PHYSICAL REVIEW D VOLUME 23, NUMBER 8 15 APRIL 1981

Lie-algebra approach to symmetry breaking

J. T. Anderson*
National University, San Diego, California
(Received 13 March 1980)

A formal Lie-algebra approach to symmetry breaking is studied in an attempt to reduce the arbitrariness of Lagrangian (Hamiltonian) models which include several free parameters and/or *ad hoc* symmetry groups. From Lie algebra it is shown that the unbroken Lagrangian vacuum symmetry can be identified from a linear function of integers which are Cartan matrix elements. In broken symmetry if the breaking operators form an algebra then the breaking symmetry (or symmetries) can be identified from linear functions of integers characteristic of the breaking symmetries. The results are applied to the Dirac Hamiltonian of a sum of flavored fermions and colored bosons in the absence of dynamical symmetry breaking. In the partially reduced quadratic Hamiltonian the breaking-operator functions are shown to consist of terms of order g^2, g, and g^0 in the color coupling constants and identified with strong (boson-boson), medium strong (boson-fermion), and fine-structure (fermion-fermion) interactions. The breaking operators include a boson helicity operator in addition to the familiar fermion helicity and "spin-orbit" terms. Within the broken vacuum defined by the conventional formalism, the field divergence yields a gauge which is a linear function of Cartan matrix integers and which specifies the vacuum symmetry. We find that the vacuum symmetry is chiral SU(3)×SU(3) and the axial-vector-current divergence gives a PCAC (partially conserved axial-vector current)-like function of the Cartan matrix integers which reduces to PCAC for SU(2)×SU(2) breaking. For the mass spectra of the nonets $J^P=0^-,1/2^+,1^-$ the integer runs through the sequence 3,0, − 1, − 2, which indicates that the breaking subgroups are the simple Lie groups. Exact axial-vector-current conservation indicates a breaking sum rule which generates octet enhancement. Finally, the second-order breaking terms are obtained from the second-order spin tensor sum of the completely reduced quartic Hamiltonian. The breaking terms include the "anomalous" $^*F_{\mu\nu}F_{\mu\nu}$ term found by Schwinger, as well as fermion and boson helicity-breaking terms. Nonvanishing of the axial-vector-current divergence indicates the presence of solitons or, for electromagnetic coupling, of magnetic monopoles as the sources of strong fields.

I. INTRODUCTION

Several approaches to hadron symmetry-breaking problems have been studied from both general viewpoints and for specific models.[1-6] The breaking terms $\mathfrak{L}'$ in a Lagrangian $\mathfrak{L}_0+\mathfrak{L}'$ are interaction terms, for example, the spin-orbit term in atoms, or the $J^\mu A_\mu$ term for fermion–vector-boson interactions or the higher-order terms in a gauge-field Lagrangian. Then for $\mathfrak{L}_0$ symmetries SU(3), SU(3)×SU(3), or SU(2)×SU(2), the $\mathfrak{L}'$ terms have served to classify various models and assumptions of $\mathfrak{L}_0$. The interaction terms are related to level or mass splittings which are functions of the breaking parameters. In the SU(3)×SU(3) decomposition and in the σ model, the bag model, and related models, $\mathfrak{L}'$ contains several free parameters which are empirically fitted to yield, for example, nonet mass spectra. The number of parameters is usually representation-dependent and the physical bases of the parameters are not clear. These and related considerations have led to the observation that symmetry-breaking models that contain several free parameters are not unique and cannot distinguish among different $\mathfrak{L}_0$ symmetries. It follows that the assumption of an $\mathfrak{L}'$ with several free parameters is not satisfactory even when the fit to data is good.

Recent gauge-field theories have provided a more systematic approach. Renormalized theories offer the hope that all operators (functions) can be represented in terms of a single parameter.[7,8] Nevertheless, the unbroken gauge symmetry is an ansatz.

