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Block-spin method for the φ^4 theory on a lattice

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A real-space renormalization-group method for the two-dimensional φ^4 field theory on a lattice is proposed. The blocks are treated using a canonical transformation of variables followed by an effective decoupling. A satisfactory picture is attained for the phases and low-lying excitations of the theory.

I. INTRODUCTION

The main reason for the large number of papers devoted to lattice field theory,[1] by high-energy physicists, is the search for new nonperturbative approaches capable of describing such phenomena as confinement, coherent states, phase transitions, etc., which are supposed to play a major role in hadron physics.[2] One of the new methods provided by the lattice strategy[3] in its various ways is the renormalization group.[4,5] A Hamiltonian blocking procedure conceptually related to the spirit of the renormalization group is due to Stoeckly and Scalapino[6] and to Drell, Weinstein, and Yankielowicz.[7] It constructs progressively and iteratively an effective ground state for the theory under study.[8,9]

These techniques have obviously severe limitations to describe a general interacting theory; for example, the description of the interaction of two fields has not been accomplished so far even in the most simple cases. However, they are simple, intuitive, and in some cases have shown themselves as powerful to reproduce with great accuracy exactly known nonperturbative numerical results.[10]

One of the main difficulties one faces with this method in analyzing an interacting theory appears at the time of decoupling the block variables. Even for a small-size block, the criterion for choosing the effective variable one retains is not obvious. Therefore it is interesting to find simple, approximate, and of course reliable recipes, which can be a safe help in analyzing more ambitious theories. With these remarks in mind I present here a model of a block-spin approach to the two-dimensional φ^4 theory.[11] It is quite simple to set up the recursion formulas of the coefficients, and it provides a very clear picture of the two phases of the system, predicting correctly the second-order character of the phase transition of the vacuum,[12] and also the particlelike and kinklike nature of the minimum excitations.

The paper is organized as follows: First, a review of the application of this technique to the free bosonic theory is given, which is useful in establishing the notation which will be used afterwards. Section III is devoted to the φ^4 theory. Two-site blocks are formed, and effectively decoupled to get the renormalization group equations. They are written explicitly in the two limiting cases where we can present them in a compact way. The numerical results for any value of parameters are given in Sec. IV, with plots of different observables. Section IV also contains the conclusions and a brief comment about the performance of the method with other theories. Finally in the Appendix some formulas used throughout the text are collected.

II. FREE BOSONIC THEORY ON THE LATTICE

The Hamiltonian of a free bosonic field on a one-dimensional spatial lattice is[7]

$$H=\Lambda\sum_{j=-N}^{N}\left[\tfrac{1}{2}p_j^{\,2}+\tfrac{1}{2}(\mu^2+2)x_j^{\,2}-x_jx_{j+1}\right] \tag{1}$$

with the canonical equal-time commutation relations $[p_j,x_{j'}]=-i\delta_{jj'}$ and periodic boundary conditions. The exact solution of this theory describes a system of noninteracting oscillators of frequency

$$\omega_\kappa=[\mu^2+4\sin^2(\tfrac{1}{2}\kappa)]^{1/2},\quad \kappa=\frac{2\pi}{2N+1}n,\quad n=0,\pm1,\pm2,\ldots,\pm N \tag{2}$$

with ground-state energy density

$$\epsilon_0=\frac{1}{2\pi}\int_0^{\pi}d\kappa[\mu^2+4\sin^2(\tfrac{1}{2}\kappa)]^{1/2}. \tag{3}$$

The correlation function $\Gamma(J)=\langle x_0 x_J\rangle$ is also obtained by transforming to momentum space. One gets

$$\Gamma(J)=\sum_{m=-N}^{N}\frac{1}{(2N+1)}\cos\left[\left(\frac{2\pi}{2N+1}\right)mJ\right]\Big/2\left\{\mu^2+4\sin^2\left[\left(\frac{2\pi}{2N+1}\right)\frac{m}{2}\right]\right\}^{1/2}. \tag{4}$$

To begin the iterative procedure, the lattice is dissected into say two-site blocks. So the Hamiltonian H is expressed as a sum of independent block Hamiltonians and the interblock terms. One of the block Hamiltonians is[13]

$$h=\tfrac{1}{2}p_1^2+\tfrac{1}{2}(\mu^2+2)x_1^2+\tfrac{1}{2}p_2^2+\tfrac{1}{2}(\mu^2+2)x_2^2-x_1x_2. \tag{5}$$

