Potential energy of three heavy quarks in the MIT bag model*

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In the Born-Oppenheimer approximation to the MIT bag model the color-singlet n-quark (and/or antiquark) system is studied. Analytic approximations to the spin-independent n-quark potential energy are developed, based on spherical and cylindrical trial bag shapes. The domain of validity of these results is established in the $q\bar{q}$ case through tests against a numerical solution. The resulting three-quark potential contains Coulomb-type and confining parts. The Coulomb-type terms are purely two-body in nature, and the confinement term is a three-body potential.

I. INTRODUCTION

Recently, Haxton and Heller[1] have studied mesons composed of a heavy quark and antiquark by treating the MIT bag model[2] in an adiabatic (Born-Oppenheimer) approximation.[3] It is assumed that the glue field changes on a time scale which is much shorter than that associated with the motion of the quarks, whose separation is treated as an adiabatic variable. It is furthermore assumed that the transverse degrees of freedom of the glue field are not excited. In the first step, they[1] solve the Yang-Mills equations for the glue field generated by the fixed quark sources, to lowest nonzero order in the quark-gluon coupling constant g. In order to satisfy the bag boundary conditions, it is necessary to determine the bag surface and the color-electrostatic field simultaneously.[4] The resulting bag energy (for each separation of the quark and antiquark) is, in the next step, regarded as a potential for describing the relative quark motion. With only two adjustable parameters (apart from the quark masses) which describe the variation of the running coupling constant with distance, a good fit to the spectra of mesons containing charmed quarks and bottom quarks was obtained.[1] It was also argued that this potential has approximate validity for mesons containing strange quarks, or one strange and one heavier (anti)quark.

In the present work we will apply the same physical picture to baryons composed of three heavy quarks. Although data on such systems are almost nonexistent at the present time—with the exception of the Ω^- which consists of three strange quarks—the formalism is nevertheless interesting, because of the *structure* of the potential-energy function which emerges. It contains readily identifiable two-body terms, and also an important three-body term. The former represent the color-Coulomb interactions and the latter represents the confining interaction. This is in marked contrast with the phenomenological attempts which have been made to describe light baryons[5,6] and multibaryons[7] with (not-so-light) quarks interacting by two-body potentials only.

In Ref. 1 the bag shapes (including cusp singularities), the color-electrostatic fields, and the bag energies were obtained by a numerical solution of the equations and boundary conditions,[8] summarized in Sec. II. It would be vastly more difficult to attempt a numerical solution for three quarks because of the much greater geometrical complexity. With the numerical solution of the $q\bar{q}$ problem in hand, however, the accuracy of approximate analytic solutions can be judged, and those same approximations can then be carried over to the qqq problem in their respective domains of validity.

For example, it is shown in Ref. 1 that a spherical approximation to the shape of the bag yields an accurate value for the energy provided the $q\bar{q}$ separation is less than 1 fm.[9] Since an analytic solution is available for a sphere, we shall use it for the qqq problem when all the separations are sufficiently small. This is the most important region for the low-lying states of systems composed of very heavy quarks and is discussed in Sec. III. The situation in which at least one quark is widely separated from the rest of the system is studied in Sec. IV. In Sec. V a global approximation is presented which joins the small- and large-separation regions. In Sec. VI we discuss some necessary extensions and problems encountered there.

GENERAL FORMULATION

In the Born-Oppenheimer approximation to the MIT bag model, one starts by taking the n quarks static. As a result the quarks enter the equations of motion only as fixed color sources, and to lowest nonzero order in g we have to solve[1,3]

$$-\vec{\nabla}^2\phi^a(\vec{x}) = \rho^a(\vec{x}) = \sum_j^n Q_j^a \delta(\vec{x} - \vec{x}_j) \quad \text{inside the bag}, \tag{1}$$

and

$$\hat{n}\vec{\nabla}\phi^a(\vec{x}) = 0 \quad \text{on the surface} \tag{2}$$

$$\tfrac{1}{2}\sum_a (\vec{\nabla}\phi^a)^2 = B \tag{3}$$

in terms of $\phi^a(\vec{x})$, the $\mu = 0$ component of the glue field $A_\mu^a(\vec{x})$. The eight ($a = 1, \ldots, 8$) color charges $Q_j^a = gF_j^a$ generate the color-SU(3) transformations of quark j. One has $F^a = \frac{1}{2}\lambda^a$ for quarks, and $F^a = -\frac{1}{2}\lambda^{aT}$ for antiquarks (T = transpose, λ^a are the usual Gell-Mann matrices). The unit surface normal is denoted by $\hat{n}$, and B is the MIT bag pressure $B^{1/4} = 0.145$ GeV.[10]