Without the assumption of the interaction terms or of the $\mathfrak{L}_0$ symmetry, it is yet possible to obtain significant results by considering the algebraic structure of a sum of Lie operators which correspond to observables of fermions and bosons. From Lie algebra it can be shown that unbroken symmetries of such systems can be identified from a linear function of integers which are Cartan matrix elements; let $A_\mu \in L^A(G^A)$ be operators of the algebra L^A of the Lie group G^A. Then operators $A_\sigma \in L^A$ exist such that

$$[A_\sigma,[A_\mu,A_\nu]]=g_{\sigma\mu}A_\nu=an_{\sigma\mu}A_\nu\,, \qquad (1)$$

where

$$n_{\sigma\mu}=\frac{2g_{\sigma\mu}}{g_{\mu\mu}} \qquad (2)$$

are the Cartan matrix elements of G^A and a is a normalization constant. The Cartan matrix elements are integers which unambiguously identify G^A. For the simple rank-2 groups, $n_{\sigma\mu}=0,-1$ and $n_{\mu\sigma}=0,-1,-2,-3$. For rank-2 groups if

$n_{\sigma\mu}=n_{\mu\sigma}=0$ then G^A is SU(2) ×SU(2) and for $n_{\sigma\mu}=n_{\mu\sigma}=-1$ the group is SU(3). These well-known results are discussed in several texts and papers.[9,10] We are here concerned with applications to the breaking part of an arbitrary Lagrangian. Let A_μ be replaced by a sum of operators $g_j A_\mu^j$. Then the right-hand side of (1) becomes a sum of terms $c_j n_{\sigma\mu}^j A_\nu^j$ belonging to the unbroken algebra and terms which are functions of the breaking operators and Cartan matrix integers which identify the breaking symmetries provided the operators form subalgebras.

We consider in this paper an approach to symmetry breaking based on Eq. (1). The result replaces the parameters in breaking terms with functions of Lie-group structure constants and invariants. The functions are linear in the Cartan matrix elements and identify the unbroken and broken symmetries. The results are applied to the problem of finding the symmetry of the Dirac Hamiltonian of several fermion and boson operators with interactions. The basic result is a first-order invariant of the medium-strong interactions which is a PCAC (partially conserved axial-vector current)-like function of Cartan matrix elements of the breaking symmetry. Vanishing of the invariant in the chiral limit together with degeneracy considerations specifies an unbroken SU(3) × SU(3) symmetry of the quadratic Hamiltonian; for SU(2) × SU(2) breaking the invariant reduces to the PCAC expression without pole dominance and saturation approximations. For the mass spectra of the low-lying nonets $J^P=0^-, \frac{1}{2}^+, 1^-$, the integer runs through the sequence 3, 0, −1, −2, which indicates that any breaking operator belongs to one of the simple Lie groups.

II. FORMULATION OF THE PROBLEM

In a broken symmetry if the breaking operators have an algebra then that algebra generates a linear function of integers which identify the breaking subgroup or subgroups and the unbroken group. Let

$$h_\alpha k_\mu^\alpha = k_\mu, \quad i g_j A_\mu^j = A_\mu$$

and (3)

$$\xi_\mu = k_\mu + A_\mu ,$$

where k_μ are fermion momentum operators of the algebra L^k of the group G^k, the α are flavor indices, $A_\mu \in L^A(G^A)$ are boson operators, and j are color indices. The g and h are dimensionless coupling constants and A_μ may be, for example, a sum of vector and axial-vector field operators. In the Dirac equation $D(\xi_\mu)\psi=0$ the vacuum symmetry is defined by the algebra $L^{\xi'}$ where ξ'_μ are obtained from

$$[\xi'_\mu, D]=0. \tag{4}$$

The ξ'_μ for which (4) holds give the vacuum-state Hamiltonian $D(\xi'_\mu)$ in the absence of Higgs fields.

Equation (4) is a set of coupled commutator equations for which the eigenvalues ξ'_μ are difficult to find and the breaking operators are not explicit. The Dirac Hamiltonian D is reducible and in the partially reduced quadratic block-diagonal form the breaking operators are contained in the linear spin terms $\sigma^{\mu\nu}[\xi_\sigma, [\xi_\mu, \xi_\nu]]$ which are of the form (1). In the completely reduced quartic diagonal form the breaking operators are quadratic spin terms which include the Pontrjagin density as well as helicity-breaking terms. Assuming (4), the symmetry of D is defined by the algebra $L^{\xi'}$ or

$$\xi_{\mu\nu} = [\xi_\mu, \xi_\nu] = F_{\mu\nu} + [A_\mu, A_\nu] + F_{\mu\nu}^\dagger + [k_\mu, k_\nu]$$
$$= G_{\mu\nu}^A + G_{\mu\nu}^k , \tag{5}$$

$$F_{\mu\nu} = k_\mu A_\nu - k_\mu A_\nu , \quad F_{\mu\nu}^\dagger = A_\mu k_\nu - A_\nu k_\mu , \tag{6}$$

