Monte Carlo study of renormalization in lattice gauge theory

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Using Monte Carlo methods with SU(2) and U(1) lattice gauge theories, we compare physical ratios of Wilson loops on different length scales. An interesting renormalization-group structure is suggested for the U(1) theory. For the SU(2) case we only see a fixed point at vanishing bare coupling. At this point asymptotic freedom is verified numerically.

I. INTRODUCTION

Monte Carlo methods have become powerful tools in the study of nonperturbative phenomena in gauge theories.[1-3] A discrete space-time lattice, used as an ultraviolet regulator in a Euclidean path integral, converts a quantized gauge theory into an equivalent classical statistical system[4] which is well suited to numerical simulation using established techniques. Possible phase transitions play a key role in all discussions of lattice gauge theory.[5,6] Indeed, the appearance of a transition in a U(1) gauge theory is essential to the existence of a massless photon in a lattice formulation of electrodynamics. Conversely, the absence of transitions in four-dimensional non-Abelian gauge models is central to our understanding of quark confinement in the continuum limit of these theories.

In earlier papers[2] we studied numerically the long-range forces between external sources with quark quantum numbers in lattice gauge theories based on the groups SU(2) and SU(3). Using the long-range force to define a renormalization scheme, we found numerical evidence that a linear potential at long distances survives in the weak-coupling continuum limit of the theory. One shortcoming of that work arose from the slow logarithmic decrease of the bare coupling with decreasing lattice size. This only allowed modestly weak coupling while still seeing the asymptotic linear potential on our lattices of only 6^4–10^4 sites. Furthermore, there are probably "roughening" transitions giving a nonanalytic behavior to this force at values of coupling where short-range correlation functions show no singularities.[7] This indicates an awkwardness with the renormalization scheme of holding the asymptotic force law fixed.

In this paper we present renormalization-group arguments based on physical quantities defined on finite length scales. This permits study of the renormalization of the bare coupling over a wider range of the cutoff parameter. Monte Carlo results for SU(2) gauge theory at weak coupling verify logarithmic dependence of the bare charge on distance scale, as predicted by asymptotic freedom. Passing from weak to strong coupling, we find no evidence for a renormalization-group fixed point, further strengthening evidence that the confinement inherent in Wilson's strong-coupling expansion[4] survives in the continuum limit.

To test these techniques on a system known to have a phase transition,[1,6] we also study the U(1) lattice gauge theory. Here asymptotic freedom is lost and a nontrivial renormalization-group structure appears. We conjecture that a continuum limit at the critical point gives an interacting field theory of photons and magnetic monopoles. Away from this critical point on the weak-coupling side, a continuum limit should yield a field theory of free photons with the monopole mass diverging when the cutoff is removed.

The essence of the procedure is to define a physical correlation function at some scale and demand that this quantity not change upon reducing the lattice spacing by a factor of 2. This relates the bare coupling constants for two values of the cutoff. Iteration then drives the bare coupling to a fixed point relevant for the continuum limit. For a non-Abelian gauge theory this yields the asymptotically free behavior of a logarithmic decrease in the bare coupling.

We only consider a single bare-coupling parameter; in this respect our analysis is less sophisticated than that proposed by Wilson[3] or, as used for spin systems, that of Swendsen.[8] Additional coupling terms could in principle reduce the finite-cutoff ambiguities at the price of substantially increasing the computing time. Our analysis will be entirely in the context of pure gauge theory without quarks. This is because we do not know how to treat fermion fields with Monte Carlo techniques.

In Sec. II we describe the general renormalization-group procedure without specifying the precise physical quantities used. Section III is where we introduce ratios of Wilson loops with different shapes but the same perimeters. We argue that ultraviolet divergences should cancel from such ratios and therefore they can serve to define a renormalized coupling constant. In Sec. IV we present our Monte Carlo measurements of these ratios. We observe asymptotic freedom for the SU(2) theory and a nontrivial structure for a U(1) gauge group. Section V contains some concluding remarks.

II. LENGTH SCALES AND RENORMALIZATION

In particle physics, the goal of renormalization is to remove ultraviolet divergences from a field theory. The bare-coupling parameters become functions of an ultraviolet cutoff in such a manner that physical quantities have a finite limit as the cutoff is removed. A renormalization scheme begins with the selection of an arbitrary set of physical measurables which is sufficiently complete to determine the bare parameters when the cutoff is in place. Then, as the cutoff is removed, the bare couplings are continuously adjusted in such a manner that these given measurables remain fixed. For a well-defined renormalizable theory, this procedure should yield unique finite limits for all physical quantities.

The observables used in a renormalization scheme are quite unconstrained. In quantum electrodynamics one usually fixes the physical electron mass and the coefficient of the long-range Coulomb force. In a confining theory, such as believed for the strong interaction, the choice is less obvious. One popular selection for nonperturbative studies of pure gauge theory without fermions is the coefficient K of the hypothesized long-distance linear potential between external sources with quark quantum numbers. Another possible choice is the mass of some physical bound state, such as the lightest glueball.