For a given bag surface S Eqs. (1) and (2) specify a Neumann problem which has a unique solution (to within an additive constant[11])

$$\phi^a(\vec{x}) = \int d^3x' G_N(\vec{x}, \vec{x}')\rho^a(\vec{x}'), \tag{4}$$

where G_N is an appropriate Green's function for the surface. The absence of a surface term in Eq. (4) is due to the boundary condition Eq. (2). The energy of this system of static sources in a bag is

$$W[S] = \int dV\left[\tfrac{1}{2}\sum_a (\vec{\nabla}\phi^a)^2 + B\right] = \int dV\left(\tfrac{1}{2}\sum_a \rho^a\phi^a + B\right), \tag{5}$$

where use has been made of Eq. (2). It depends on the positions of the quarks, of course, and because of the one-to-one correspondence between the surface S and the fields ϕ^a, W can be regarded as a surface functional. Given ϕ^a, Eq. (3) determines which surface is consistent with a bag solution. This problem can also be stated as an energy variational problem.[12] The condition that W, subject to Eqs. (1) and (2), be stable with respect to arbitrary surface variations, indeed yields Eq. (3).

The variational formulation is particularly useful for complicated cases like the heavy qqq system. Here, determination of the correct bag surface S_B would be a tremendous task. Instead, it is much more practical to approximate S_B by a trial surface $S(a_i)$ which is a function of a small number of parameters a_i. One then requires that W be minimal with respect to variations of $S(a_i)$, or equivalently

$$\partial W/\partial a_i = 0, \tag{6}$$

which yields the optimal parameters $\bar{a}_i$. The corresponding fields ϕ^a will not satisfy Eq. (3), since these variations are not the most general ones conceivable. Rather, one satisfies it "on average."

From Eqs. (1) and (4) it is clear that the prescription given above can be applied to any number of quarks and antiquarks in a color-singlet state.

Finally, the potential energy of the n-quark system is identified with the bag energy, evaluated for the optimal parameters

$$V(\vec{x}_1, \ldots, \vec{x}_n) = W[S(\bar{a}_i)]. \tag{7}$$

This potential satisfies translation invariance.

III. SPHERICAL APPROXIMATION FOR SMALL SEPARATIONS

Consider a color-singlet system of n fixed quarks and/or antiquarks, at positions $\vec{x}_i$, inside a bag. When the distance between every pair of quarks is small compared to the scale set by the bag constant $B^{-1/4} \cong 1.4$ fm, we expect the surface to be far away from all the quarks. This can be seen as follows.[12] According to Eq. (3), the surface is that set of points where the pressure of the color-electric fields $\sim \vec{E}^2$, balances the vacuum pressure B. For a color-singlet state of quarks sufficiently close together, the dominant contribution to the color-electric field will be the dipole term: $E \sim gx/r^3$, where r is the average distance from the quarks. At the surface ($r = R$) one then has $x^2 \sim BR^6/\alpha_s$, or $R \sim x^{1/3}$, where x gives the effective size of the quark distribution. When $x \ll B^{-1/4}$, also $x \ll R$. This relation suggests that the energy of such a system will be rather insensitive to the details of the bag shape. In particular, a sphere (which has some practical advantages over the actual shape which may contain topological singularities such as cusps[1]) can be expected to be a good approximation.

Using the Green's function for a sphere of radius R, the solution to Eqs. (1) and (2) is[13]

$$\phi^a(\vec{x}) = \frac{g}{4\pi R}\sum_j F_j^a\left[\frac{R}{|\vec{x} - \vec{x}_j|} + \frac{R^2}{|R^2\hat{x} - x\vec{x}_j|} - 1 - \ln\left(\frac{R^2 - \vec{x}\cdot\vec{x}_j + |R^2\hat{x} - \vec{x}_j x|}{2R^2}\right)\right] + \text{constant}, \tag{8}$$

under the restriction that one consider only color-singlet systems, for which $\sum_j F_j^a = 0$. From Eqs. (1), (5), (6), and (8), the energy of the spherical bag can be obtained, but the minimization of $W[R]$ with respect to R requires a numerical solution of a transcendental equation.

If one omits all homogeneous terms in Eq. (8) but the dipole ($l = 1$) term, it reduces to

$$\phi_d^a(\vec{x}) = \frac{g}{4\pi}\sum_j F_j^a\left(\frac{1}{|\vec{x} - \vec{x}_j|} + \frac{2\vec{x}\cdot\vec{x}_j}{R^3}\right) \tag{9}$$

and from Eqs. (1) and 5 the energy becomes

$$W^d[R]=\tfrac{4}{3}\pi BR^3+\alpha_s\sum_{i>j}\sum\frac{F_i\cdot F_j}{|\vec{x}_i-\vec{x}_j|}+\alpha_s\sum_i\sum_j\frac{F_i\cdot F_j\vec{x}_i\cdot\vec{x}_j}{R^3}, \tag{10}$$

where $\alpha_s\equiv g^2/4\pi$, $F_i\cdot F_j=\sum_a F_i^a F_j^a$, and the infinite self-energy terms have been omitted. From Eq. (6) the bag radius is determined to be

$$R_d^{\,6}=\frac{3\alpha_s}{4\pi B}\sum_i\sum_j F_i\cdot F_j\vec{x}_i\cdot\vec{x}_j \tag{11}$$

and the potential energy of the n-quark system in one spherical bag in the dipole approximation has a simple analytic expression

$$V^d(\vec{x}_1,\ldots,\vec{x}_n)=\alpha_s\sum_{i>j}\sum\frac{F_i\cdot F_j}{|\vec{x}_i-\vec{x}_j|}+2\left(\frac{4\pi B\alpha_s}{3}\sum_i\sum_j F_i\cdot F_j\vec{x}_i\cdot\vec{x}_j\right)^{1/2}. \tag{12}$$