$$F_{\mu\nu} + F_{\mu\nu}^\dagger = [k_\mu, A_\nu] + [A_\mu, k_\nu] . \tag{7}$$

The commutators in (7) are of order g and are defined interaction operators

$$[k_\mu, A_\nu] = c_{\mu\nu}^\rho C_\rho \in X^C , \tag{7$'$}$$

where $c_{\mu\nu}^\rho$ are not necessarily Lie-group structure constants and X^C is a set of operators which do not necessarily form an algebra. The right-hand side of (5) forms a hierarchy of terms in powers of the coupling constant to order g^2. If the g^2 terms belong to the vacuum-symmetry algebra then the g terms are breaking terms associated with medium-strong interactions[11] and the g^0 terms are responsible for the fine structure. If the operators of order g have an algebra then the medium-strong interactions can be identified with a definite symmetry. Then we have the basic questions: (1) Does the broken symmetry have an algebra? (2) If so, then do the C and $C^\dagger$ operators form subalgebras? If so then the medium-strong interactions can be identified with a definite symmetry group.

Although these questions cannot be answered generally, for conserved systems the first question can be answered in the affirmative. For example, a system of interacting currents J_μ^α, for which $k_\alpha^\mu J_\mu^\alpha = 0$ has a symmetry group. Similarly, a conserved Hamiltonian with interactions has a symmetry group. The answer to the second question is affirmative if the series

$$[C_\mu, k_\nu], [C_\sigma, [C_\mu, k_\nu]], \ldots \tag{8}$$

terminates[10] at an operator in $X^C \times X^{C\dagger}$.

III. VACUUM CONSIDERATIONS

The linear Dirac Hamiltonian is ideal for studying interactions as additional fermions and/or fields add linearly and each matrix element is a linear sum of fermion and field operators. Then for any number of fermions and bosons the vacuum states of the total momentum and the vacuum-state Hamiltonian are defined by (4). This representation consists of four coupled first-order commutator equations from which ξ_μ' is difficult to obtain; moreover, the breaking terms, although present in the linear form, are not explicit. In the quadratic Hamiltonian, first-order breaking terms appear in the spin tensor sum but the vacuum representation still consists of four commutator expressions which are coupled in pairs. Moreover, the second-order spin terms which include the $^*F_{\mu\nu}F_{\mu\nu}$ term do not appear in the quadratic Hamiltonian. The second-order spin terms do appear in the quartic representation which can be completely reduced. In this form the vacuum-state representation is a single expression defined by (4). As the chiral and helicity structures are not clear in the quartic form we consider both the quadratic and quartic forms of

$$D_{\text{quartic}} = D^+_{\text{quadratic}}\, D^-_{\text{quadratic}}\,, \tag{9}$$

where $D^-_{\text{quadratic}}$ is defined by Eq. (11). The vacuum is then obtained from

$$D^+[\xi_\mu, D^-] + [\xi_\mu, D^+]D^- = 0\,, \tag{10}$$

where $D^\pm$ denote the $\pm$ sum of spin terms [Eq. (11)]. We now consider the breaking operators of the quadratic forms.

IV. BREAKING OPERATORS OF THE QUADRATIC HAMILTONIAN

The partially reduced quadratic Hamiltonian in Euclidean space can be written in a block-diagonal representation

$$D = \xi^\mu\xi_\mu + (M+\Phi)^2 - b\sigma^{\mu\nu}\xi_{\mu\nu} + \text{dynamical breaking terms}\,, \tag{11}$$

where b is a constant. The Minkowski-space representation adds diagonal terms $\{(M+\Phi), \xi_0\}$ and dynamical-breaking terms, which will not be considered here. The non-dynamical-breaking terms in (11) are the familiar helicity and spin-orbit operators which commute with the Hamiltonian. The vacuum state $D(\xi_\mu')$ is obtained from (4) for $\Phi = 0$ and the vacuum-breaking operators of the quadratic Hamiltonian are obtained from the spin terms or

$$\sigma^{\mu\nu}[\xi_\sigma, \xi_{\mu\nu}] = \sigma^{\mu\nu}\{n^k_{\sigma\mu}k_\nu' + n^A_{\sigma\mu}A_\nu' + [k_\sigma, [A_\mu, A_\nu]] + [A_\sigma, [k_\mu, k_\nu]] + [\xi_\sigma, (F_{\mu\nu} + F^\dagger_{\mu\nu})]\}\,. \tag{12}$$

The first two terms include Cartan matrix integers which identify the unbroken groups G^k and G^A. The remaining terms include breaking operators which generate Cartan matrix integers which identify the breaking symmetries.