Let us define two new variables, an average variable and a relative one,

$$x=\tfrac{1}{2}(x_1+x_2),\quad x'=x_1-x_2. \tag{6}$$

In terms of them we have

$$x_1=x+\tfrac{1}{2}x',\quad x_2=x-\tfrac{1}{2}x',\qquad \partial_{x_1}=\tfrac{1}{2}\partial_x+\partial_{x'},\quad \partial_{x_2}=\tfrac{1}{2}\partial_x-\partial_{x'}, \tag{7}$$

and h becomes decoupled,

$$h=\tfrac{1}{2}\left\{\tfrac{1}{2}p^2+\tfrac{1}{2}[4(\mu^2+2-1)]x^2\right\}+2\left\{\tfrac{1}{2}p'^2+\tfrac{1}{2}[\tfrac{1}{4}(\mu^2+2+1)]x'^2\right\}. \tag{8}$$

Our approximation now is to replace any power of the relative coordinate x' by its ground-state expectation value. In this way we pass in the general Hamiltonian to a new problem where the number of degrees of freedom has been reduced by a factor of $\frac{1}{2}$. The new Hamiltonian is

$$H^{(1)}=\sum_l \tfrac{1}{2}\left\{\tfrac{1}{2}p_l^2+\tfrac{1}{2}[4(\mu^2+2-1)]x_l^2-2x_lx_{l+1}\right\}+\sum_l \tfrac{1}{2}(\mu^2+3)^{1/2}. \tag{9}$$

Its ground state has approximately the same information that the vacuum of H had.

Iterating this process n times we get

$$H^{(n)}=\sum_{l'}\left[d^{(n)}+(\tfrac{1}{2})^n\left(\tfrac{1}{2}p_{l'}^2+\tfrac{1}{2}\omega^{(n)2}x_{l'}^2-2^n x_{l'}x_{l'+1}\right)\right]$$
$$=\sum_{l'}\left[d^{(n)}+(\tfrac{1}{2})^n\left(\tfrac{1}{2}p_{l'}^2+\tfrac{1}{2}4^nZ^{(n)2}x_{l'}^2-2^n x_{l'}x_{l'+1}\right)\right] \tag{10}$$

with

$$d^{(n+1)}=2d^{(n)}+\frac{1}{2}\left(\mu^2+\frac{3}{2^n}\right)^{1/2},\qquad Z^{(n+1)2}=\mu^2+(\tfrac{1}{2})^n. \tag{11}$$

In

$$\mathcal{E}_g=\lim_{n\to\infty}\frac{d^{(n)}}{2^n}, \tag{12}$$

one can easily recognize the prediction of the method for the ground-state energy density.[7] For $\mu^2\to 0$, which is the most unfavorable case, $\mathcal{E}_g=0.67$ to be compared with the exact result of (3), $\mathcal{E}_0=0.64$. On the other hand, aside from the global factor $(\frac{1}{2})^n$, as we iterate, the truncated Hamiltonian becomes that of a set of quantum oscillators with quadratic coefficient $4^n\mu^2$ coupled between neighbors with a strength 2^n, i.e., they become decoupled at will. So, by inspection, the effective vacuum we get consists of the state of having all the oscillators in their ground state. This state is unique, and the perturbation of the coupling terms is irrelevant. Thus the minimum excitation of the theory consists in exciting one of the oscillators. The mass gap will be $(\frac{1}{2})^n(4^n\mu^2)^{1/2}=\mu$, which is the exactly correct result.

To compute the two-point correlation function $\Gamma(J)=\langle x(0)x(J)\rangle$ with this approximate technique, we follow, adapting it to our problem, the method developed for Ising spins in Ref. 14. The result is

$$\Gamma(J)=(\tfrac{1}{2})^{q+1}\left\{-\frac{1}{[\mu^2+(3/2^{q-1})]}+\frac{1}{2}\left[\sum_{j=q+1}^{\infty}(\tfrac{1}{2})^{j-q-1}\frac{1}{[\mu^2+(3/2^{j-1})]^{1/2}}\right]\right\}, \tag{13}$$

where $J=2^q$, q being the number of blocking steps which are necessary to bridge the distance J. In (12) we have used the formula (A2) of the Appendix for the primed nth oscillator, i.e., the one we discard in each step. Equations (13) and (4) provide similar decaying to zero functions.

III. φ^4 THEORY ON THE LATTICE

The φ^4 theory with nearest-neighbor interactions is described on the lattice by the Hamiltonian

$$H=\sum_j\left[\tfrac{1}{2}p_j^2+\tfrac{1}{2}(\mu^2+2)x_j^2+\lambda x_j^4-x_jx_{j+1}\right]. \tag{14}$$