The first terms on the right-hand side of Eq. (12) represent the two-body color-Coulomb interactions. The volume energy and the homogeneous part of the color-electrostatic energy contribute equally to the final term in Eq. (12) which is an *n-body confining potential*. In this paper we will be concerned only with $n=2$ and $n=3$ quark systems in detail, but in this section we formulate our results for arbitrary n in order to stress the extent to which they can be treated in a similar fashion.[14]

For the cases of $q\bar{q}$ and qqq in color-singlet states, Eq. (12) becomes

$$V_{q\bar{q}}^d(\vec{x}_q,\vec{x}_{\bar{q}})=-\frac{4\alpha_s}{3|\vec{x}_q-\vec{x}_{\bar{q}}|}+(\tfrac{2}{3})^{1/2}k\,|\vec{x}_q-\vec{x}_{\bar{q}}| \tag{13}$$

and

$$V_{q^3}^d(\vec{x}_1,\vec{x}_2,\vec{x}_3)=-\tfrac{2}{3}\sum_{i>j}\sum\frac{\alpha_s}{|\vec{x}_i-\vec{x}_j|}+k\rho, \tag{14}$$

where the slope of the linear term is

$$k=(8\pi B\alpha_s F_q^{\,2})^{1/2} \tag{15}$$

and the relations $F_q^{\,2}=\frac{4}{3}$, $F_q\cdot F_{\bar{q}}=-\frac{4}{3}$, and $F_q\cdot F_{q'}=-\frac{2}{3}$ have been used. Note that the q^3 confining term in the dipole approximation has a very simple form, depending only on the hyperradius $\rho\equiv[\frac{1}{3}\sum_{i>j}(\vec{x}_i-\vec{x}_j)^2]^{1/2}$.

Comparing Eqs. (14) and (13), we see that the q^3 Coulomb-type terms are a direct generalization of the corresponding $q\bar{q}$ term, with the substitution $F_qF_{q'}$ for $F_qF_{\bar{q}}$. The q^3 confinement term is a genuine three-body potential which results from Eq. (6), or more generally Eq. (3), since the glue field and the correct surface depend upon the positions of all the particles. This phenomenon is due to the confinement property of the MIT bag model, and may be a quite general feature of the n-heavy-quark potential.

The accuracy of the spherical-bag approximation can be tested in the $q\bar{q}$ case. In Fig. 1 we have plotted the numerical solution,[15] and the

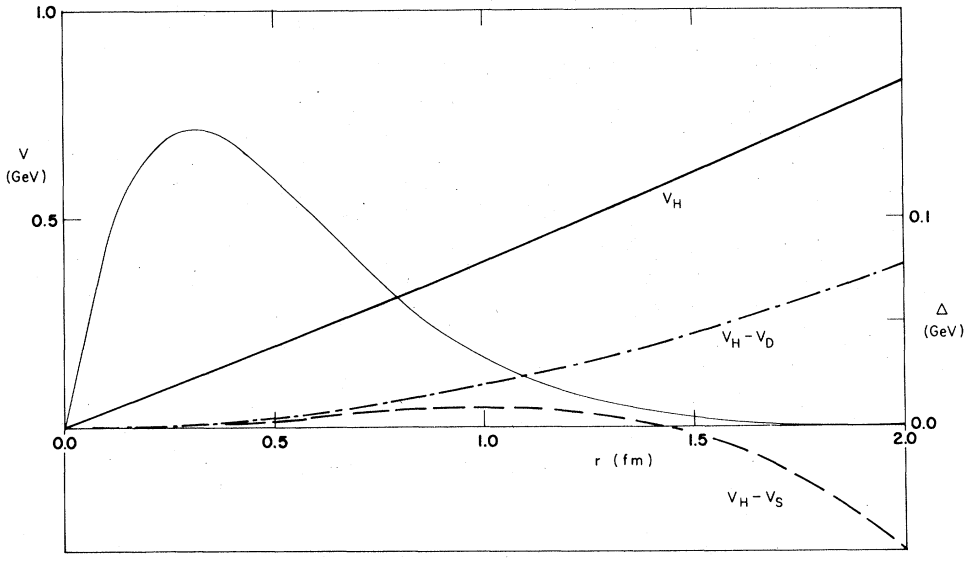

FIG. 1. Comparison of the heavy-quark–antiquark potential V_H, as obtained from Haxton (Ref. 15) (solid), with the full spherical-bag result V_S and the dipole approximation to the spherical result, V_D. V_H minus the Coulomb potential is shown on the left-hand scale. The differences V_H-V_S (dashed), and V_H-V_D (dot-dashed) are shown on the right-hand scale. Also the $1s$ $c\bar{c}$ ground-state wave function is shown (thin solid curve) in arbitrary units. Here we used $\alpha_s=0.55$ and $B^{1/4}=0.145$ GeV (Ref. 9).