The unbroken symmetry of G^ξ is that of the vacuum algebra defined by (4). However, it is evident from (12) that the algebra $L^\xi \times X^C \times X^{C\dagger}$ may generate a hierarchy of subalgebras according to powers of g. With $g = n^2/\alpha$ for soliton or Schwinger monopole coupling, the g^2 commutators represent strong boson-boson interactions, the g commutators represent medium-strong fermion-boson interactions, and the g^0 commutators represent fine-structure fermion-fermion interactions. Similarly, the invariants of L^ξ form a hierarchy within which $g^2A^\mu A_\mu$ belongs to the strong-coupling subalgebra L^A and the invariants of L^C are $A^\mu k_\mu$ and $k^\mu A_\mu$ and vanish in the case of exact current conservation.

We now show that within the broken vacuum defined by (4) and (12), the field divergence, which is an invariant of the interaction subalgebra L^C, generates a linear gauge function which specifies the medium-strong breaking symmetry and the scale factor. Then (4) determines a gauge which is a linear function of the integer Cartan matrix elements and is equivalent to the current divergence in the quartic Hamiltonian. The completely reduced quadratic D for operators defined on Euclidean space reads

$$\begin{aligned}
D &= \xi^\mu\xi_\mu + (M+\Phi)^2 - b\sigma^{\mu\nu}[\xi_\mu, \xi_\nu] \\
&\to \xi^\mu\xi_\mu + (M+\Phi)^2 + \tfrac{1}{2}S[\xi_+, \xi_-]\,, \\
S &= \begin{cases} +1 : \psi_+ = (\psi_2, \psi_4)\,, \\ -1 : \psi_- = (\psi_1, \psi_3)\,, \end{cases} \\
&[\xi_0, \xi_+]\psi_+ = 1 = [\xi_3, \xi_+]\psi_+\,, \\
&[\xi_0, \xi_-]\psi_+ = 0 = [\xi_3, \xi_-]\psi_+\,, \\
&[\xi_0, \xi_+]\psi_- = 0 = [\xi_3, \xi_+]\psi_-\,, \\
&[\xi_0, \xi_-]\psi_- = 1 = [\xi_3, \xi_-]\psi_-\,, \\
&[\xi_0, \xi_3] = 0\,,
\end{aligned} \tag{13}$$

together with the chiral algebra[12] for $\psi_\pm$ and $\Phi = \lambda_i\phi^i$, is a sum of scalar fields which have the symmetry of G^{ξ_0}. Equations (13) and (4) give

$$-(k^\mu A^j_\mu + A^{j\mu}k_\mu)\xi_\mu = \frac{b}{g_j(1-h)}\sigma^{\mu\nu}[\xi_\mu, [\xi_\mu, \xi_\nu]] - \frac{1}{2g}\frac{S}{1-h}[\xi_\mu, [\xi_+, \xi_-]]\,. \tag{14}$$

The second line of (14) is in correct form for expansion in Cartan-Weyl operators $k_\pm \to E_{\pm\alpha}$ and $A_\pm \to E_{\pm\beta}$. Then the $[\xi_+, \xi_-]$ commutators give a sum of diagonal operators and we find

$$k^\mu A^j_\mu = \frac{M}{2g_j}\frac{b'}{1-h} S(1+n^c)\varphi\ , \tag{15}$$

where b' is a constant, $n^c = \bar{n}^c$ are Cartan matrix elements of the breaking subgroups, and $\bar{n}^c$ are the transposed elements. The unbroken symmetry can now be identified for $1+n^c=0$. In the chiral limit $(15)=0$ so that the symmetry is unbroken if G^C is SU(3). Then the chiral symmetry is SU(3) $\times$ SU(3). The breaking symmetry (or symmetries) can now be identified by the values of n^c and $\bar{n}^c$. For the mass spectra of the low-lying nonets $J^P = 0^-, \frac{1}{2}^+, 1^-$ the integer runs through the sequence $3, 0, -1, -2$, which indicates that any breaking operator belongs to one of the simple Lie algebras.[12] In other words, the mass spectrum appears to select a different simple group for each mass state. However, current conservation with (15) imposes an SU(3) "envelope" and implies a sum rule which generates octet enhancement within the breaking.

