to such vibration modes, which remain solely in the limit of vanishing radius R and which are stronger than the other ones because of $L/R\gg1$ [compare(2.22)]. These are the longitudinal oscillations defined by

$$u_\varphi\equiv0,\quad u_r=u_r(r,z,t),\quad u_z=u_z(r,z,t), \tag{2.11a}$$

which implies

$$\sigma_{r\varphi}=\sigma_{z\varphi}\equiv0. \tag{2.11b}$$

With the boundary conditions

$$t_{rr}(r=R,z,t)=t_{rz}(r=R,z,t)=t_{zz}(r,z=0,t)=t_{zz}(r,z=L,t)=0, \tag{2.12a}$$

wherein the abbreviation

$$t_{lm}=\sigma_{lm}-p\delta_{lm} \tag{2.12b}$$

represents the complete stress tensor, it follows from (2.9) exactly that[11]

$$\begin{aligned}u_r&=-\sum_n A_n[\alpha_n J_1(\alpha_n r)+k_n\delta_n J_1(\beta_n r)]\times\sin(k_n z)\cos(\omega_n t+\epsilon_n),\\ u_\varphi&\equiv0,\\ u_z&=\sum_n A_n[k_n J_0(\alpha_n r)-\beta_n\delta_n J_0(\beta_n r)]\times\cos(k_n z)\cos(\omega_n t+\epsilon_n).\end{aligned} \tag{2.13}$$

with the Bessel functions J_0 and J_1 and the relations

$$\begin{aligned}\alpha_n^2&=\left(\frac{\omega_n}{c_1}\right)^2-k_n^2,\quad c_1^2=(\lambda+2\mu)/\rho_0,\\ \beta_n^2&=\left(\frac{\omega_n}{c_2}\right)^2-k_n^2,\quad c_2^2=\mu/\rho_0,\\ \delta_n&=2k_n\alpha_n J_1(\alpha_n R)/(\beta_n^2-k_n^2)J_1(\beta_n R),\\ k_n&=n\pi/L,\ n=1,2,3,\ldots\ .\end{aligned} \tag{2.14}$$

The values of the eigenfrequencies are given by the secular equation (Pochhammer's frequency equation)

$$(k_n^2-\beta_n^2)^2J_0(\alpha_n R)J_1(\beta_n R)+4\alpha_n\beta_n k_n^2 J_1(\alpha_n R)J_0(\beta_n R)-\frac{2\alpha_n}{R}\frac{\omega_n^2}{c_2^2}J_1(\alpha_n R)J_1(\beta_n R)=0\ . \tag{2.15}$$

The constants A_n and ϵ_n may be determined to satisfy the initial conditions. The use of the nonrelativistic limit means that the velocities of sound are small compared with the velocity of light:

$$c_1\ll1,\quad c_2\ll1\ . \tag{2.16}$$

From (2.13), (2.12b), (2.4a), and (2.4b) it follows for the stress component t_{rz} immediately that

$$t_{rz}=-\sum_n A_n\mu[2k_n\alpha_n J_1(\alpha_n r)-\delta_n(\beta_n^2-k_n^2)J_1(\beta_n r)]\times\cos(k_n z)\cos(\omega_n t+\epsilon_n)\ . \tag{2.17}$$

Evidently expression (2.17) satisfies the relation (2.12a) but the full boundary conditions of free vibrations (2.3) are fulfilled by (2.11b), (2.12a), and (2.17) only, if (2.17) vanishes at the positions $z=0$ and $z=L$. But this is not the case exactly with the exception of the points $r=0$ and $r=R$. This means that the longitudinal oscillations (2.13) are not those of the freely vibrating finite circular cylinder. In spite of this we can go on with the foregoing considerations if we restrict ourselves to a thin cylinder.