amount the full spherical-bag result for $V_{q\bar{q}}$ differs from it. We also plotted the difference one gets using only the color-electric-dipole field. For comparison, the ground-state wave function for the $c\bar{c}$ system has been plotted also. Clearly, there is little difference between the three potentials in the region relevant for this charmonium state, and they yield energy eigenvalues for this state which are within 5 MeV of each other.[9] We are confident, therefore, that the use of the dipole approximation will yield a reasonable description for the lowest few states of the three-heavy-quark system, also. To improve the numerical accuracy, especially for the excited states, one would like a more accurate description of the potential beyond 1 fm. This is achieved in the next section by first examining the behavior of the bag at large separations. Although large separations are not very interesting in themselves, because of the possibility of light-quark–antiquark pair creation, this study will enable us to extend the domain of our potential approximation.

IV. THE TUBULAR APPROXIMATION FOR LARGE SEPARATIONS

As $r = |\vec{x}_q - \vec{x}_{\bar{q}}|$ increases, the dipole approximation to the $q\bar{q}$ system gradually gets worse (Fig. 1). The color-electric field $\vec{E}^a = -\vec{\nabla}\phi^a$ thus can no longer be regarded as that of a dipole. Under influence of the bag pressure B, the field lines become increasingly collimated as the quark and the antiquark move further apart. When $r \to \infty$, another simple shape with axial symmetry emerges: the cylinder. In this limit the $q\bar{q}$ system is a tube of color-electric flux, which connects the opposite color charges at the tube ends [Eq. (1)].[3,16] The field $\vec{E}^a$ is uniform and tangential to the surface [Eq. (2)]. Its strength is proportional to the strength of the color charges (Gauss's law) and together with Eq. (3) the radius R of the cylinder is determined to be[17]

$$R_\infty = (2\alpha_s F_q^2/\pi B)^{1/4} \tag{16}$$

for the $q\bar{q}$ system. The energy per unit length, stored in this flux tube, is k [Eq. (15)], as calculated from Eq. (5).

The above remarks suggest that when r is large ($\gg R_\infty$) the bag shape may be approximated by a finite cylinder of suitable length L and a radius $R \cong R_\infty$. This approximation will reproduce correctly the dominant part of the potential energy which grows linearly with r, but it may be clear that it will be less successful in describing the situation at the ends of the bag, near the color charges, where the fields $\vec{E}^a$ will no longer be uniform (Fig. 2). One can generalize these considerations easily to the q^3 system. Starting from

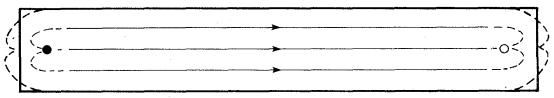

FIG. 2. Solid line: tubular approximation to the quark-antiquark bag. The dashed lines indicate the distortion near the quarks.

three quarks close together, and then moving one of them away from the other two, one finds a confinement term highly similar to the $q\bar{q}$ one: kr, with $r = |\vec{x}_q - \vec{x}_{q2}|$. The next step, to arrive at the most general q^3 configuration, is to separate the two quarks in the diquark. Considering only the two simplest forms, one obtains a tripod (Y) [Fig. 3(a)] or a triangle (Δ) [Fig. 3(b)] shape. In case of the Y, each quark sits at the end of its own tube which connects it to a common junction. For each quark the junction seemingly has the color properties of an antiquark, although there is no net color charge at the junction. The confining term in the potential energy of this system is then kr_M where r_M is the sum of the lengths of the legs, with the position of the junction $\vec{x}$ arranged in such a way that this total length is minimal:

$$\vec{\nabla}\left(\sum_i |\vec{x} - \vec{x}_i|\right) = 0 . \tag{17}$$

Equation (17) states that the angles between the legs of the tripod must be 120° and so r_M will clearly depend on the positions of all three quarks simultaneously:

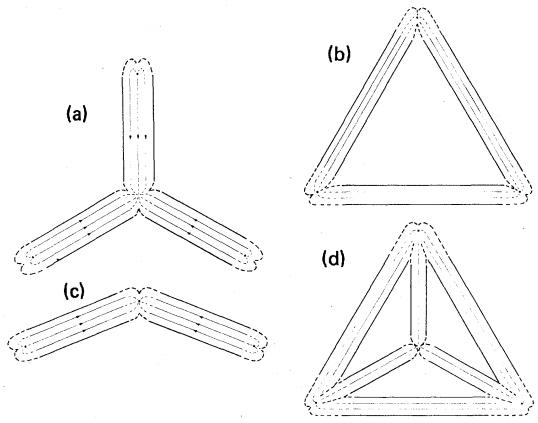

FIG. 3. Shape of a bag, containing three widely separated quarks [(a), (b), and (d)] when none of the angles in the quark triangle exceeds 120°, (c) when one of the angles exceeds 120°. The solid (dashed) lines indicate the part of the geometry of which the energy can(not) be computed reliably, using the tubular approximation.

$$r_M = (\tfrac{3}{2})^{1/2}\rho\left\{1+\sqrt{3}\left[4\frac{x_{12}^2x_{13}^2+x_{13}^2x_{23}^2+x_{12}^2x_{23}^2}{(x_{12}^2+x_{13}^2+x_{23}^2)^2}-1\right]^{1/2}\right\}^{1/2} \tag{18a}$$

in terms of $x_{ij} = |\vec{x}_{ij}| = |\vec{x}_i - \vec{x}_j|$. When one of the angles ϑ in the quark triangle becomes larger than 120°, Eq. (18a) no longer is valid, and a two-legged bag shape emerges [Fig. 3(c)] for which

$$r_M = x_{13} + x_{23} \tag{18b}$$

in case $\vartheta = \langle(\vec{x}_{31}, \vec{x}_{32})\rangle$. Finally, a special case of this is three quarks on a line, within one cylinder, where $r_M = x_{12}$. Examination of Eqs. (18a) and (18b) for fixed ρ shows that the more collinear configurations have smaller r_M, and therefore smaller "non-Coulombic" potential energies.

In case of the Δ shape [Fig. 3(b)], in which each pair of quarks is joined directly by a tube of flux, one finds the same energy (kr_M) and the same domain of validity as for the Y shape. The way in which this result comes about is somewhat different though. For a Δ shape the length of the legs is fixed, but not the amount of flux going through them. It is then by adjusting the flux in the legs, or equivalently their diameter, in accordance with Eqs. (1) to (3), that the energy of the Δ configuration is minimized. Going one step further one finds that the intermediate bag shape which is partly Y, partly Δ [Fig. 3(d)] will also have the same energy and domain of validity, independent of the relative amounts of flux going through the Y or Δ. We have no way of distinguishing between the various geometries. We will assume that the correct energy can be obtained from Eqs. 18(a) and 18(b), and that more involved geometries will not yield lower energies.

Considering quark systems with $n \geq 4$, one has to be aware of the fact that now other than triplet and singlet color charges may occur. The effect of such charges, generated by composite quark subsystems, is most clearly seen in case they terminate a single flux tube, which then contains a definite color flux. Both the energy per unit length in the flux tube and its diameter are changed. The new values for these quantities can be obtained from Eqs. (15) and (16), respectively, by replacing F_q^a with the proper charge $F_c^a = \sum_j F_j^a$, the sum ranging over the quarks in the subsystem. This replacement is, of course, also the proper one, when the subsystem is located at a junction.

The Coulomb (=inhomogeneous) part of the potential energy does not depend on the shape of the bag and will be identical to that found for the spherical bag. Its contribution to the potential energy will only be relevant for quarks close together, of course.

The only remaining contribution to the bag energy appears to be that associated with the ends of the tubes, something which does not grow with the overall scale of the system. Calling this term V_0, we write the tubular approximation to the potential as

$$V^t(\vec{x}_1, \ldots, \vec{x}_n) = \alpha_s \sum\sum_{i>j} \frac{F_i \cdot F_j}{|\vec{x}_i - \vec{x}_j|} + kr_M + V_0 \tag{19}$$

and expect this to be useful when at least one quark is far away from the rest of the system.

The suitability of Eq. (19) can be tested in the special case that all the quarks are collinear and $n \leq 3$ so that only color-triplet charges are present. One can then try a finite cylinder as bag shape, using again the Green's-function technique. One puts the quarks on the cylinder axis, which is also the z axis. The quark positions then are completely specified by their z coordinates z_i. Equations (1) and (2) will give the glue potential ϕ^a, and Eq. (5) the energy W.

In addition to eliminating the cylinder parameters R and L, according to Eq. (6), one also has to make sure that for a given quark configuration these quarks are placed optimally inside the cylinder. This is done by replacing $z_i \to z_i + z_0$, and determining for which value of z_0 W is minimal. Typically, the cylinder will have to be placed symmetrically around the two color charges which are closest to its top and bottom surfaces, since their positions dominate W. Such a color charge may be a quark, an antiquark, or a diquark. Only when another quark is within a distance $0 < d < R_\infty$ will this symmetric arrangement be affected slightly. Denoting the distance between the two outermost color charges by r, we consider the limit $r \gg R_\infty$. (In this geometry $r_M = r$.) Equation (6), for $a_i = R$ yields Eq. (16), and for $a_i = L$ yields $L \cong r + 0.55R_\infty$.