With $j_\mu = m^2 A_\mu$ or normalizing to ρ we obtain the PCAC result at $n^c=0$ or SU(2) $\times$ SU(2) breaking symmetry. Equation (15) then specifies the SU(3) $\times$ SU(3) unbroken symmetry in the conserved-current limit, gives no massless excitations, and is independent of pole dominance and saturation approximations.

The current divergence obtained from the quadratic Hamiltonian does not include the Adler-Bell-Jackiw anomaly which is a second-order spin term in the quartic representation. The total current divergence reads

$$k^\mu j^j_\mu = \frac{M}{2g_j}\frac{b'}{1-h} m^2 S(1+n)\varphi + d{}^*\xi_{\mu\nu}\,\xi_{\mu\nu}\ , \tag{16}$$

where ${}^*\xi_{\mu\nu}$ is the dual tensor. In Sec. V we consider the second-order breaking terms.

V. BREAKING OPERATORS OF THE QUARTIC HAMILTONIAN

The quartic Hamiltonian can be written in a completely reduced representation

$$D_{\text{quartic}} = [\xi^\mu \xi_\mu + (M+\Phi)^2 + b\sigma^{\mu\nu}\xi_{\mu\nu}] \times [\xi^\mu\xi_\mu + (M+\Phi)^2 - b\sigma^{\mu\nu}\xi_{\mu\nu}] \tag{17}$$

and the sum of spin terms commutes with $D^\pm$. The breaking operators in (17) are related to the second-order spin tensor sum by

$$\tfrac{1}{4}(\sigma_{\mu\nu}\xi_{\mu\nu})^2 = \tfrac{1}{2}\xi_{\mu\nu}{}^2 + \tfrac{1}{2}\gamma_5{}^*\xi_{\mu\nu}\,\xi_{\mu\nu}\ , \tag{18}$$

where ${}^*\xi_{\mu\nu}$ is the dual tensor and

$$\begin{aligned}{}^*\xi_{\mu\nu}\,\xi_{\mu\nu} = {}& \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}(n^k_{\sigma\mu}k'_\nu k_\rho + n^A_{\sigma\mu}A'_\nu A_\rho) \\ &+ {}^*k_{\mu\nu}A_{\mu\nu} + {}^*A_{\mu\nu}k_{\mu\nu} \\ &+ {}^*(k_{\mu\nu}+A_{\mu\nu})(F_{\mu\nu}+F^\dagger_{\mu\nu}) \\ &+ {}^*(F_{\mu\nu}+F^\dagger_{\mu\nu})(k_{\mu\nu}+A_{\mu\nu}) \\ &+ {}^*(F_{\mu\nu}+F^\dagger_{\mu\nu})(F_{\mu\nu}+F^\dagger_{\mu\nu})\ .\end{aligned} \tag{19}$$

The right-hand side includes a fermion helicity-breaking operator ${}^*k_{\mu\nu}k_{\mu\nu}$ and a boson helicity-breaking operator ${}^*A_{\mu\nu}A_{\mu\nu}$ which generate the integer Cartan matrix elements in the first two terms; the remaining terms represent the interactions of fermion and boson helicity operators, spin-orbit operators, and the Adler-Bell-Jackiw anomaly, which appears as a second-order spin-orbit term for Abelian G^k. We consider (19) for the Abelian special cases and in the chiral limit.

For Abelian G^A, Eq. (19) reads

$$\begin{aligned}{}^*\xi_{\mu\nu}\,\xi_{\mu\nu} = {}& {}^*k_{\mu\nu}k_{\mu\nu} + {}^*k_{\mu\nu}(F_{\mu\nu}+F^\dagger_{\mu\nu}) \\ &+ {}^*(F_{\mu\nu}+F^\dagger_{\mu\nu})k_{\mu\nu} \\ &+ {}^*(F_{\mu\nu}+F^\dagger_{\mu\nu})(F_{\mu\nu}+F^\dagger_{\mu\nu})\end{aligned} \tag{20}$$

which relates the axial-vector-current divergence to the Pontrjagin density for $k_\mu = \partial_\mu$. In the chiral limit, (20) becomes

$$\begin{aligned}{}^*k_{\mu\nu}F_{\mu\nu} = k_{\mu\nu}{}^*F_{\mu\nu} &= -\tfrac{1}{2}({}^*k_{\mu\nu}k_{\mu\nu} + 4{}^*F_{\mu\nu}F_{\mu\nu}) \\ &= 0\ .\end{aligned} \tag{21}$$

