Comment on gauge theories without anomalies

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We study the problem of the triangular anomaly for different irreducible representations of the SU(N) group and show that the anomaly of a given representation of the SU(N) group can be expressed as the sum of the anomalies of the irreducible representation of SU(N-K) which are contained in it. Using this property we study different irreducible representations of SU(N) for $N<16$ and dimension <1000.

One problem facing the unified gauge theory based on non-Abelian groups is the possible occurrence of the triangular anomalies which may spoil the renormalizability of the theory. Banks and Georgi and, independently, Okubo[1] gave a general formula for calculation of the anomaly number of a general irreducible representation of the SU(N) group. Adopting notations of Banks and Georgi their results are stated as follows. Given the Young tableaux of a certain irreducible representation of SU(N), let (q_i-1) be the number of columns with i boxes. In terms of the q_i's, the anomaly of the given irreducible representation is

$$A(q,N)=D(q,N)\sum_{i,j,k=1}^{N-1} a_{ijk}\,q_i\,q_j\,q_k\ , \tag{1}$$

where $D(q,N)$ is the dimension of the representation and a_{ijk} is completely symmetric in i,j,k and for $i\leqslant j\leqslant k$

$$a_{ijk}\equiv\frac{2(N-3)!}{(N+2)}\,i\,(N-2j)(N-k)\ . \tag{2}$$

Some useful results for special cases are the following.

(a) The completely symmetric representation with m boxes in the Young tableau has the anomaly number

$$A(m,N)=\frac{(N+m)!\,(N+2m)}{(N+2)!\,(m-1)!}\ . \tag{3}$$

(b) The completely antisymmetric case has

$$A(m,N)=\frac{(N-3)!(N-2m)}{(N-m-1)!(m-1)!}\ . \tag{4}$$

(c) For SU(4),

$$A(q,4)=\frac{D(q,4)}{60}\,(q_1-q_3)(q_1+q_3)(q_1+2q_2+q_3)\ . \tag{5}$$

In the case of irreducible representations of SU(N) ($N>5$) formula (1) is rather cumbersome for practical evaluation. The purpose of this paper is to give a prescription for obtaining the anomaly number for a given irreducible representation which, for $n>5$, is more convenient than using the general formula (1).

Our prescription is based on the following theorem. For an irreducible representation (q,N) of the SU(N) group, the anomaly number $A(q,N)$ is

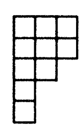
$$A(q,N)=\sum_{\alpha} C_\alpha A(q_\alpha,\ N-k)\ , \tag{6}$$

where C_α is the multiplicity of representations $(q_\alpha,\ N-k)$ of SU($N-k$) which are contained in (q,N). We can prove this theorem in the following way.

Let G_i be any non-Abelian proper subgroup of group G. Then if the T_a are the generators of G, the T_a will clearly be block diagonal in the representation space, since T_a will not mix one irreducible representation of G_i with another. Hence Eq. (6) follows directly from definition of the anomaly.

So, in practice it would be much more convenient to go from SU(N) to a smaller group such as SU(4) and calculate the anomaly from formula (6), provided, of course, one knows the multiplicities.

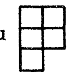
We illustrate the method by example. Consider an irreducible representation,

(5, 3, 2) =  of SU(7) .

It contains[2] the following representations of SU(6):

(5, 3, 2) (5, 3, 1) (5, 2, 2) (4, 3, 2) (5, 2, 1)
(a) (b) (c) (d) (e)

(4, 3, 1) (4, 2, 2) (4, 2, 1)
(f) (g) (h)

TABLE I. In this Table we use the following notation: the set of numbers $\underset{4}{(32)}$ stands for the Young tableau , which is contained 4 times in the given representation.

Representations of SU(5)

Representation	Dimension	Anomalies	Representations of SU(3) contained in given representation				
(21)	40	16	(21) 1	(20) 2	(11) 2	(10) 4	
(22)	50	15	(22) 1	(21) 2	(11) 3	(20) 1	(10) 2
(31)	45	6	(10) 3	(21) 2	(20) 4	(11) 1	
(32)	75	0	(20) 3	(10) 3	(22) 2	(21) 4	(11) 2
(41)	24	0	(10) 2	(21) 1	(20) 2		

Representations of SU(6)

Representation	Dimension	Anomalies	Representations of SU(5) contained in given representation			
(21)	70	27	(21) 1	(20) 1	(11) 1	(10) 1
(22)	105	40	(22) 1	(21) 1	(11) 1	
(31)	105	22	(31) 1	(30) 1	(21) 1	(20) 1
(32)	210	37	(32) 1	(31) 1	(22) 1	(21) 1
(33)	175	0	(33) 1	(32) 1	(22) 1	
(41)	84	4	(41) 1	(40) 1	(31) 1	(30) 1
(42)	189	0	(42) 1	(41) 1	(32) 1	(31) 1
(51)	35	0	(10) 1	(41) 1	(40) 1	

Representations of SU(7)

Representation	Dimension	Anomalies	Representations of SU(3) contained in given representation								
(21)	112	40	(21) 1	(20) 2	(11) 2	(10) 4					
(22)	196	77	(22) 1	(21) 2	(11) 3	(20) 1	(10) 2				
(31)	210	51	(31) 1	(30) 2	(21) 2	(20) 4	(11) 1	(10) 2			
(32)	490	126	(32) 1	(31) 2	(22) 2	(21) 4	(30) 1	(20) 2	(11) 2	(10) 1	
(33)	490	77	(33) 1	(32) 2	(22) 3	(31) 1	(21) 2	(11) 1			
(41)	224	24	(41) 1	(40) 2	(31) 2	(30) 4	(21) 1	(20) 2			
(42)	588	63	(42) 1	(41) 2	(32) 2	(31) 4	(40) 1	(30) 2	(22) 1	(21) 2	(20) 1