For the case of a color-singlet $q\bar{q}$ state, the resulting potential energy has precisely the form of Eq. (19), with $V_0 = -0.86\alpha_s/R_\infty$.

V. GLOBAL APPROXIMATION TO THE POTENTIAL

The information which can be gathered from studying the spherical and tubular approximations suffices to construct a representation of the q-$\bar{q}$ potential which is both accurate enough for doing spectroscopy, and simple enough to serve as a model for other, less tractable systems. Note the following: the cylinder bag yields the correct slope for $r \to \infty$, but not quite the right intercept. One obtains $V_0 \cong -0.11$ GeV from the numerical

result and $V_0 = -0.083$ GeV from the analytic formula. We note that $V^t_{q\bar{q}}$ would intersect $V^d_{q\bar{q}}$ at $r = R_\infty$, if we set $V_0 = -0.094$ GeV. This suggests the following global approximation to $V_{q\bar{q}}$: use the dipole approximation for $r \leq R_\infty$, and the tubular approximation for $r \geq R_\infty$, but with V_0 adjusted to make the potential continuous:

$$V^G_{q\bar{q}}(r) = V^d_{q\bar{q}}(r), \quad r \leq R_\infty,$$

$$= -\frac{4}{3}\frac{\alpha_s}{r} + kr + V^0_{q\bar{q}}, \quad r \geq R_\infty, \tag{20}$$

where $V^0_{q\bar{q}} = -0.98\alpha_s/R_\infty$. This potential has been plotted in Fig. 4, as V_G (solid curve) for the values of the parameters given in Ref. 9. The Coulomb term has been omitted. For comparison the difference of this potential and the numerical result V_H of Haxton,[15] has been plotted as $V_H - V_G$. As can be seen, V_G will represent $V_{q\bar{q}}$ accurately below 0.5 fm and reasonably well all the way out to ~2 fm which, in practice, is more than ample.

The potential energy of a color-singlet q^3 state in the cylinder also has the form of Eq. (19) when the two outermost quarks are widely separated, but now V_0 depends upon the position of the third quark z_3. V_0 is the same as for the $q\bar{q}$ case if there is a diquark at one and a quark at the other, but $V_0 = -2.33\alpha_s/R_\infty$ if z_3 is midway between the end quarks.

V_0 changes most rapidly as the diquark splits up into two separate quarks and is virtually constant when $|z_i - z_3| > R_\infty$, $i = 1$ and 2. The change in V_0 accompanies the transition of a one-leg quark-diquark bag to a two-legged three-quark one. However, the remarks made above regarding the limitations of the cylinder approximation as a bag shape apply here equally well to the quantity V_0. Furthermore, V_0 becomes incalculable analytically in case of the Y- or Δ-shaped bags, where one has to account for the contribution to the energy of the end points and the junction [Fig. 3(a)] or the corners [Fig. 3(b)]. The best we can do then is to fix V_0 by generalizing Eq. (20), and match $V^t_{q^3}$ to $V^d_{q^3}$ at a suitable point. For example, one possible prescription based on the observation that the antiquark-quark and the diquark-quark systems are very similar, would be to set

$$V^G_{q^3}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = V^d_{q^3}(\vec{x}_1, \vec{x}_2, \vec{x}_3), \quad \rho \leq R_\infty(\tfrac{2}{3})^{1/2},$$

$$= -\frac{2}{3}\sum_{i>j}\sum \frac{\alpha_s}{|\vec{x}_i - \vec{x}_j|} + kr_M$$

$$+ V^0_{q^3}(\vec{x}_1, \vec{x}_2, \vec{x}_3), \quad \rho \geq R_\infty(\tfrac{2}{3})^{1/2}, \tag{21}$$

with $V^0_{q^3}$ chosen to make the potential continuous for every geometry. Since ρ is fixed $V^0_{q^3}$ will depend on two variables only.

VI. COMMENTS AND QUESTIONS

In Ref. 1 it was found that the fixed MIT coupling constant[10] $\alpha_s = 2.2$ yields a potential which is much too strong at small separations. This problem was