The $k_{\mu\nu}{}^*F_{\mu\nu}$ term gives the axial-vector-current divergence and ${}^*F_{\mu\nu}F_{\mu\nu}$ is the anomalous term. The ${}^*k_{\mu\nu}k_{\mu\nu}$ term is a color neutral helicity-breaking term which vanishes for Abelian G^k and for SU(2) $\times$ SU(2) breaking for which $n^k_{\sigma\mu}=0$. Then ${}^*F_{\mu\nu}F_{\mu\nu}=0$ in the chiral limit. In electromagnetism the left-hand side of (21) is just the divergence of the pseudovector current which vanishes if there are no magnetic monopoles.

For Abelian G^k and G^A, Eq. (18) becomes

$$\tfrac{1}{2}(\sigma_{\mu\nu}F_{\mu\nu})^2 = F_{\mu\nu}{}^2 + \gamma_5{}^*F_{\mu\nu}F_{\mu\nu} \tag{22}$$

which gives the Schwinger result[13] for $k_\mu = \partial_\mu$. In the chiral limit the sum of breaking terms Eq. (19) vanishes identically: $1+n^c=0$ and $k^\mu A_\mu = A^\mu k_\mu = 0$ by the chiral algebra so that ${}^*F_{\mu\nu}F_{\mu\nu} = 0$.

A. Lagrangian vacuum

The Hamiltonian $-D(\xi'_\mu)$ which results from (17) for $\dot{\xi}_\mu = 0$ is equivalent to the Lagrangian vacuum broken by ${}^*\xi_{\mu\nu}\,\xi_{\mu\nu}$ or the quadratic spin terms. The breaking terms can be identified with definite symmetries provided the group ${}^*G^\xi \times G^\xi$ exists and there is an overall symmetry $G^\xi \times G^\xi$. Then (18) reduces to

$$(\tfrac{1}{2}\sigma_{\mu\nu}\,\xi_{\mu\nu})^2=\tfrac{1}{2}a'\xi_\rho^{\,2}+\tfrac{1}{8}g_{\mu\mu}\gamma_5\epsilon_{\mu\nu\rho\sigma}n_{\nu\sigma}\xi_\mu\xi_\rho\,, \qquad (23)$$

where $a'=(d^\rho_{\mu\nu})^2$, $d^\rho_{\mu\nu}$ are structure constants of G^ξ, $g_{\mu\mu}$ is the diagonal Killing form, and $n_{\nu\sigma}$ together with $n_{\sigma\nu}$ identify the symmetry of the breaking subgroups of $*G^\xi\times G^\xi$. However, (23) implies that $*G^\xi\times G^\xi$ is chirally symmetric, that A_μ is axial, and

$$\tfrac{1}{2}\gamma_5{*F_{\mu\nu}}F_{\mu\nu}=\tfrac{1}{8}g_{\mu\mu}\gamma_5\epsilon_{\mu\nu\rho\sigma}n_{\nu\sigma}A_\mu A_\rho\,. \qquad (24)$$

If $*G^\xi\times G^\xi$ is SU(2) × SU(2) then (24) vanishes. Consider instead the spin terms in the more familiar form

$$(\tfrac{1}{2}\sigma_{\mu\nu}k_{\mu\nu})^2=\tfrac{1}{2}k_{\mu\nu}^{\,2}+\tfrac{1}{2}\gamma_5{*k_{\mu\nu}}k_{\mu\nu}\,, \qquad (25)$$

$$(\tfrac{1}{2}\sigma_{\mu\nu}A_{\mu\nu})^2=\tfrac{1}{2}A_{\mu\nu}^{\,2}+\tfrac{1}{2}\gamma_5{*A_{\mu\nu}}A_{\mu\nu}\,, \qquad (26)$$

$$(\tfrac{1}{2}\sigma_{\mu\nu}F_{\mu\nu})^2=\tfrac{1}{2}F_{\mu\nu}^{\,2}+\tfrac{1}{2}\gamma_5{*F_{\mu\nu}}F_{\mu\nu}\,, \qquad (27)$$