For this reason we expand the Bessel functions for small arguments into power series:

$$J_0(x)=1-\tfrac14x^2+\tfrac1{64}x^4-\cdots\ , \tag{2.18a}$$

$$J_1(x)=\tfrac12x-\tfrac1{16}x^3+\cdots\ . \tag{2.18b}$$

Insertion of (2.18b) into (2.17) shows with respect to (2.14) that the remaining boundary conditions

$$t_{rz}(r,z=0,t)=t_{rz}(r,z=L,t)=0 \tag{2.19}$$

are valid, if we approximate $J_1(x) \simeq \frac{1}{2}x$; this means

$$|x| = |\alpha_n| r \leq |\alpha_n| R \ll 1 , \qquad |x| = |\beta_n| r \leq |\beta_n| R \ll 1 . \tag{2.20}$$

In the same approximation Eq. (2.15) can be solved with respect to the eigenfrequencies and gives [cf. (2.14)]

$$\omega_n = k_n c_0 = \frac{n\pi}{L} c_0, \quad c_0^{\,2} = E/\rho_0, \quad E = \frac{3\lambda + 2\mu}{\lambda + \mu}\mu \tag{2.21}$$

(where E is Young's modulus). Here and with (2.14) the conditions (2.20) result in

$$n\frac{R}{L} \ll \frac{1}{\pi}\left(\frac{\lambda + \mu}{2\lambda + \mu}\right)^{1/2} \rightarrow \omega_n R \ll c_2\left(\frac{3\lambda + 2\mu}{2\lambda + \mu}\right)^{1/2}, \tag{2.22}$$

which we consider as a condition for the ratio R/L of the cylinder; this means that for any fixed value of n the ratio R/L must be chosen so small that (2.22) is fulfilled (thin cylinder).[12]

Under this restriction the longitudinal oscillations (2.13) can be considered as free oscillations of the circular cylinder. In the approximation used above they take the form with regard to (2.14) and (2.21)

$$u_r = \sum_n A_n k_n^{\,2} \frac{\lambda}{\lambda + \mu} \frac{3\lambda + 2\mu}{\lambda + 2\mu} \frac{r}{2} \sin(k_n z) \cos(\omega_n t + \epsilon_n) ,$$

$$u_\varphi \equiv 0 , \tag{2.23}$$

$$u_z = \sum_n A_n k_n \left\{ 1 - \left(\frac{\alpha_n r}{2}\right)^2 + \frac{2\lambda}{\lambda + 2\mu}\left[1 - \left(\frac{\beta_n r}{2}\right)^2\right]\right\} \times \cos(k_n z) \cos(\omega_n t + \epsilon_n) .$$

Here and with regard to (2.4a), (2.4b), (2.5), (2.6), (2.7), and (2.8) the energy-momentum stress tensor (2.1) for the free longitudinal oscillations of the thin cylinder is known in the nonrelativistic limit [cf. (2.16)].

III. GRAVITATIONAL RADIATION OF A SINGLE VIBRATION MODE

At first the gravitational wave field produced by a single free oscillation mode of the thin elastic cylinder will be calculated. As one finds easily from the results of Sec. II the following ratios between T^{00}, T^{0i}, and T^{lm} are valid:

$$T^{00} : T^{0i} : T^{lm} \sim 1 : c_0 : c_0^{\,2} \tag{3.1}$$

($c_0 \ll 1$), so that in formula (1.4) only the first term on the right-hand side is to be taken into account in the nonrelativistic limit. In this way we get from (2.1), (2.6), and (2.7)

$$\hat{h}^{jk}(\vec{x}, t) = -\frac{2}{\hat{d}}\frac{d^2}{dt^2}\int [(\rho_{(0)}u^i)_{|i}]_{\text{ret}} x'^j x'^k d^3x' \tag{3.2}$$

and with respect to (2.10) after partial integration ("retarded quadrupole formula")

$$\hat{h}^{jk}(\vec{x}, t) = \frac{2}{\hat{d}}\frac{d^2}{dt^2}\int \rho_{(0)}(u^j x'^k + u^k x'^j)_{\text{ret}} d^3x' . \tag{3.3}$$

By this procedure we have permuted the retardation and the derivative in (3.2); but the error produced in this way is smaller than (3.3) by the factor c_0 and can therefore be neglected.