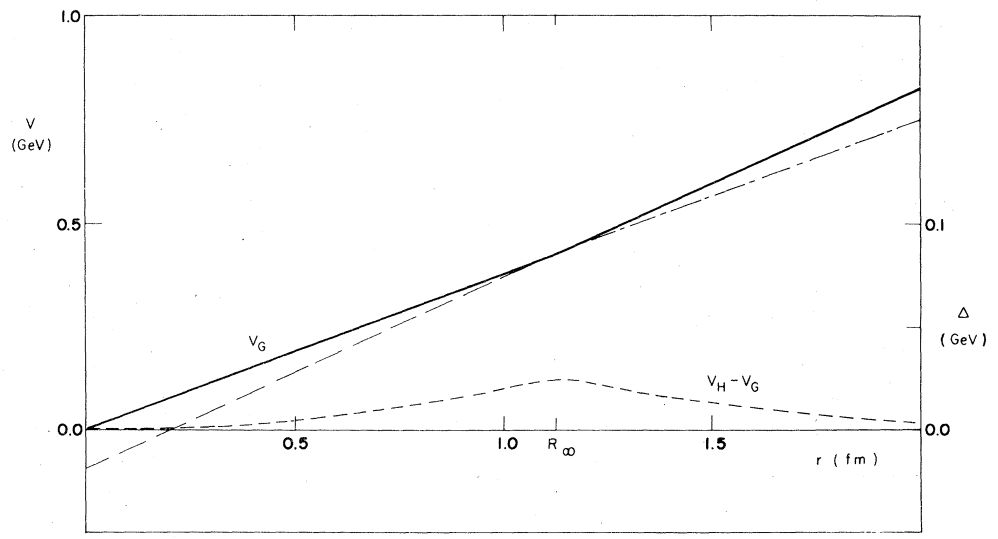

FIG. 4. The global approximation, V_G (solid), to the q-$\bar{q}$ potential is shown on the left-hand scale. It consists of the dipole approximation to spherical-bag result for $r < R_\infty$ (dot-dashed for $r > R_\infty$), and the tubular result for $r > R_\infty$ (dashed for $r < R_\infty$). The Coulomb term has been omitted. The small-dashed curve represents the difference between the numerical result (Ref. 15) and the global approximation $V_H - V_G$ (right-hand scale).

partly resolved by noting that the quantum-chromodynamics (QCD) coupling constant depends on the scale r, relevant to the problem. For very short distances its leading behavior can be calculated[18]

$$\alpha_s(r) = \frac{4\pi}{11 - 2n_f/3} \frac{1}{\ln(1/\Lambda^2 r^2)} \tag{22}$$

in terms of the number of quark flavors n_f and the QCD scale parameter Λ. The MIT value for α_s, quoted above, was determined for light, relativistic quarks which can be found with comparable probability throughout the bag, and are at an average separation of about 1 fm from each other. The quarks in $c\bar{c}$ and $b\bar{b}$ bound states are highly localized, and their separations range from 0 to beyond 1 fm. Their interaction at short distances should therefore be sensitive to the variation of α_s. In Ref. 1 r was identified with the q-$\bar{q}$ separation and the quark-gluon coupling constant g occurring in Eq. (1) was simply replaced by $g(r) \equiv [4\pi\alpha_s(r)]^{1/2}$, with α_s parametrized as

$$\alpha_s(r) = \frac{4\pi}{11 - 2n_f/3} \frac{1}{\ln[(1/\Lambda^2 r^2) + \gamma]} \tag{23}$$

in order to give the correct small-r behavior and avoid singularities for positive r. Λ and γ were treated as free parameters. A good reproduction of the $c\bar{c}$ and $b\bar{b}$ spectra was obtained with $\Lambda = 0.24$ GeV and α_s (1 fm) = 1.0 (corresponding to $\gamma = 3.36$).[19]

The identification of r with the q-$\bar{q}$ separation is fairly unambiguous. For q-$\bar{q}$ separations of the order of 0.5 fm and less the quarks still are well inside the (spherical) bag and this separation then is the only available distance scale. It is not clear, at present, how to incorporate the variation of coupling constant into the q^3 system, where the choice of (a) distance scale(s) is much more ambiguous. One might guess that the coupling constant in the Coulomb-type terms will depend on the two-body separations. But the coupling-constant scale dependence in the other terms (k, R_∞) is not at all clear. More systematic ways of investigating these problems are being studied at present. Once this issue has been resolved, we will be able to treat the heavy-q^3 system on the same footing as the heavy-$q\bar{q}$ system.[1] At that point also the geometry-dependent part of the zero-point energy of the confined fields should be included in the potential energy of the q^3 system.[20]

To proceed with the second step of the Born-Oppenheimer approximation, a variety of techniques to solve the resulting three-body Schrödinger equation is available. We find that both separation of the Schrödinger equation into coupled Faddeev equations combined with a partial-wave expansion,[21] and the use of a hyperspherical expansion[22] converge rapidly for the ground-state configuration of the q^3 potential of Eq. (21).

VII. SUMMARY

In the previous sections we calculated the potential energies of a $q\bar{q}$ [Eq. (20)] and a qqq [Eq. (21)] system, in an adiabatic approximation to the MIT bag model. It was shown that although these systems can be treated on the same footing, the resulting q^3 potential cannot be obtained by naive generalization of the $q\bar{q}$ result. The reason, we think, is that the interactions between the quarks depend, in the presence of confinement, on the configuration of the entire system.