provided

$$(k_{\mu\nu}+{*k_{\mu\nu}})G^A_{\mu\nu}+(A_{\mu\nu}+{*A_{\mu\nu}})G^k_{\mu\nu}+(F_{\mu\nu}+{*F_{\mu\nu}})(k_{\mu\nu}+A_{\mu\nu})=0 \qquad (28)$$

for $F_{\mu\nu}=F^\dagger_{\mu\nu}$. In this form the helicity and current breaking are symmetric and can be identified with the breaking subgroups $*G^k\times G^k$, $*G^A\times G^A$, and $*G^k\times G^A$, as the integers obtained from

$$*k_{\mu\nu}k_{\mu\nu}=\tfrac{1}{4}g_{\mu\mu}\epsilon_{\mu\nu\rho\sigma}n_{\nu\sigma}k_\mu k_\rho \qquad (29)$$

together with $n_{\sigma\nu}$ identify, for example, the breaking subgroup $*G^k\times G^k$. In spherical symmetry, $k_{\mu\nu}=2(L_{\mu\nu}+K_{\mu\nu})$ and

$$\tfrac{1}{4}{*k_{\mu\nu}}k_{\mu\nu}=D^4=x\partial_x+y\partial_y+z\partial_z+t\partial_t\,, \qquad (30)$$

where D^4 is the Euclidean dilatation operator. The operators $*k_{\mu\nu}$ represent rotations in the opposite direction; the algebra $*L^k$ is therefore the same as the $k_{\mu\nu}$ algebra but the operator indices are anticyclic. The couplings of $*G^k$ and G^k are both of order g^0 so there is no medium-strong breaking of G^k. The dilatation operator D^4 is characteristic of the conformal and Weyl groups and indicates $k_\mu\to L_\mu+K_\mu+D_\mu$ which generate operators of the form $x\partial_y+y\partial_x$ and spectrum-generating subgroups such as SU(1, 1) and SU(2, 1).

Nonvanishing of the axial-vector field divergence in (19) indicates the presence of solitons or for electromagnetism of magnetic monopoles. Noting that $A_\mu\to g_jA^j_\mu$ with $g=\alpha$ and $*g=n^2/\alpha$ for Schwinger quantization the strength of the electromagnetic breaking can be found by means of

$$A_{\mu\nu}=a^\rho_{\mu\nu}\,\alpha_j{}^2A^j_\rho\phi\to\frac{\alpha^2}{r^2}\,F(r) \qquad (31)$$

in spherical symmetry and

$$*A_{\mu\nu}={*a^\rho_{\mu\nu}}\,\frac{n^2}{\alpha_j{}^2}\,{*A^j_\rho}{*\phi}\to\frac{n^2}{\alpha^2}\,\frac{1}{r^2}\,{*F(r)}\,. \qquad (32)$$

The coupling of $A_{\mu\nu}$ is of order α^2 and $*A_{\mu\nu}$ is of order α^{-2} so that the fields of the breaking subgroup $*G^A\times G^A$ are of order α^0 compared to the unbroken group. A similar result for the field tensor breaking term $*F_{\mu\nu}F_{\mu\nu}$ leads to the conclusion that the strength of the vacuum coupling must be α^{-1} and the strong field sources are magnetic monopoles or solitons. This conclusion, which has been reached by many authors, is also supported by the equation of motion

$$[\xi^\mu\xi_\mu+(M+\Phi)^2]\xi_\mu-ik^\nu(k_{\mu\nu}+G_{\mu\nu})-i\gamma_5k^\nu({*k_{\mu\nu}}+{*G_{\mu\nu}})=0 \qquad (33)$$

in which the dual field tensors are of order α^{-1} and α_0 compared to α and α^2. The pseudovector current is therefore of order α^{-1} compared to α.

VI. SUMMARY AND CONCLUDING REMARKS

In this paper we have introduced a Lie-algebra approach to symmetry breaking in an attempt to reduce the arbitrariness inherent in models having several free parameters and/or an *ad hoc* vacuum symmetry. In addition to σ models we hope in this way to improve vacuum-symmetry identification and formulations of symmetry breaking in bag models as well as non-Abelian Higgs field and non-Abelian vector-gluon theories. The method essentially replaces the free parameters with functions of structure constants and invariants. Then vanishing of the sum of vacuum-breaking terms gives an algebraic function or functions which may identify the vacuum symmetry as well as the breaking symmetries. The condition is that the breaking operators form subalgebras of the broken symmetry and it is argued that this condition is satisfied for conserved systems: As a conserved system has a definite symmetry, the breaking operators have subalgebras at least by adjoining some vacuum-state operators.

