Homogeneous cosmological model in general scalar-tensor theory

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Some general results on spatially homogeneous cosmological models are obtained in the general scalar-tensor theory proposed by Nordtvedt and Barker. Cosmological equations for a Bianchi type-I empty universe are solved and the behavior of the model is discussed. It is found that the universe expands from the initial singularity of zero volume and then contracts back. There is a significant difference between the nature of the singularity in this model and that in the corresponding Kasner universe in Einstein's theory.

I. INTRODUCTION

In view of the recent experimental evidence it is argued that if the Brans-Dicke theory of gravitation[1] is to be a correct theory, the value of the parameter ω in this theory has to be as large as, or even more than, 30.[2] With such a large value for ω it is difficult to distinguish between the Brans-Dicke theory of gravitation and the general theory of relativity, at least from their consequences. On the other hand, since there is no *a priori* reason to exlude the introduction of any long-range scalar field in the evolution of the universe, which might be quite important at some epoch, one may explore the possibility of a general scalar-tensor theory with ω as a time-dependent function. Within the framework of Nordtvedt's[3] general scalar-tensor theory Barker[4] proposed a particular ω-ϕ relation in the form $\omega=(4-3\phi)/2(\phi-1)$, which has a consequence that the local gravitational constant in the Newtonian approximation does not change with time. In fact, there is no strong experimental evidence so far in favor of varying G. This theory is quite promising in the sense that ω may be large at the present time to be consistent at least up to the post-Newtonian approximation to the general theory of relativity, but in the past or in the future perhaps ω might be small enough to give quite different results. Barker[4] obtained an analytical solution to the empty universe with $k=0$ in the Robertson-Walker model, which is isotropic as well as spatially homogeneous. He at the same time expressed doubt about the existence of any other analytical solution for the cosmological model in this theory. There is, however, no real basis for such a speculation and indeed there exists an exact cosmological solution for the empty Bianchi type-I[5] universe, which is anisotropic but spatially homogeneous and admits three Abelian groups of translations along three spatial directions. It is a more generalized situation, of which Barker's model is only a special case. This model starts from the singularity of zero proper volume, increases to a maximum, and then subsequently collapses in finite time. It is interesting to note that in this case near the singularity there is either a collapse or explosion in all three spatial directions, unlike a disk-like singularity for Kasner's empty universe[6] in Einstein's theory. The anisotropy as well as the expansion scalar are indefinitely large near the singularity.

In Sec. II we obtained two interesting results valid for any spatially homogeneous cosmological model in the framework of the general scalar-tensor theory of Nordtvedt. In Sec. III we consider a Bianchi type-I line element and set up the field equations for such a metric in the general scalar-tensor theory and in Sec, IV we obtained solutions for empty space with Barker's choice $w=(4-3\phi)/2(\phi-1)$. Finally, in Sec. V we analyzed the behavior of our model throughout its evolution.

II. HOMOGENEOUS COSMOLOGICAL MODEL IN THE GENERAL SCALAR-TENSOR THEORY

The field equations in the general scalar-tensor theory of Nordtvedt[3] can be expressed as

$$G_{\mu\nu}=-\frac{8\pi G_0}{c^4\phi}T_{\mu\nu}-\frac{\omega}{\phi^2}(\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}) -\frac{1}{\phi}(\phi_{,\mu;\nu}-g_{\mu\nu}\Box\phi) \tag{2.1}$$

and

$$\Box\phi=\frac{1}{(3+2\omega)}\left(\frac{8\pi G_0}{c^4\phi}-\phi_{,\lambda}\phi^{,\lambda}\frac{d\omega}{d\phi}\right). \tag{2.2}$$