We have shown that a simple and accurate approximation to the central part of the heavy-quark–antiquark potential energy for small distances is provided by the sum of the color-Coulomb potential and the dipole approximation to the spherical-bag confining potential [Eq. (13)]. When the same approximation is applied to the q^3 system, the confining term—which is a three-body potential—has a very simple form, linear in the hyperradius [Eq. (14)].

By joining the dipole approximation for small distances with the tubular approximation [Eq. (19)] for large distances, a global approximation to the potential is obtained. This is accurate out to 2 fm for the q-$\bar{q}$ system, and we presume that is also accurate for the q^3 system.

The proper way to incorporate asymptotic freedom into the confined q^3 system remains to be ascertained.[23]

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[1]W. C. Haxton and L. Heller, Phys. Rev. D 22, 1198 (1980).

[2]A. Chodos *et al.*, Phys. Rev. D 9, 3471 (1974).

[3]P. Hasenfratz *et al.*, in *Charm, Color and the J/ψ*, proceedings of the Xth Recontre de Moriond, Méribel-les-

Allues, France, 1975, edited by J. Trân Thanh Vân (CNRS, Paris, 1975); P. Hasenfratz and J. Kuti, Phys. Rep. 40C, 75 (1978).

[4]This procedure differs from that for relativistic quarks in a fixed-cavity approximation (Ref. 10). In the latter case the quark fields determine the shape of the bag, whereas here there is only the glue field. As a result the spherical-bag shape is an exact one for relativistic quarks in their ground state, whereas it is only approximate for heavy nonrelativistic quarks. The resulting description of the heavy quarks, on the other hand, is translation invariant.

[5]For a review, see O. W. Greenberg, Annu. Rev. Nucl. Part. Sci. 28, 327 (1978).

[6]In some cases three-body-potential terms will be found among the "hyper-fine splitting" and other correction terms. For example, N. Isgur and G. Karl [Phys. Rev. D 18, 4187 (1978)] assume two-body, spin-independent harmonic confining potentials as a zeroth-order approximation, but find that a small potential term like the spin-orbit one will have three-body components at the next level of refinement.

[7]D. Liberman, Phys. Rev. D 16, 1542 (1978); D. Robson, Nucl. Phys. A308, 327 (1978).

[8]For the corresponding two-dimensional problem, an analytic solution is given in R. C. Giles, Phys. Rev. D 18, 513 (1978).

[9]This statement is somewhat dependent upon the actual value of the parameters used. A more precise statement is that the spherical approximation is accurate for $r \ll R_\infty$, where R_∞ is defined in Sec. IV. If we assume exclusively for the purpose of illustration the values $B^{1/4} = 0.145$ GeV and $\alpha_s = 0.55$, then $R_\infty = 1.12$ fm. Note, however, the discussion of α_s in Sec. VI and Ref. 1.

[10]T. DeGrand *et al.*, Phys. Rev. D 12, 2060 (1975).

[11]See, e.g., J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).

[12]K. Johnson, in *Current Trends in the Theory of Fields*, proceedings of the Symposium, Tallahassee, 1978, edited by J. E. Lanutti and P. K. Williams (AIP, New York, 1978), p. 112.

[13]The result for one quark and one antiquark appears in L. Heller and K. Johnson, Phys. Lett. 84B, 501 (1979). See, also, appendix B in T. D. Lee, Phys. Rev. D 19, 1802 (1979).

[14]For $n \geq 4$ there are some complications, because more than one color singlet can be constructed. If the quarks are all inside one sphere, then $W[R]$ will become a matrix in color-singlet space. In addition, the dipole approximation may break down already at a scale much smaller than $B^{-1/4}$ if color-singlet subsystems begin to separate.

[15]W. C. Haxton, private communication. This calculation is similar to those presented in Ref. 1, but for fixed coupling constant (Ref. 9) $\alpha_s = 0.55$.

[16]P. Gnädig *et al.*, Phys. Lett. 64B, 62 (1976); K. Johnson and C. B. Thorn, Phys. Rev. D 13, 1934 (1976).

[17]Equation (16) can be verified by a calculation of this quantity in a finite-cylinder approximation to the bag. See discussion below.

[18]T. Appelquist *et al.*, Annu. Rev. Nucl. Part. Sci. 28, 387 (1978).

[19]If one insists on constraining $\alpha(1\text{ fm})$ to the value 2.2, then the low-lying states can still be fit (with $\Lambda = 0.375$ GeV), but the higher states are pushed up too high in energy.

[20]K. Johnson, Stanford Report No. SLAC-PUB-2436, 1979 (unpublished).

[21]G. L. Payne *et al.*, Phys. Rev. C 22, 823 (1980).

[22]Yu. A. Simonov, Yad. Fiz. 3, 630 (1966) [Sov. J. Nucl. Phys. 3, 461 (1966)].

[23]After the completion of this work, we received reports of related work by P. Hasenfratz *et al.* [Phys. Lett. 95B, 299 (1980); 94B, 401 (1980)]. These authors also find that the sphere, and the Y and Δ shapes, are accurate approximations for small and large separations, respectively.