The symmetry of the Dirac Hamiltonian of a sum of flavored fermions and colored bosons is considered in the partially-reduced quadratic and irreducible quartic representations. In the quadratic form we find that the unbroken symmetry consists of a hierarchy of subalgebras in powers of the coupling g. For soliton or magnetic monopole fields the g^2 commutators and invariants represent strong boson-boson interactions, the g commutators and invariants represent medium-strong fermion-boson interactions, and the g^0 terms represent the fine-structure fermion-fermion interactions. The broken vacuum defined in the conventional Hamiltonian formalism is then considered in order to find the breaking functions within the hierarchy. We find that within the broken vacuum the field divergence yields a gauge which is a linear function of integer Cartan matrix elements and which specifies the vacuum symmetry. The result is a PCAC-like expression which vanishes for $SU(3)\times SU(3)$ vacuum symmetry and reduces to PCAC for $SU(2)\times SU(2)$ breaking symmetry. In this case the current divergence cannot vanish without the Adler-Bell-Jackiw term which is a second-order spin term in the quartic Hamiltonian. The breaking terms in the quartic representation are second-order spin terms which include fermion and boson helicity-breaking terms in addition to the spin-orbit breaking term $*F_{\mu\nu}F_{\mu\nu}$. The Abelian special cases and the chiral limit are discussed.

ACKNOWLEDGMENT

The author has benefited from conversations with P. Ramond, H. Kleinert, F. Gilman, Michael Anderson, and D. W. G. Leith at various stages of development of this manuscript.

---

*Permanent address: Box 94, La Jolla, California 92037.

[1]L. Michel and L. Radicatti, Ann. Phys. (N.Y.) 66, 785 (1972); L. Michel, in *Lecture Notes in Physics, Group Representation in Mathematics and Physics* (Springer, Berlin, 1968); M. Gell-Mann, in *The Eightfold Way*, edited by M. Gell-Mann and Y. Ne'eman (Benjamin, New York, 1964).

[2]J. Bernstein, Rev. Mod. Phys. 46, 7 (1974); G. Cicogna, F. Strocchi, and R. Caffarelli, Phys. Rev. D 6, 301 (1972); S. Eliezer, *ibid.* 11, 3611 (1975).

[3]The earliest σ models and $\overline{3}3+3\overline{3}$ breaking were reported by M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960); M. Levy, *ibid.* 52A, 23 (1967); M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968); S. Georgi and S. Glashow, Phys. Rev. Lett. 28, 1494 (1972); a comprehensive review with many references is H. Pagels, Phys. Rep. 21C, 16 (1977).

[4]The bag models have been studied by M. Creutz, Phys. Rev. D 10, 1749 (1974); M. Creutz and K. S. Soh, *ibid.* 12, 443 (1975); A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. K. Weisskopf, *ibid.* 9, 3471 (1974); A. Chodos, R. L. Jaffe, K. Johnson, and C. B. Thorn, *ibid.* 10, 2599 (1977); W. Bardeen, M. Chanowitz, S. Drell, M. Weinstein, and T. Yan, *ibid.* 11, 1014 (1975).

[5]The Higgs models have been reviewed by R. Jackiw, Rev. Mod. Phys. 49, 681 (1977) and by Pagels (Ref. 3) and Bernstein (Ref. 2).

[6]The vector gluon and related models have been reviewed by R. Jackiw (Ref. 5).

[7]C. G. Callan, Jr., Phys. Rev. D 2, 1541 (1970).

[8]C. Callan, R. Dashen, and D. Gross, Phys. Rev. D 17, 2717 (1978).

[9]D. Kleima, W. Holborn III, and L. Biedenharn, in *Group Theory and its Applications*, edited by E. M. Loebl (Academic, New York, 1968).

[10]B. G. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974).

[11]Y. Ne'eman, Phys. Rev. 134, B1355 (1964); J. J. Sakurai, Phys. Rev. Lett. 9, 472 (1962); Phys. Rev. 132, 434 (1963); S. Glashow, Phys. Rev. Lett. 11, 48 (1963).

[12]J. T. Anderson, Lett. Nuovo Cimento 14, 158 (1975); 18, 109 (1977).

[13]J. Schwinger, Phys. Rev. 82, 664 (1951).