In the post-Newtonian approximation the gravitational constant is

$$G=\frac{G_0}{\phi}\;\frac{4+2\omega}{3+2\omega}\;.$$

In the above G_0 is an arbitrary constant having no effect on the physical result. We shall choose as did Barker[4] $G_0=G_{\text{today}}$, so that ϕ is a dimensionless scalar field. The semicolons represent covariant derivatives. It can be shown that we have, as a consequence of (2.1) and (2.2), the relation

$$T^{\mu\nu}{}_{;\nu}=0\,. \tag{2.3}$$

Now if we take the energy-momentum tensor of a perfect fluid as

$$T_{\mu\nu}=(\rho+p)v_\mu v_\nu+pg_{\mu\nu},\quad v^\mu v_\mu=-1 \tag{2.4}$$

we have

$$\dot v^\mu=\frac{p_{,\nu}(g^{\mu\nu}-v^\mu v^\nu)}{(\rho+p)} \tag{2.5}$$

and

$$\theta=v^\mu{}_{;\mu}=-\frac{\dot\rho}{(\rho+p)}\,, \tag{2.6}$$

where a dot indicates the covariant derivative along the world line. Now if our universe is spatially homogeneous and we choose t lines as the world lines of matter (comoving coordinates) with the homogeneous varieties as the t-constant spaces, our line element can be written after a suitable time transformation as

$$ds^2=-dt^2+2g_{0i}\,dt\,dx^i+g_{ik}dx^idx^k\,. \tag{2.7}$$

On account of spatial homogeneity $\phi_{,i}=p_{,i}=\rho_{,i}=0$ and because of comoving coordinates $\dot\phi=d\phi/dt$. Here we have Latin indices with values 1–3 and Greek indices with values 1–4. It is thus possible to write (2.2) as

$$(g^{\mu0}\dot\phi\sqrt{-g})_{,\mu}=\frac{\sqrt{-g}}{(2\omega+3)}\left[\frac{8\pi G_0}{c^4}(\rho-3p)-\phi_{,\lambda}\phi^{,\lambda}\frac{d\omega}{d\phi}\right]. \tag{2.8}$$

For a stationary universe the left-hand side of (2.8) is zero for obvious reasons and one thus has

$$\frac{8\pi G_0}{c^4}(\rho-3p)=g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\frac{d\omega}{d\phi}\,. \tag{2.9}$$

In Nordtvedt's theory we have $d\omega/d\phi\neq0$ and also $g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\neq0$ for a nonvanishing scalar field and thus one can arrive at the result $(\rho-3p)\neq0$. This result leads us to conclude that spatially homogeneous stationary perfect-fluid cosmological models in Nordtvedt's general scalar-tensor theory cannot include the radiation-filled universe or the empty universe at the limit.

If the rotation vanishes the homogeneous varieties are orthogonal to t lines and one can always reduce g_{0i} in (2.7) to zero by a suitable coordinate transformation. Then equation (2.8) gives

$$\dot\phi l^3=\int\frac{l^3}{(2\omega+3)}\left[\frac{8\pi G_0}{c^4}(\rho-3p)-\phi_{,\lambda}\phi^{,\lambda}\frac{d\omega}{d\phi}\,dt\right], \tag{2.10}$$

where l^3 stands for $\sqrt{-g}$. Again if we have a spatially homogeneous universe with both ρ and p constant on the homogeneous varieties then[7]

$$l=S(t)\cdot W(x^i)\,. \tag{2.11}$$

It is therefore possible to write (2.10) in view of (2.11) as

$$\dot\phi S^3(t)=\frac{8\pi G_0}{c^4}\int\frac{(\rho-3p)S^3(t)}{(2\omega+3)}dt-\int\frac{(S^3\dot\phi)d\phi/dt}{(2\omega+3)}dt\,. \tag{2.12}$$

when $\rho=3p$ (radiation universe) or $\rho\to0$, $p\to0$ (empty universe) we have from (2.12)

$$\dot\phi S^3=-\int\frac{\dot\phi S^3}{(2\omega+3)}d\omega\,. \tag{2.13}$$

The equation (2.13) is equivalent to a relation

$$\frac{dx}{d\omega}=-\frac{x}{(2w+3)}\,,$$

where x stands for $(\dot\phi S^3)$ and this in turn, when integrated, leads us to

$$x=\dot\phi S^3=\frac{A}{(2\omega+3)^{1/2}}\,, \tag{2.14}$$

A being an integration constant. It may be noted that the relation (2.14) holds for any spatially homogeneous universe, which is either radiation filled or empty in Nordtvedt's general scalar-tensor theory with varying ω.