In analogy with the free case, we begin dissecting (14) into two-site blocks, with a Hamiltonian given by

$$h=\tfrac{1}{2}p_1^2+\tfrac{1}{2}(\mu^2+2)x_1^2+\lambda x_1^4+\tfrac{1}{2}p_2^2+\tfrac{1}{2}(\mu^2+2)x_2^2+\lambda x_2^4-x_1x_2 \tag{15}$$

and define x and x' as we did in (6). Now h is not decoupled,

$$h=\tfrac{1}{2}\{\tfrac{1}{2}p^2+\tfrac{1}{2}[4(\mu^2+2-1)]x^2+4\lambda x^4\}$$
$$+2\{\tfrac{1}{2}p'^2+\tfrac{1}{2}[\tfrac{1}{4}(\mu^2+2+1)]x'^2+\tfrac{1}{16}\lambda x'^4\}+3\lambda x^2x'^2 \tag{16}$$

and we need an effective procedure to get it. An expeditive, however too crude, way would be to substitute the x'^2 factor of the coupling term by its ground-state expectation value (using as ground state that of the primed oscillator). With an example we can easily convince ourselves that this is not adequate. Let us suppose that the initial parameters correspond to what will be deeply ordered phase, i.e., large and negative μ^2 for a fixed λ. With the former recipe we would have

$$h\to\tfrac{1}{2}\{\tfrac{1}{2}p^2+\tfrac{1}{2}[4(\mu^2+2-1)]x^2+4\lambda x^4\}+3\lambda\langle x'^2\rangle x^2$$
$$+2\left\langle\tfrac{1}{2}p'^2+\tfrac{1}{2}[\tfrac{1}{4}(\mu^2+2+1)]x'^2+\frac{\lambda}{16}x'^4\right\rangle . \tag{17}$$

As by hypothesis we are deeply in the magnetized region, $\langle x'^2\rangle$ may be well approximated by its classical value (see the Appendix),

$$\langle x'^2\rangle=\frac{M^2-3}{\lambda},\quad M^2=|\mu^2|, \tag{18}$$

$$h\to\text{const}+\tfrac{1}{2}\{\tfrac{1}{2}p^2+\tfrac{1}{2}[4(2M^2-6)x^2]+4\lambda x^4\} \tag{19}$$

and we see that the oscillator of the average coordinate has changed the geometry of its potential to a single-minimum well (in other words the quadratic coefficient is positive now). This is clearly nonrealistic. Using a simple classical argument we will see that the basic effect of the coupling term in (16) is not to destroy the two-minima geometry of the coordinate x but to do that only with the x' coordinate.

Forgetting for a minute the kinetic energy of (16), let us see which one of the four following classical situations minimizes the energy:

(i) Both particles are lying at the bottom of their respective wells. The energy would be

$$E=\frac{1}{2}\left\{-\frac{[4(M^2-1)]^2}{16\times4\lambda}\right\}=-\frac{(M^2-1)^2}{8\lambda},$$
$$E'=2\left\{-\frac{[\frac{1}{4}(M^2-3)]^4}{16(\lambda/16)}\right\}=-\frac{(M^2-3)^2}{8\lambda},$$
$$V=3\lambda\frac{4(M^2-1)}{16\lambda}\,\frac{\frac{1}{4}(M^2-3)}{\lambda/4}=\frac{3(M^2-1)(M^2-3)}{4\lambda}, \tag{20}$$
$$E_{\rm i}=-\frac{(M^2-1)^2+(M^2-3)^2}{8\lambda}+\frac{3(M^2-1)(M^2-3)}{4\lambda}.$$

(ii) One particle lies on $x=0$ and the other at the bottom of one of the two minima of x',

$$E=0,\quad V=0,$$
$$E'=-\frac{(M^2-3)^2}{8\lambda}, \tag{21}$$
$$E_{\rm ii}=-\frac{(M^2-3)^2}{8\lambda}.$$

(iii) One particle at one of the bottoms of the x well and the other at $x'=0$,

$$E=-\frac{(M^2-1)^2}{8\lambda},\quad E'=V=0, \tag{22}$$
$$E_{\rm iii}=-\frac{(M^2-1)^2}{8\lambda}.$$

(iv) Both particles are at zero,

$$E_{\rm iv}=0.$$

For M^2 large the energy is minimized by the third situation, so that, roughly speaking, in the ground state of (16) the x particle sees a double well, and the x' a single well. From these simple remarks we conclude that the manipulation of (17) is not good.

To decouple (16) we will proceed more symmetrically between x and x'. The coupling term is divided into two equal pieces $\frac{3}{2}\lambda x^2x'^2$. One of them serves to modify the x' oscillator with a term $\frac{3}{2}\lambda\langle x^2\rangle x'^2$, where $\langle x^2\rangle$ has been computed in the original x oscillator. The other piece $\frac{3}{2}\lambda x^2x'^2$ is used to modify the average oscillator x using the already modified x' oscillator.