After transformation of (2.23) into Cartesian coordinates and insertion into (3.3) one finds that the leading component of $\hat{h}^{jk}$ is $\hat{h}^{zz}$, whereas the remaining ones are smaller by the factor $[\lambda/(\lambda + \mu)](k_n R)^2$ and can be thrown away in view of (2.22). Hence we obtain for the only relevant component

$$\hat{h}^{zz} = -\frac{4}{\hat{d}}\omega_n^{\,2} A_n k_n \frac{3\lambda + 2\mu}{\lambda + 2\mu} \times \int \rho_{(0)} z' \cos(k_n z') \cos(\omega_n t' + \epsilon_n) d^3x' \tag{3.4a}$$

with the retarded time

$$t' = t - \hat{d} + z' \cos\vartheta + r' \sin\vartheta \cos(\varphi - \varphi') , \tag{3.4b}$$

wherein $\hat{d}, \vartheta, \varphi$ are the spherical coordinates of the field point $\vec{x}$. Because $\omega_n r' \leq \omega_n R \ll c_2 \ll 1$ according to (2.16) and (2.22), the combination of (3.4a) and (3.4b) results in

$$\hat{h}^{zz} = -\frac{4}{\hat{d}}\frac{M}{L}\omega_n^{\,2} A_n k_n \frac{3\lambda + 2\mu}{\lambda + 2\mu} \times \int_0^L z' \cos(k_n z') \cos[\omega_n (t - \hat{d} + z' \cos\vartheta) + \epsilon_n] dz' \tag{3.5a}$$

with the total mass of the cylinder

$$M = \pi R^2 L \rho_0 . \tag{3.5b}$$

Evaluating the integral in (3.5a) we obtain under consideration of (2.21) and of $c_0 \ll 1$

$$\hat{h}^{zz} = \frac{4}{\hat{d}}\frac{M}{L} c_0 A_n \omega_n \frac{3\lambda + 2\mu}{\lambda + 2\mu}\{[1 - (-1)^n(\cos(\omega_n L \cos\vartheta) - (\omega_n L \cos\vartheta)\sin(\omega_n L \cos\vartheta))] \cos(\omega_n (t - \hat{d}) + \epsilon_n) + (-1)^n[\sin(\omega_n L \cos\vartheta) + (\omega_n L \cos\vartheta)\cos(\omega_n L \cos\vartheta)] \sin(\omega_n (t - \hat{d}) + \epsilon_n)\} . \tag{3.6}$$

Now we are able to determine the radial gravitational radiation flux density produced by the single mode discussed above. Insertion of (3.6) into (1.7) and (1.7) into (1.5) gives

$$T^{0r}_{(\mathrm{GW})}=\frac{\omega_n^2}{64\pi}\sin^4\vartheta\langle(\hat{h}^{zz})^2\rangle=\frac{\omega_n^4}{4\pi}\frac{M^2c_0^2}{\hat{d}^2L^2}A_n^2\left(\frac{3\lambda+2\mu}{\lambda+2\mu}\right)^2\sin^4\vartheta\{1-(-1)^n[\cos(\omega_nL\cos\vartheta)-(\omega_nL\cos\vartheta)\sin(\omega_nL\cos\vartheta)]+(\omega_nL\cos\vartheta)^2/2\}. \tag{3.7}$$

Integration over the total sphere results in the energy loss according to (1.6):

$$-\frac{dE_n}{dt}=\frac{8}{15}\frac{\omega_n^4M^2c_0^2}{L^2}A_n^2\left(\frac{3\lambda+2\mu}{\lambda+2\mu}\right)^2\left\{1-(-1)^n\frac{15}{\omega_n^2L^2}\left[\left(5-\frac{12}{\omega_n^2L^2}\right)\frac{\sin\omega_nL}{\omega_nL}+\left(\frac{12}{\omega_n^2L^2}-1\right)\cos\omega_nL\right]+\tfrac{1}{14}\omega_n^2L^2\right\}. \tag{3.8}$$

Finally we substitute the amplitude A_n through the energy E_n of the single oscillation mode of the thin cylinder. Consistent determination of

$$E_n=\int T^{00}d^3x'-M \tag{3.9}$$

up to the first nonvanishing term gives with the use of (2.1) and (2.23)

$$E_n=\frac{M}{4}A_n^2\omega_n^4c_0^{-2}\left(\frac{3\lambda+2\mu}{\lambda+2\mu}\right)^2. \tag{3.10}$$