III. FIELD EQUATIONS FOR BIANCHI TYPE-I COSMOLOGICAL MODEL

The line element for the spatially homogeneous Bianchi type-I cosmological model can be written as

$$ds^2=-dt^2+e^{2\gamma}dx^2+e^{2\theta}dy^2+e^{2\psi}dz^2\,. \tag{3.1}$$

The field equations (2.1) in empty space ($\rho=p=0$) for the metric (3.1) can be explicitly written as

$$G^0{}_0=\frac{9}{2}\left(\frac{\dot R}{R}\right)^2-\tfrac12(\dot\gamma^2+\dot\theta^2+\dot\psi^2)=\frac{\omega}{2}\left(\frac{\dot\phi}{\phi}\right)^2+\frac{\ddot\phi}{\phi}+\frac{\Box\phi}{\phi}\,, \tag{3.2}$$

$$G^1{}_1=\ddot\theta+\ddot\psi+\frac{3}{2}\frac{\dot R}{R}(\dot\theta+\dot\psi-\dot\gamma)+\tfrac12(\dot\gamma^2+\dot\theta^2+\dot\psi^2)=-\frac{\omega}{2}\left(\frac{\dot\phi}{\phi}\right)^2+\dot\gamma\frac{\dot\phi}{\phi}+\frac{\Box\phi}{\phi}\,, \tag{3.3}$$

$$G^2{}_2=\ddot\gamma+\ddot\psi+\frac{3}{2}\frac{\dot R}{R}(\dot\gamma+\dot\psi-\dot\theta)+\tfrac12(\dot\gamma^2+\dot\theta^2+\dot\psi^2)=-\frac{\omega}{2}\left(\frac{\dot\phi}{\phi}\right)^2+\dot\theta\frac{\dot\phi}{\phi}+\frac{\Box\phi}{\phi}\,, \tag{3.4}$$

$$G^3{}_3 = \ddot\gamma + \ddot\theta + \frac{3}{2}\frac{\dot R}{R}(\dot\gamma + \dot\theta - \dot\psi) + \tfrac{1}{2}(\dot\gamma^2 + \dot\theta^2 + \dot\psi^2)$$
$$= -\frac{\omega}{2}\left(\frac{\dot\phi}{\phi}\right)^2 + \dot\psi\frac{\dot\phi}{\phi} + \frac{\Box\phi}{\phi} \qquad (3.5)$$

and the wave equation (2.2) is given by

$$\Box\phi \equiv -\ddot\phi - 3\frac{\dot R}{R}\dot\phi = \frac{1}{(3+2\omega)}\dot\phi^2\frac{d\omega}{d\phi}\,. \qquad (3.6)$$

In this and the following sections the "dot" symbol will now stand for the time derivative and R^3 will stand for $\exp(\gamma + \theta + \gamma)$.