With this basic philosophy let us work out all the expressions for the deeply ordered phase. Here

$$\langle x^2\rangle=\frac{M^2-1}{4\lambda}, \tag{23}$$

$$h\to\tfrac{1}{2}\{\tfrac{1}{2}p^2+\tfrac{1}{2}[4(\mu^2+1)]x^2+4\lambda x^4\}+\tfrac{3}{2}\lambda\frac{M^2-1}{4\lambda}x'^2+\tfrac{3}{2}\lambda x^2x'^2+2\{\tfrac{1}{2}p'^2+\tfrac{1}{2}[\tfrac{1}{4}(\mu^2+3)]x'^2+\tfrac{1}{16}\lambda x'^4\}$$
$$=\tfrac{1}{2}\{\tfrac{1}{2}p^2-\tfrac{1}{2}[4(M^2-1)]x^2+4\lambda x^4\}+\tfrac{3}{2}\lambda x^2x'^2+2\{\tfrac{1}{2}p'^2+\tfrac{1}{2}[\tfrac{1}{8}(M^2+3)]x'^2+\tfrac{1}{16}\lambda x'^4\}$$
$$-\tfrac{1}{2}\{\tfrac{1}{2}p^2-\tfrac{1}{2}[4(M^2-1)]x^2+4\lambda x^4\}+\tfrac{3}{2}\lambda\frac{1}{2[(M^2+3)/8]^{1/2}}x^2+\text{const}$$
$$=\tfrac{1}{2}\left\{\tfrac{1}{2}p^2-\tfrac{1}{2}\left[4\left(M^2-1-\frac{3}{\sqrt{2}}\,\frac{\lambda}{(M^2+3)^{1/2}}\right)x^2\right]+4\lambda x^4\right\}+\text{const}. \tag{24}$$

Hence the new global Hamiltonian would be

$$H^{(1)}=\sum_l\left(\tfrac{1}{2}\right)^1\left\{\frac{p_l^{\,2}}{2}+\frac{x_l^{\,2}}{2}\left[4^1\left(\mu^2+2-1+\frac{3\lambda}{[2(M^2-3)]^{1/2}}\right)\right]+4^1\lambda x_l^{\,4}-2^1x_lx_{l+1}\right\}+\text{const}. \tag{25}$$

Iterating this process n times one obtains

$$\begin{aligned}H^{(n)}&=\sum_{l'}(\tfrac{1}{2})^n\{\tfrac{1}{2}p_{l'}^{\,2}+\tfrac{1}{2}\omega^{(n)2}x_{l'}^{\,2}\\&\quad+\Delta^{(n)}x_{l'}^{\,4}-2^nx_{l'}x_{l'+1}\}+\text{const}\\&=\sum_{l'}(\tfrac{1}{2})^n\left\{\frac{p_{l'}^{\,2}}{2}+\tfrac{1}{2}4^nz^{(n)2}x_{l'}^{\,2}+4^n\lambda x_{l'}^{\,4}\right.\\&\quad\left.-2^nx_{l'}x_{l'+1}\right\}+\text{const}\end{aligned} \tag{26}$$

with

$$Z^{(0)2}=\mu^2+2,$$

$$Z^{(n+1)2}=Z^{(n)2}-(\tfrac{1}{2})^n+\frac{3\lambda}{\sqrt{2}\;2^n[|Z^{(n)2}|+5(\tfrac{1}{2})^n]^{1/2}}, \tag{27}$$

$$Z^{(n+1)2}=\mu^2+\frac{3\lambda}{\sqrt{2}}\sum_{l=0}^{n}\frac{1}{2^l}\frac{1}{[|Z^{(n)2}|+5(\tfrac{1}{2})^l]^{1/2}}. \tag{28}$$

So we see that in the effective Hamiltonian that governs the deeply ordered region, aside from the global factor $(\frac{1}{2})^n$ which will set the magnitude of the excitations, the anharmonic oscillators become progressively decoupled, and we can consider the coupling terms as a perturbation of the single-site Hamiltonians. For a negative and large enough μ^2 the series $Z^{2(n)}$ converges for a large n to a finite number Z^2, which is negative, and which obviously depends on the initial μ^2 and λ. These anharmonic oscillators have two minima; the position of them is a constant equal to $|Z^2|/4\lambda$ and their depth increases at will with a factor of 4 for each blocking step. In this situation, the two lowest-energy eigenstates become degenerate, and so our φ^4 field theory would have a largely degenerate vacuum, corresponding to any situation of having the different particles indistinctly in one of the two minima of their respective double-well potentials. The pertubation coming from the coupling terms removes obviously this large degeneracy giving a twice-degenerate vacuum, corresponding to the states with either all the particles at the left or all at the right. That is we are describing a magnetized phase, with an order parameter

$$\langle x\rangle=\tfrac{1}{2}\left(\frac{|Z^2|}{\lambda}\right)^{1/2}. \tag{29}$$

With respect to the low excitations, it is obvious that here the most economical one corresponds to setting up a kink or domain wall. Its mass would be

$$M_{\text{kink}}=(\tfrac{1}{2})^n[2^{n+1}\langle x\rangle^2]=\frac{|Z^2|}{2\lambda} \tag{30}$$