TABLE I. (*Continued*)

Representations of SU(7)

Representation	Dimension	Anomalies	Representations of SU(5) contained in given representation							
(43)	784	0	(43)	(42)	(33)	(32)	(41)	(31)	(22)	(21)
			1	2	2	4	1	2	2	1
(51)	140	1	(10)	(41)	(40)	(31)	(30)			
			1	2	4	1	2			
(52)	392	0	(20)	(10)	(42)	(41)	(40)	(32)	(31)	(30)
			1	2	2	4	2	1	2	1
(61)	48	0	(10)	(41)	(40)					
			2	1	2					

Representations of SU(8)

Representation	Dimension	Anomalies	Representations of SU(5) contained in given representation							
(21)	168	55	(21)	(20)	(11)	(10)				
			1	3	3	9				
(22)	336	128	(22)	(21)	(11)	(20)	(10)			
			1	3	6	3	8			
(31)	378	96	(31)	(30)	(21)	(20)	(11)	(10)		
			1	3	3	9	3	9		
(32)	1008	294	(32)	(31)	(22)	(21)	(30)	(20)	(11)	(10)
			1	3	3	9	3	9	8	9
(41)	504	75	(41)	(40)	(31)	(30)	(21)	(20)	(11)	(10)
			1	3	3	9	3	9	1	3
(51)	420	20	(10)	(41)	(40)	(31)	(30)	(21)	(20)	
			1	3	9	3	9	1	3	
(61)	216	−3	(10)	(41)	(40)	(31)	(30)			
			3	3	9	1	3			
(62)	720	0	(20)	(10)	(42)	(41)	(40)	(32)	(31)	(30)
			3	9	3	9	9	1	3	3
(71)	63	0	(10)	(41)	(40)					
			3	1	3					

Representations of SU(9)

Representation	Dimension	Anomalies	Representations of SU(5) contained in given representation							
(21)	240	72	(21)	(20)	(11)	(10)				
			1	4	4	16				
(22)	540	195	(22)	(21)	(11)	(20)	(10)			
			1	4	10	6	20			
(31)	630	160	(31)	(30)	(21)	(20)	(11)	(10)		
			1	4	4	16	6	24		
(41)	1008	176	(41)	(40)	(31)	(30)	(21)	(20)	(11)	(10)
			1	4	4	16	6	24	4	16
(51)	1050	90	(10)	(41)	(40)	(31)	(30)	(21)	(20)	(11)
			5	4	16	6	24	4	16	1
(61)	720	8	(10)	(41)	(40)	(31)	(30)	(21)	(20)	
			4	6	24	4	16	1	4	
(71)	315	−8	(10)	(41)	(40)	(31)	(30)			
			6	4	16	1	4			
(81)	80	0	(10)	(41)	(40)					
			4	1	4					

TABLE I. (*Continued*)

Representations of SU(10)

Representation	Dimension	Anomalies	Representations of SU(5) contained in given representation						
(21)	330	91	(21)	(20)	(11)	(10)			
			1	5	5	25			
(22)	825	280	(22)	(21)	(11)	(20)	(10)		
			1	5	15	10	40		
(31)	990	246	(31)	(30)	(21)	(20)	(11)	(10)	
			1	5	5	25	10	50	
(71)	1155	−14	(10)	(41)	(40)	(31)	(30)	(21)	(20)
			10	10	50	5	25	1	5
(81)	440	−14	(10)	(41)	(40)	(31)	(30)		
			10	5	25	1	5		

Representations of SU(11)

Representation	Dimension	Anomalies	Representations of SU(5) contained in given representation				
(21)	440	112	(21)	(20)	(11)	(10)	
			1	6	6	36	
(91)	594	−21	(10)	(41)	(40)	(31)	(30)
			15	6	36	1	6

Representations of SU(12)

Representation	Dimension	Anomalies	Representation of SU(5) contained in given representation				
(21)	527	135	(21)	(20)	(11)	(10)	
			1	7	7	49	
(101)	780	−29	(10)	(41)	(40)	(31)	(30)
			21	7	49	1	7

Representations of SU(13)

Representation	Dimension	Anomalies	Representations of SU(5) contained in given representation				
(21)	728	160	(21)	(20)	(11)	(10)	
			1	8	8	64	
(111)	1001	−38	(10)	(41)	(40)	(31)	(30)
			28	8	64	1	8

Representation of SU(14)

Representation	Dimension	Anomalies	Representations of SU(5) contained in given representation			
(21)	910	187	(21)	(20)	(11)	(10)
			1	9	9	81

Now, notice that representations (a) and (f) of SU(6) are conjugate to each other, so they have equal and opposite anomaly numbers. Representations (b) and (d) are self-conjugate and will not contribute to the value of anomaly number. We can proceed further, taking only representations (c), (e), (g), and (h), and obtaining finally representations of SU(4).

We find the above method particularly useful in searching for anomaly-free linear combinations of representations which may be interesting as representations for fundamental fermions.

Recently[3] several grand unified models have been proposed, based on SU(N) groups with N greater than five and with fermions assigned to some anomaly-free set of the totally antisymmetric representation. However, the use of only totally antisymmetric representations is a limitation because *a priori* no convincing argument seems to exist at the moment against the existence of six-dimensional, eight-dimensional, etc., irreducible representations of color SU(3).

The method suggested here was used to study different irreducible representations of the SU(N)

groups for $N < 16$ with dimension $< 10^3$.

In Table I we listed all representations that we studied. We assumed that all grand unified theories based on SU(N) gauge groups should contain standard SU(5) theory,[4] so we also listed representations of the SU(5) contained in the representations of higher groups we investigated.

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