Equation (3.6) can be integrated to give

$$\dot\phi = \frac{A}{(2\omega+3)^{1/2}}\frac{1}{R^3}\,, \qquad (3.7)$$

which is equivalent to the general expression (2.14) obtained previously. The trace of (2.1) for the metric (3.1) can be written as

$$3\left(\frac{\dot R}{R}\right)^2 + \frac{6\ddot R}{R} + \dot\gamma^2 + \dot\theta^2 + \dot\gamma^2 = -\omega\left(\frac{\dot\phi}{\phi}\right)^2 + 3\frac{\Box\phi}{\phi}\,. \qquad (3.8)$$

Combining (3.8) with (3.2) one has

$$12\left(\frac{\dot R}{R}\right)^2 + 6\frac{\ddot R}{R} = 2\frac{\ddot\phi}{\phi} + 5\frac{\Box\phi}{\phi}\,. \qquad (3.9)$$

Defining

$$f \equiv \frac{1}{(3+2\omega)}\frac{d\omega}{dt} \qquad (3.10)$$

and using (3.6) in (3.9) we arrive at the equation

$$-\frac{\dddot\phi}{\dot\phi} + 2\left(\frac{\ddot\phi}{\dot\phi}\right)^2 - \frac{\ddot\phi}{\phi} + f\left(2\frac{\ddot\phi}{\dot\phi} - \frac{5}{2}\frac{\dot\phi}{\phi}\right) - \dot f + f^2 = 0\,. \qquad (3.11)$$

The substitution $\phi/\dot\phi = u$ transforms (3.11) into a linear second-order differential equation for u,

$$\ddot u - 2f\dot u + (f^2 - \dot f)u - \tfrac{1}{2}f = 0\,. \qquad (3.12)$$

The general solution of (3.12) for u is

$$u = (3+2\omega)^{1/2}\left[c_1 t + c_2 - \int\frac{dt}{2(3+2\omega)^{1/2}}\right], \qquad (3.13)$$

where c_1 and c_2 are constants of integration. Further, from the field equations (3.3) to (3.5), after subtracting one from the other and integrating once in each case, it is not difficult to obtain the following relations:

$$\frac{1}{\phi} = (\dot\theta - \dot\psi)D_1R^3 = (\dot\gamma - \dot\psi)D_2R^3 = (\dot\gamma - \dot\theta)D_3R^3\,. \qquad (3.14)$$

The relations (3.14) are obtained on the assumption that all of $\dot\gamma$, $\dot\theta$, and $\dot\psi$ are different from one another, or in other words, the universe is anisotropic. The constants D_1, D_2, D_3 are arbitrary up to the condition

$$D_2D_3 + D_1D_2 = D_1D_3\,. \qquad (3.15)$$

IV. EXACT SOLUTIONS

It is interesting to note that for empty space with $T_{\mu\nu} = 0$ the wave equation (2.2) is a consequence of the field equations (2.1). So for the Bianchi type-I model one has four independent field equations (3.2)–(3.5) and five unknowns γ, θ, ψ, ϕ, and ω. One has, therefore, the freedom to reduce the number of independent variables to four by assuming a relationship between any two of them. Barker[4] assumed a relationship between ω and ϕ, so that the local gravitational constant G in the Newtonian approximation remains independent of time. We make here the same choice so that

$$\omega = \frac{4-3\phi}{2(\phi-1)} \qquad (4.1)$$

and

$$G = G_0\,. \qquad (4.2)$$

Differentiating (3.13) once with respect to time and using (4.1) we have

$$(\phi-1)^{1/2}\left(\frac{3}{2} - \frac{\phi\ddot\phi}{\dot\phi^2}\right) + \frac{1}{2}\frac{\phi}{(\phi-1)^{1/2}} = c_1\,. \qquad (4.3)$$

Upon substitution of Ω^2 for $(\phi - 1)$, Eq. (4.3) takes a simpler form,

$$\frac{\ddot\Omega}{\dot\Omega} + \left(\frac{2c_1 - 3\Omega}{\Omega^2+1}\right)\dot\Omega = 0\,, \qquad (4.4)$$

which on first integration yields

$$\dot\Omega = c_3(\Omega^2+1)^{3/2}\exp(2c_1\arctan\Omega) \qquad (4.5)$$

and the second integration gives us the solution

$$\frac{\exp[2c_1\arctan\Omega]}{(4c_1{}^2+1)}[2c_1\cos(\arctan\Omega) + \sin(\arctan\Omega)] = c_3t + c_4\,. \qquad (4.6)$$

c_3 and c_4 in (4.6) are integration constants. The equation (4.6) is in transcendental form preventing us from obtaining algebraically ϕ as a function of time. Nonetheless, one can analyze the solution without actually giving its explicit form and that is what we do in the following section.