Let us see now the expressions we get for the iteration in the deeply disordered region, i.e., μ^2 is large and positive. Here we are allowed to do the harmonic approximation for the different $\langle x^2\rangle$ values. From (16),

$$\langle x^2\rangle=\frac{1}{2[4(\mu^2+1)]^{1/2}}, \tag{31}$$

$$\begin{aligned}h&-\tfrac{1}{2}\{\tfrac{1}{2}p^2+\tfrac{1}{2}[4(\mu^2+2-1)]x^2+4\lambda x^4\}+\tfrac{3}{2}\lambda x^2x'^2+\tfrac{3}{2}\lambda x'^2\frac{1}{2[4(\mu^2+1)]^{1/2}}\\&+2\{\tfrac{1}{2}p'^2+\tfrac{1}{2}[\tfrac{1}{4}(\mu^2+3)]x'^2+\tfrac{1}{16}\lambda x'^4\}\\&=\tfrac{1}{2}\{\tfrac{1}{2}p^2+\tfrac{1}{2}[4(\mu^2+2-1)]x^2+4\lambda x^4\}+\tfrac{3}{2}\lambda x^2x'^2+2\left\{\tfrac{1}{2}p'^2+\tfrac{1}{2}\left[\tfrac{1}{4}\left(\mu^2+3+\frac{3\lambda}{[4(\mu^2+1)]^{1/2}}\right)\right]x'^2+\tfrac{1}{16}\lambda x'^4\right\}\\&-\tfrac{1}{2}\left\{\tfrac{1}{2}p^2+\tfrac{1}{2}\left[4\left(\mu^2+1+\tfrac{3}{2}\lambda\frac{1}{[\mu^2+3+3\lambda/4(\mu^2+1)]^{1/2}}\right)\right]x^2+4\lambda x^4\right.\quad+\text{const}.\end{aligned} \tag{32}$$

Hence the new global Hamiltonian would be

$$H^{(1)}=\sum_l(\tfrac{1}{2})^1\left\{\frac{1}{2}p_l^{\,2}+\frac{1}{2}x_l^{\,2}\left[4^1\left(\mu^2+1+\frac{3\lambda}{2\{\mu^2+3+3\lambda/[4(\mu^2+1)]^{1/2}\}}\right)\right]+4^1\lambda x^4-2^1x_lx_{l+1}\right\}+\text{const}. \tag{33}$$

Iterating n times,

$$\begin{aligned}H^{(n)}&=\sum_{l'}(\tfrac{1}{2})^n\{\tfrac{1}{2}p_{l'}^{\,2}+\tfrac{1}{2}\omega^{(n)2}x_{l'}^{\,2}+\Delta^{(n)}x_{l'}^{\,4}-2^nx_{l'}x_{l'+1}\}+\text{const}\\&=\sum_{l'}(\tfrac{1}{2})^n\{\tfrac{1}{2}p_{l'}^{\,2}+\tfrac{1}{2}4^nZ^{(n)2}x_{l'}^{\,2}+4^n\lambda x_{l'}^{\,4}-2^nx_{l'}x_{l'+1}\}+\text{const}\end{aligned} \tag{34}$$

with

$$Z^{(0)2} = \mu^2 + 2 ,$$

$$Z^{(n+1)2} = Z^{(n)2} - \left(\tfrac{1}{2}\right)^n + \frac{3\lambda}{2^{n+1}\left\{Z^{(n)2} + \left(\tfrac{1}{2}\right)^n + 3\lambda/2^{n+1}\left[Z^{(n)2} - \left(\tfrac{1}{2}\right)^n\right]^{1/2}\right\}} , \tag{35}$$

$$Z^{(n+1)2} = \mu^2 + \frac{3\lambda}{2}\sum_{l=0}^{n}\frac{1}{2^l}\frac{1}{\left\{Z^{(l)2} + \left(\tfrac{1}{2}\right)^l + 3\lambda/2^{l+1}\left[Z^{(l)2} - \left(\tfrac{1}{2}\right)^l\right]^{1/2}\right\}} . \tag{36}$$

The iteration process leads again to progressively decoupled anharmonic oscillators. The difference with the above lies in the sign of Z^2—here this limit is positive. Therefore, the vacuum of the theory is given by the superposition of the ground states of each single-well anharmonic oscillator. This state is unique, there is no magnetization, and the perturbative coupling terms play no role. The minimally excited state corresponds to having one oscillator minimally excited and the rest in their ground states. This is a sort of particlelike excitation; its mass would be

$$M_{\text{particle}} = \left(\tfrac{1}{2}\right)^n (4^n Z^2)^{1/2} = (Z^2)^{1/2} . \tag{37}$$