Elimination of the amplitude A_n in (3.7) and (3.8) with the help of (3.10) results in

$$T^{0r}_{(\mathrm{GW})}=\frac{Mc_0^4}{\pi\hat{d}^2L^2}E_n\sin^4\vartheta\{1-(-1)^n[\cos(\omega_nL\cos\vartheta)-(\omega_nL\cos\vartheta)\sin(\omega_nL\cos\vartheta)]+(\omega_nL\cos\vartheta)^2/2\} \tag{3.11}$$

and

$$-\frac{dE_n}{dt}=\frac{32}{15}\frac{Mc_0^4}{L^2}E_n\left\{1-(-1)^n\frac{15}{\omega_n^2L^2}\left[\left(5-\frac{12}{\omega_n^2L^2}\right)\frac{\sin\omega_nL}{\omega_nL}+\left(\frac{12}{\omega_n^2L^2}-1\right)\cos\omega_nL\right]+\tfrac{1}{14}\omega_n^2L^2\right\}. \tag{3.12}$$

Evidently the right-hand side of (3.12) has, except for the factor E_n, the meaning of the reciprocal lifetime of the mode n with respect to gravitational radiation.

In the *low*-frequency limit $\omega_nL\ll 1$ the expressions (3.11) and (3.12) go over into

$$T^{0r}_{(\mathrm{GW})}=\frac{Mc_0^4}{\pi\hat{d}^2L^2}E_n\sin^4\vartheta[1-(-1)^n] \tag{3.13}$$

and

$$-\frac{dE_n}{dt}=\frac{32}{15}\frac{Mc_0^4}{L^2}E_n[1-(-1)^n], \tag{3.14}$$

which are identical with those results obtained by the usual quadrupole radiation formula (see, e.g., Weinberg[13]). In contrast to this in the *high*-frequency limit $\omega_nL\gg 1$ we obtain

$$T^{0r}_{(\mathrm{GW})}=\frac{Mc_0^4}{8\pi\hat{d}^2}\omega_n^2E_n\sin^2\vartheta\sin^22\vartheta \tag{3.15}$$

and

$$-\frac{dE_n}{dt}=\tfrac{16}{105}Mc_0^4\omega_n^2E_n, \tag{3.16}$$

which result is based essentially on the retardation of formula (1.4) or (3.3), respectively. As one can easily see, the retarded quadrupole formula (3.3) gives a radiation power proportional to c^{-7} (c velocity of light) in contrast to the factor c^{-5} in the case of the usual quadrupole formula.

Whereas in the low-frequency limit only the odd modes contribute to the radiation, in the case of high frequencies all modes radiate gravitationally. Furthermore, the angular characteristic is different in both cases whereby the high-frequency characteristic is much more pronounced than the low-frequency one.

IV. THE RADIATION POWER FOR THERMAL EXCITATION OF THE VIBRATION MODES

At first we quantize the energy of the oscillations according to

$$E_n=\hbar\omega_n l,\quad l=1,2,3,\ldots \tag{4.1}$$

omitting the zero-point energy. Then the mean value of the energy of the oscillation with the frequency ω_n in the case of thermal equilibrium is given by the well-known relation

$$\overline{E}_n=\hbar\omega_n/(e^{\hbar\omega_n/kT}-1). \tag{4.2}$$

As shown in the Introduction, for realistic temperatures only the high-frequency limit makes practical sense. Therefore we substitute E_n only in Eqs. (3.15) and (3.16) through (4.2). Simultaneously we add up the contributions of all single modes and replace subsequently the sum over the single eigenfrequencies ω_n by the integral; this means in view of (2.21) that

$$\sum_{\omega_n}\to\frac{L}{\pi c_0}\int d\omega.$$

In this way we obtain for the radial gravitational energy flux density produced by the thermal phonons of the thin cylinder

$$S^r(\vartheta)\equiv\sum_{\omega_n}T^{0r}_{(\mathrm{GW})}(\overline{E}_n)=\frac{MLc_0^{\,3}}{8\pi^2\hat{d}^2}\sin^2\vartheta\,\sin^2 2\vartheta\int_0^\infty\frac{\hbar\omega^3 d\omega}{e^{\hbar\omega/kT}-1} \tag{4.3}$$

and for the total radiation power

$$P=-\frac{dE}{dt}=2\pi\int_0^\pi S^r(\vartheta)\hat{d}^2\sin\vartheta d\vartheta=\frac{16}{105\pi}MLc_0^{\,3}\int_0^\infty\frac{\hbar\omega^3 d\omega}{e^{\hbar\omega/kT}-1}. \tag{4.4}$$