We now express the time derivatives of the metric coefficients in terms of the scalar field ϕ and its derivatives. In principle it is possible to obtain the explicit forms for the metric as functions of time in view of the relation (4.6). Using (3.14), (4.1), and (4.3) in the wave equation (3.6) it is not difficult to obtain

$$3\dot{\psi}=\frac{c_1\dot{\phi}}{\phi(\phi-1)^{1/2}}-\frac{3}{2}\frac{\dot{\phi}}{\phi}-\frac{\dot{\phi}}{\phi(\phi-1)^{1/2}}\frac{1}{A}\left(\frac{1}{D_1}+\frac{1}{D_2}\right). \tag{4.7}$$

Again Eq. (3.2) along with (3.14) and (4.1) yields a second-order polynomial in $\dot{\psi}$, and after solving for it we obtain

$$3\dot{\psi}=-\frac{3}{2}\frac{\dot{\phi}}{\phi}-\frac{\dot{\phi}}{\phi(\phi-1)^{1/2}}\frac{1}{A}\left(\frac{1}{D_1}+\frac{1}{D_2}\right)\pm\frac{1}{2}\left\{\left[\frac{2\dot{\phi}}{\phi(\phi-1)^{1/2}}\frac{1}{A}\left(\frac{1}{D_1}+\frac{1}{D_2}\right)+3\frac{\dot{\phi}}{\phi}\right]^2 -12\left[\left(\frac{\dot{\phi}}{\phi(\phi-1)^{1/2}A}\right)^2\frac{1}{D_1D_2}-\left(\frac{4-3\phi}{2\phi-2}\right)\frac{1}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 +\frac{\dot{\phi}^2}{\phi^2(\phi-1)^{1/2}}\frac{1}{A}\left(\frac{1}{D_1}+\frac{1}{D_2}\right)\right]\right\}^{1/2}. \tag{4.8}$$

Comparing (4.7) and (4.8) we obtain the relationship between the constants

$$c_1^{\,2}=\frac{1}{A^2}\left(\frac{1}{D_1^{\,2}}+\frac{1}{D_2^{\,2}}-\frac{1}{D_1D_2}\right)+\frac{3}{4}. \tag{4.9}$$

It is clear from (4.9) that one cannot have $c_1=0$ for all the constants A, D_1, and D_2 to be real quantities.

V. THE BEHAVIOR OF THE MODEL

Now since we have substituted Ω^2 for $(\phi-1)$ in Sec. IV, it is evident that $\Omega\to\infty$ when the scalar field $\phi\to\infty$ and at this stage $\arctan\Omega\to\pi_2$, which is a finite quantity. Hence from (4.5), when $\Omega\to\infty$, we have

$$\dot{\Omega}\sim\Omega^3. \tag{5.1}$$

Again from (3.7) using (4.1) and expressing ϕ in terms of Ω one has

$$2\dot{\Omega}=\frac{A}{R^3}. \tag{5.2}$$

So when $\Omega\to\infty$ we have, in view of (5.1), $R^3\sim 1/\Omega^3$ and so the proper volume contracts to zero. At this stage the curvature invariants such as Ricci scalar $g^{\mu\nu}R_{\mu\nu}$ tend towards infinity. The expansion scalar and the anisotropy can also be calculated and their behaviors are worth investigating. Defining the expansion scalar θ and the anisotropy $|\sigma|$ as[8]