To end this section we present the formulas which allow us to compute the correlation functions in the two extreme regions. For $\Gamma(J) = \langle x_0 x_J \rangle$, if $J = 2^q$ we find

$$\Gamma(J) = \frac{1}{4}\left(-\langle x'^2\rangle_q + \sum_{j=q+1}^{N}\langle x'^2\rangle_j\right) + \langle x^2\rangle_N \tag{38}$$

with $N \to \infty$. $\langle x^2\rangle_j$ stands for the expectation value in the ground state of the x oscillator, in the nth step of iteration. In the deeply ordered region, (38) is computed using

$$\langle x'^2\rangle_j = \frac{1}{2}\frac{1}{2^{j-1}\left[|Z^{(j)2}|/2 + \frac{5}{2}\left(\tfrac{1}{2}\right)^j\right]^{1/2}} ,$$
$$\langle x^2\rangle_N = \tfrac{1}{4}\frac{|Z^2|}{\lambda} . \tag{39}$$

On the other hand, in the deeply disordered region we use instead

$$\langle x'^2\rangle_j = \frac{1}{2\times 2^{j-1}\left\{Z^{(j)2} + \left(\tfrac{1}{2}\right)^j + 3\lambda/2^{j+1}\left[Z^{(j)2} - \left(\tfrac{1}{2}\right)^j\right]^{1/2}\right\}^{1/2}} , \tag{40}$$
$$\langle x^2\rangle_N = 0 .$$

The formulas presented so far for the computation of $Z^2(\lambda, \mu^2)$ have been computed in two extreme regions, which has allowed us to approximate the $\langle x^2\rangle$ of an anharmonic oscillator either by the classical value or by the harmonic approximation. In general these approximations are not good, and one should compute the $\langle x^2\rangle$ values more carefully. For this quantum-mechanical computation we rely on the Hartree-type linearization of the anharmonic oscillator[15] as explained in the Appendix. The numerical results for any initial value of the parameters appear in the next section. They are improved in accuracy. However, one should realize that the picture we have seen in the two extreme cases holds at any time. That is, if the final Z^2 value is negative the vacuum is doubly degenerate and $\langle x\rangle$ is different from 0. If it is positive the vacuum is unique. Furthermore, for any Z^2 final value and for any of our approximations, (29), (30), and (37) are correct. This may be easily seen, for example, for (37). Using (A7),

$$M_{\text{particle}} = \left(\frac{1}{2}\right)^n\left[(4^n\lambda)^{1/3} g\left(1, \frac{4^n Z^2}{(4^n\lambda)^{2/3}}\right)\right]$$
$$= \left(\frac{1}{2}\right)^n\left[(4^n\lambda)^{1/3}\frac{2^n (Z^2)^{1/2}}{(4^n\lambda)^{1/3}}\right] = \sqrt{Z^2} . \tag{41}$$

IV. NUMERICAL RESULTS AND CONCLUSIONS

As commented above, the use of the two extreme approximations (A2) and (A4) to compute $\langle x^2\rangle$ for anharmonic oscillators is clearly bad for the intermediate region, where we will stick to the prediction of the Hartree approximation (A5), maintaining always the blocking philosophy explained at length in Sec. III. The results collected here are for $\lambda = 1$, which can be obviously changed at will.

The prediction for the magnetization $\langle x\rangle$ is plotted in Fig. 1, showing the second-order nature of this phase transition.[12] In Fig. 2 the mass gap of the theory is represented. The left part

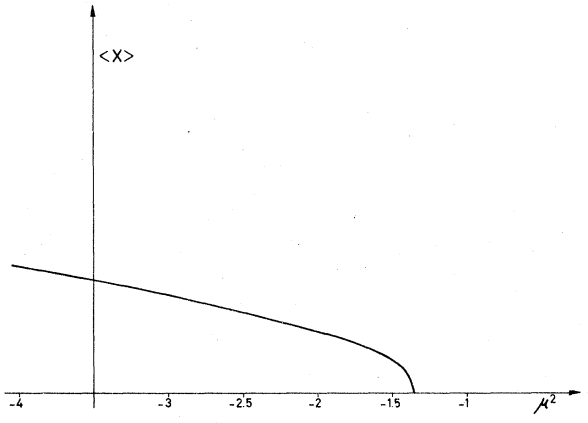

FIG. 1. A plot of the magnetization $\langle x\rangle$ vs μ^2 for $\lambda = 1$.

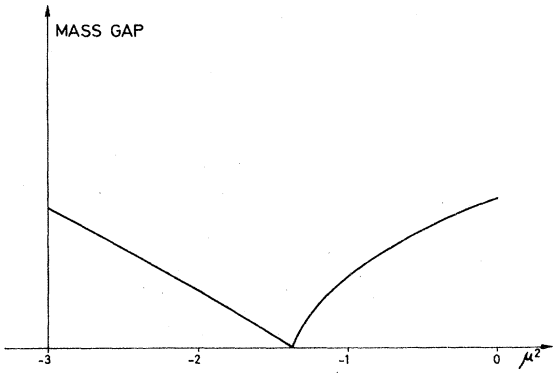

FIG. 2. A plot of the mass gap to the first excited state vs μ^2 for $\lambda=1$. The unit of energy is Λ.