For a realistic solid the upper integration limit in (4.3) and (4.4) is correct only in the case that $T\ll T_D$ (Debye temperature); otherwise it is determined by the Debye frequency ω_D. Performance of the integrals in (4.3) and (4.4) for $T\ll T_D$ gives

$$S^r(\vartheta)=\frac{MLc_0^{\,3}}{2\hat{d}^2}\sigma T^4\sin^2\vartheta\,\sin^2 2\vartheta \tag{4.5}$$

and

$$P=\frac{64\pi}{105}MLc_0^{\,3}\sigma T^4 \tag{4.6}$$

[$\sigma=(\pi^2/60)k^4/\hbar^3$ Stefan-Boltzmann constant].

Evidently we have the result that the spectral energy distribution of the gravitational radiation of the thermal phonons of the thin elastic cylinder has a Planck blackbody character in contrast to the result of Halpern *et al.*, and that there exists a Stefan-Boltzmann law. As one can prove easily, this is not the case for the low-frequency limit [compare (3.13) and (3.14)], which is identical with the low-temperature limit. Furthermore, we find a directional dependence of the radiation, which is different from the result of Halpern *et al.*, where the radiation of the single mode ω_n is emitted inside the very small solid angle of the order of $1/\omega_n L$ centered by the axis of the cylinder.

By division of P by $2\pi RL$ we obtain the radiation power S per unit of the surface of the thin cylinder:

$$S=\tfrac{32}{105}(M/R)c_0^{\,3}\sigma T^4. \tag{4.7a}$$

Comparison with the exact blackbody radiation formula shows that the emission (4.7a) is smaller than that of a blackbody by the "opalescence" factor

$$q=\tfrac{32}{105}(M/R)c_0^{\,3}. \tag{4.7b}$$

The order of (4.7b) is given by the ratio of the Schwarzschild radius and the radius of the cylinder multiplied with the third power of the ratio of the velocity of sound and the velocity of light. Therefore in the case of usual solid materials the value of (4.7b) is so small that the order of the emission power (4.7a) lies under the observational limit. Only in the case of dense astrophysical objects is the value of the opalescence factor q of such an order that the thermal gravitational radiation power would be comparable with the electromagnetic one; however, the application of the results (4.3) up to (4.7a) to astrophysical objects seems to be questionable because of the very special shape of the material body.

In spite of this objection we give finally a rough estimation of the thermal gravitational luminosity of a "young" neutron star, inside of which there exists a neutron fluid or even a neutron lattice. Assuming for the mass $M=M_\odot$ correlated with a radius $a\simeq 10$ km and a density $\rho_0\simeq 10^{14}$ g/cm^3, one finds that the velocity of sound $c_0\simeq 0.1c$ and the Debye temperature $T_D\simeq 5\times10^{11}$ K. For an inner temperature of the neutron star of the order of $T=10^9$ K (see, e.g., Helfand *et al.*[14]) the condition $T\ll T_D$ is fulfilled and the Stefan-Boltzmann laws (4.6) and (4.7a) can be applied. To do this we approximate roughly the linear dimensions R and L of the cylinder by the radius a of the star. Then the opalescence factor (4.7b) has the magnitude

$$q\simeq 5\times10^{-5}. \tag{4.8}$$

This means that the star is transparent for gravitational radiation, so that the radiation calculated according to (4.6) and (4.7a) with the *inner* temperature T leaves the star immediately. So we find at the surface a gravitational radiation flux density

$$S\simeq 10^{27}\ \mathrm{erg/sec\,cm^2} \tag{4.9a}$$

and a total gravitational luminosity

$$P \simeq 10^{40} \text{ erg/sec}. \tag{4.9b}$$

In the case of an effective surface temperature of the neutron star of $T_{eff} \simeq 10^7$ K the electromagnetic radiation power amounts to

$$P_{em} \simeq 10^{37} \text{ erg/sec}. \tag{4.10}$$

In view of the sensitive dependence of the gravitational radiation intensity on the inner temperature of the star, the results (4.9b) and (4.10) may be considered of the same order, so that the thermal gravitational radiation should not be neglected for the cooling process of a neutron star.