$$\theta=\frac{3\dot{R}}{R} \tag{5.3}$$

and

$$\sigma^2=\frac{1}{12}\left[\left(\frac{\dot{g}_{11}}{g_{11}}-\frac{\dot{g}_{22}}{g_{22}}\right)^2+\left(\frac{\dot{g}_{22}}{g_{22}}-\frac{\dot{g}_{33}}{g_{33}}\right)^2+\left(\frac{\dot{g}_{33}}{g_{33}}-\frac{\dot{g}_{11}}{g_{11}}\right)^2\right], \tag{5.4}$$

one has near the singularity ($\Omega\to\infty$), in view of (5.2) and (3.14),

$$\theta\sim\Omega^2 \text{ and } |\sigma|\sim\Omega.$$

So both these scalars become indefinitely large, but the ratio of anisotropy to expansion is vanishingly small at the singularity. An important difference in the behavior of our model from that in the corresponding Bianchi type-I model in the Brans-Dicke theory[9] is that we have the singularity for $\phi\to\infty$, while in the Brans-Dicke theory there is a singularity either when $\phi\to\infty$ or when $\phi\to 0$.

If we now write Eq. (4.5) in terms of ϕ, it appears as

$$\dot{\phi}=2(\phi-1)^{1/2}\phi^{3/2}c_3\exp[-2c_1\arctan(\phi-1)^{1/2}]. \tag{5.5}$$

It is clear from (5.5) that $\dot{\phi}=0$ when $\phi=1$ and so $\arctan(\phi-1)^{1/2}=0$. The scalar field ϕ has a minimum with the parameter $\omega\to\infty$ at this point, after which it again increases to infinity. On the other hand, since $\dot{\Omega}$ has no zero at any stage, there is no turning point for Ω. If Ω starts from infinity, it can decrease to zero and subsequently reaches minus infinity, which, however, corresponds to indefinitely large positive magnitude for ϕ. When ϕ has a minimum that is $\phi=1$, we have $\Omega=0$. In this limit $\dot{\Omega}$ approaches c_3 and in view of (5.2) R^3 remains finite indicating that the proper volume has a finite magnitude. At this stage, since the parameter ω becomes infinitely large, there is little difference from the general theory of relativity. Again from (4.4), (4.5), and (5.2) we have

$$\frac{3\dot{R}}{R}=c_3(2c_1-3\Omega)(\Omega^2+1)^{1/2}\exp(-2c_1\arctan\Omega). \tag{5.6}$$

If we assume that the universe explodes from the initial singularity, where $R^3\to 0$ and $\Omega\to\infty$, we are to choose our constant c_3 to be of negative magnitude. It is interesting to note that the scalar field ϕ has a turning point when $\phi=1$, that is, at the epoch when $\Omega=0$, but the expansion of the universe does not halt at this stage. The universe has a maximum volume only when $\Omega=\frac{2}{3}c_1$ and, since in view of (4.9) c_1 is not allowed to assume zero value, the minimum of the scalar field never occurs at the same instant as the maximum of the model. When $\dot{R}=0$ we have $\Omega>0$ if $c_1>0$ and $\Omega<0$ if $c_1<0$. So the expansion halts before or after the scalar field ϕ reaches its minimum according to whether the arbitrary constant c_1 is positive or negative.

Finally, it is not difficult to observe that near the singularity ($\Omega\to\infty$), we have from (4.7) $\dot{\psi}\sim\Omega^2$, whereas the differences $(\dot{\theta}-\dot{\psi})$ or $(\dot{\gamma}-\dot{\psi})$ or $(\dot{\gamma}-\dot{\theta})$ go to infinity as only the first power of Ω. It means that $\dot{\theta}$, $\dot{\gamma}$, $\dot{\psi}$ all will have the same sign at the singularity and so the model can collapse to a point singularity unlike in Einstein's case, where the corresponding Kasner's empty-space universe collapses to a disk-like singularity. In the latter case one of the dimensions increases indefinitely while the other two dimensions collapse.

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