of the curve corresponds to the kink, and the right one to the particlelike excitation. Both leave from 0. Finally in Fig. 3 we have drawn the results for the two-point correlation function in two extreme situations. 3(a) is in the ordered phase and 3(b) in the disordered one. We have omitted the computation of the ground-state energy density, which is stored in the successively accumulated constant terms. Obviously any of the predictions of the method may be easily improved in accuracy by using more exact expressions for the basic $\langle x^2\rangle$ value of the anharmonic oscillator. That would be certainly important for the numerical computation of critical parameters. However, we believe that our method at the level presented here is remarkably good for its simplicity and economy, leading to a quite simple picture of the asymptotic regions reached after iterating.

In a coming paper we study the higher dimensions of this model and others with continuous symmetries. However, we shall advance here the main qualitative features of the asymptotic behavior one gets after iterating.

For the φ^4 O(2) model in (1+1) dimensions, i.e., for a chain of two-dimensional anharmonic oscillators,

$$H=\sum_j\left[\tfrac{1}{2}\vec{p}_j^{\,2}+\tfrac{1}{2}\mu^2\vec{r}_j^{\,2}+\lambda\vec{r}_j^{\,4}+\tfrac{1}{2}(\vec{r}_j-\vec{r}_{j+1})^2\right] \qquad (42)$$

with $\vec{r}=(x,y)$, two phases are obtained. They are characterized by the sign of Z^2. When $Z^2>0$ we are again in a disordered phase with a finite mass gap. For $Z^2<0$, we get that the low-energy physics of this phase is equivalent to that of a set of quantum rotators with constant length, where the kinetic energy becomes as small as we wish in comparison with the coupling terms. In spite of this relative smallness we know that perturbation theory is not adequate.[16] A simple recursive method built *ad hoc* for this special situation[17] tells us that we are in massless phase with $\langle\vec{r}\rangle=0$.

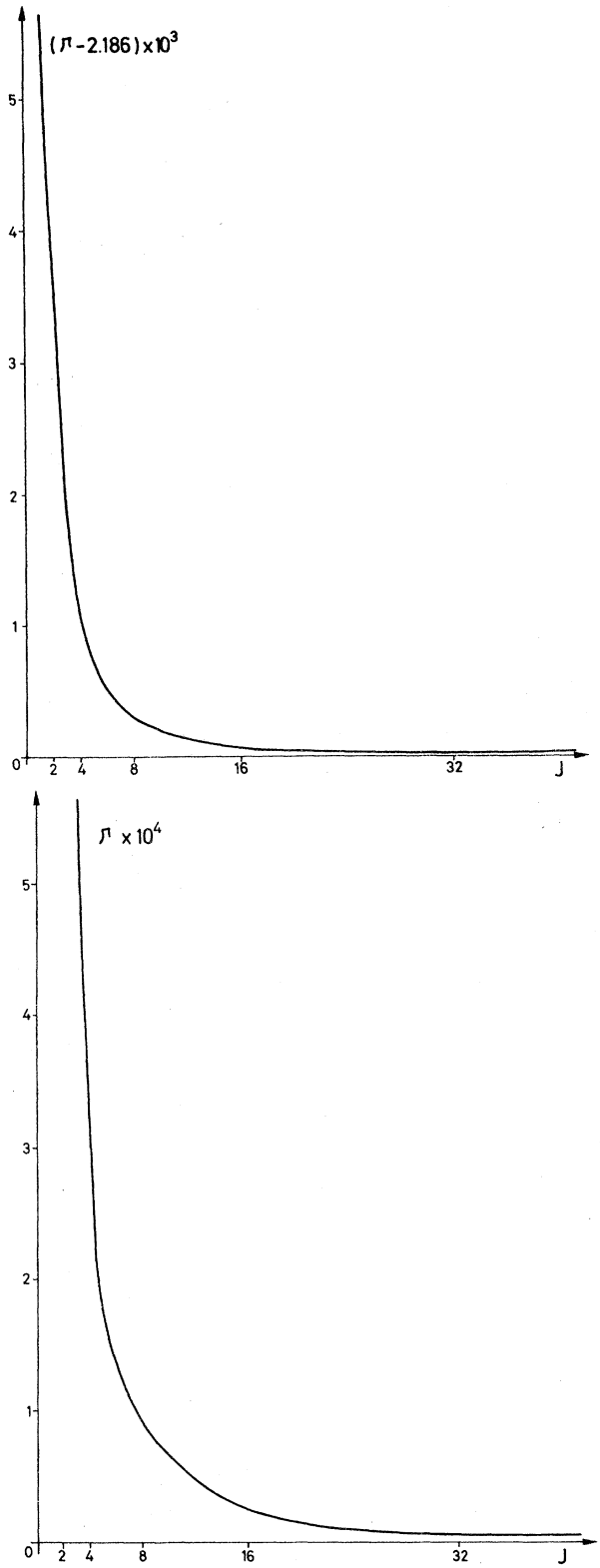

FIG. 3. A plot of the correlation function $\Gamma(J)=\langle x(0)x(J)\rangle$ vs J. (a) $\lambda=1$, $\mu^2=-10$. (b) $\lambda=1$, $\mu^2=+10$. In both cases we have used the formulas (37)–(41).