V. ABSORPTION CROSS SECTION

Finally we take the occasion to say that the foregoing considerations allow also the calculation of the resonance integral of the absorption cross section for a single mode of frequency ω_n of the thin cylinder in the high-frequency limit. Avoiding the addition of the contributions of all frequencies in (4.4) we obtain the energy emission of the single frequency mode ω_n in the case of thermal excitation:

$$\frac{dE_n^{(e)}}{dt} = \frac{16}{105} M c_0^4 \frac{\hbar \omega_n^3}{e^{\hbar\omega_n/kT} - 1}. \tag{5.1}$$

On the other hand, the energy absorption of the mode ω_n in the case of incident blackbody gravitational radiation is given by

$$\frac{dE_n^{(a)}}{dt} = \int \sigma(\omega, \omega_n) d\omega \frac{\hbar\omega_n^3/\pi^2}{e^{\hbar\omega_n/kT} - 1}. \tag{5.2}$$

In the thermal equilibrium the expressions (5.1) and (5.2) must be equal, which results in

$$\int \sigma(\omega, \omega_n) d\omega = \frac{16\pi^2}{105} M c_0^4, \tag{5.3}$$

whereas the corresponding result for the low-frequency limit given by Misner *et al.*[6] (p. 1035) is proportional to $Mc_0^4\omega_n^{-2}L^{-2}$ [for the odd modes only, cf. (3.14)]. As to be expected from the blackbody character of the emission power (4.4), the resonance integral of the absorption cross section does not depend on the frequency ω_n in the high-frequency limit.

With the help of "Exercise 37.10" in the textbook of Misner *et al.*[6] it is easy to show that Eq. (5.3) follows also from absorption calculations with the retarded Riemann tensor in the geodesic deviation equation. This result suggests that *generally* in the high-frequency limit the tidal force in the geodesic deviation equation is to be modified by retardation of the Riemann tensor only.

VI. FINAL REMARKS

It remains to estimate the precise range of physical realization of our investigations in the case of thermal excitation of the phonons. Insertion of (1.1) into (2.22) results in the condition for a "thin cylinder":

$$R \ll \frac{\hbar c_2}{kT}\left(\frac{3\lambda + 2\mu}{2\lambda + \mu}\right)^{1/2}. \tag{6.1}$$

Taking additionally the high-temperature condition $\lambda\!\!\!^{-}_{max} \ll L$ [compare (1.1)],

$$L \gg \frac{\hbar}{kT}, \tag{6.2}$$

the order of the dimensions of R and L of the cylinder is determined by the temperature T alone. Combining (6.1) and (6.2) we get finally

$$\frac{R}{L} \ll c_2\left(\frac{3\lambda + 2\mu}{2\lambda + \mu}\right)^{1/2} (\ll 1). \tag{6.3}$$

Evidently the last relation follows also immediately from (2.22) and $\omega_n^{-1} = \lambda\!\!\!^{-}_n \ll L$; it is the condition for the thin cylinder in the high-frequency limit.

Whereas the condition (6.2) is not a strong restriction for a large temperature range and (6.3) can still be satisfied, the condition (6.1) makes serious difficulties for not too low temperatures in the case of usual materials. Because their velocity of sound does not exceed the value of 10^6 cm/sec essentially, the right-hand side of (6.1) is only of the order of 10^{-5} cm even for $T \simeq 10$ K. Also in the case of neutron matter of a neutron star the condition (6.1) can be hardly satisfied. In spite of this we have risked an application of our result to a young neutron star (at the end of Sec. IV), which is suggested by the fact that the results (4.7a) and (4.7b) seem to be of a quite general structure.

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[5]Greek indices run from 0 to 3, Latin indices from 1 to 3. The partial derivative with respect to the coordinate x^α is denoted by $|\alpha$. Lifting and lowering of indices is performed with the Minkowski metric.

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[12]This condition is the same under which the usual cylinder calculations are meaningful, see, e.g., G. Pizzella, Riv. Nuovo Cimento 5, 369 (1975).

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