For the φ^4 O(3) model, i.e., a dynamics given by (42) with $\vec{r}=(x,y,z)$, for any initial value of μ^2 and λ we get $Z^2>0$. Thus there is only one phase. Technically this appears when one makes the basic manipulation of (24). Then the positive coefficient of x^2 one gets from the $\vec{r}'$ oscillator is always bigger than the original negative coefficient. This happens simply because now the quantum oscillators are tridimensional. Thus we get a massive theory for any value of the coupling.[18]

With respect to the (2+1)-dimensional case, we have begun exploring these models, extended in a hexagonal lattice. Each block contains four sites, and we retain the average variable. One always finds two phases, even for the O(3) case. The signal is, as usual, the sign of Z^2. Here for the $Z^2<0$ phase, perturbation theory is correct and, therefore, the corresponding global symmetry is spontaneously broken.

We believe that armed with a blocking procedure like this and the equivalent for fermions of Ref. 8 we will be able to describe satisfactorily the interaction of two fields and to analyze the physics of compound boson-fermion systems on the lattice.

ACKNOWLEDGMENTS

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APPENDIX

Here we collect some formulas of quantum oscillators, frequently used in the text. For the free case we needed the expectation value of x^2 in the ground state of harmonic oscillator. For the Hamiltonian

$$h=\tfrac{1}{2}p^2+\tfrac{1}{2}m^2x^2, \tag{A1}$$

$$\langle x^2\rangle=\frac{1}{2m}. \tag{A2}$$

For the φ^4 theory we need equivalently $\langle x^2\rangle$ in an anharmonic oscillator. For the Hamiltonian

$$h=\tfrac{1}{2}p^2+\tfrac{1}{2}m^2x^2+\lambda x^4, \tag{A3}$$

if we are deeply in the double-well region, i.e., $m^2<0$ and $|m^2|=M^2\gg 0$, we assign the classical (geometrical) value

$$\langle x^2\rangle=\frac{M^2}{4\lambda}. \tag{A4}$$

On any other situation of parameters we stick to the prediction of the Hartree-type linearization of the anharmonic oscillator.[15] Here $\langle x^2\rangle=a^2$ verifies the condition

$$a^2=\frac{1}{2(m^2+12\lambda a^2)^{1/2}}. \tag{A5}$$

Equation (A5) for a large and positive m^2 eventually reaches the harmonic value (A2).

In Fig. 4 we plot $\langle x^2\rangle$ vs m^2 for $\lambda=1$, showing the intercept of the two different regimes. This interpolating procedure is apparently crude but quite sufficient for our purposes here.

Once one knows the value of $\langle x^2\rangle$ for $\lambda=1$, and any quadratic coefficient, any other situation is also known by using the scaling relation

$$\langle x^2(\lambda,m^2)\rangle=\frac{1}{\lambda^{1/3}}\langle x^2(1,m^2/\lambda^{2/3})\rangle. \tag{A6}$$

The difference in energy between the two lowest-energy eigenvalues also verifies a scaling relation

$$g(\lambda,m^2)=\lambda^{1/3}g(1,m^2/\lambda^{2/3}). \tag{A7}$$

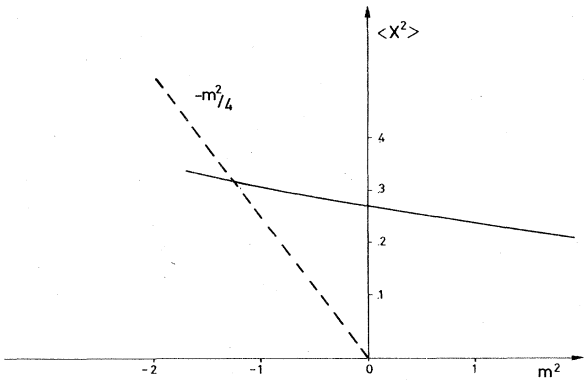

FIG. 4. A plot of $\langle x^2\rangle$ in the anharmonic oscillator with a Hamiltonian $h=\frac{1}{2}p^2+\frac{1}{2}m^2x^2+\lambda x^4$, for $\lambda=1$. The dashed line corresponds in the double-well region to the classical value $\langle x^2\rangle=-m^2/4\lambda$. The continuous line is the prediction of the Hartree aproximation. For $\lambda=1$, the intercept of both regimes happens for $m^2\approx-1.3$.

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