All of the quantities mentioned in the previous paragraph are defined in terms of long-range effects. This is clear for the long-distance potentials, but it also applies to a particle mass as this determines how the particle propagates over an extended range. It is, however, also convenient to consider physical observables involving only finite scales. For example, in traditional perturbative renormalization-group discussions one studies vertex functions in momentum space with all legs off shell at some given momentum scale μ. Alternatively, one might be interested in some interparticle force at a finite range r. By varying these parameters μ or r, one studies the interrelationships of physics on different length scales.

From now on we will restrict our discussion to a theory, such as quarkless gauge theory, which has only one bare dimensionless coupling parameter, g_0. A general physical observable P is a function of g_0, as well as the cutoff scale of length a, and the scale r on which P is to be measured,

$$P=P(r,a,g_0(a)) . \tag{2.1}$$

Here we have explicitly shown the cutoff dependence of the bare coupling $g_0(a)$. The precise form of this dependence will depend on the renormalization scheme used. For simplicity, assume that P is dimensionless; if it were not, just multiply by enough powers of r to make it so. For example, from an interparticle force $F(r)$ construct $P=r^2F$.

As a becomes small and we approach the continuum limit, P should lose cutoff dependence. Thus we expect

$$P\left(r,\frac{a}{2},g_0\left(\frac{a}{2}\right)\right)=P(r,a,g_0(a))+O(a^2) , \tag{2.2}$$

where we have arbitrarily compared cutoffs differing by a factor of 2. In general there are two classes of dimensional parameters which set the scale for the order-a^2 corrections in Eq. (2.2). First, of course, is the scale r used to define P. In addition we must consider the long-range physical parameters characterizing the continuum theory. In particular, regardless of how large r is, we expect corrections to Eq. (2.2) or order a^2m^2 where m is a typical mass in the physical spectrum. The important point is that it is dangerous to regard the lattice theory as phenomenologically useful when the spacing a is larger than either the scale under consideration or the characteristic scales of the continuum theory.

If one adopts the renormalization scheme of holding $P(r,a,g_0(a))$ fixed at a given scale r, then at that scale there are by definition no $O(a^2)$ corrections in Eq. (2.2). However, in the following we consider physics on different scales and so at some point these corrections must plague us. In the Monte Carlo analysis of Sec. III, our lattice of 10^4 sites forces us to keep r only a few times the lattice spacing and thus these cutoff corrections could be substantial for any a. Nonetheless we proceed with optimism and neglect these terms. The ultimate test of this lies with larger lattices, but our results in verifying asymptotic freedom are encouraging.

Since P is dimensionless we can scale a factor of 2 from r and a in Eq. (2.2) to give

$$P\left(2r,a,g_0\left(\frac{a}{2}\right)\right)\approx P(r,a,g_0(a)) , \tag{2.3}$$

where the approximate equality represents the neglected finite-cutoff corrections. This equation shows the correlation between the bare coupling for two values of cutoff and the measured observable at two different length scales. The process that gave Eq. (2.3) is now iterated, with r replaced by $2^{-n}r$, to give the pivotal formula

$$P\left(2r, a, g_0\left(\frac{a}{2^{n+1}}\right)\right) \approx P\left(r, a, g_0\left(\frac{a}{2^n}\right)\right). \tag{2.4}$$

To study the renormalization of g_0, we use this formula as follows. Assume that for some fixed values of r and a we use some technique such as Monte Carlo simulation to calculate the two functions of bare coupling

$$F(g_0) \equiv P(r, a, g_0), \tag{2.5}$$

$$G(g_0) \equiv P(2r, a, g_0). \tag{2.6}$$

Suppose further that we pick r so that in the physical continuum limit P has the value P_0:

$$\lim_{a \to 0} P(r, a, g_0(a)) = P_0. \tag{2.7}$$

Then on a graph of $F(g_0)$ versus g_0, we find $g_0(a)$ as the value of g_0 where

$$F(g_0(a)) = P_0. \tag{2.8}$$

From Eq. (2.3) we find the coupling at half this cutoff:

$$G\left(g_0\left(\frac{a}{2}\right)\right) = P_0. \tag{2.9}$$

Knowing $g_0(a/2)$, we define P_1 by

$$P_1 = F\left(g_0\left(\frac{a}{2}\right)\right). \tag{2.10}$$

Equation (2.4) now tells us how to find $g_0(a/4)$:

$$G\left(g_0\left(\frac{a}{4}\right)\right) = P_1. \tag{2.11}$$

Iterating gives

$$P_n \equiv F\left(g_0\left(\frac{a}{2^n}\right)\right), \tag{2.12}$$

$$G\left(g_0\left(\frac{a}{2^{n+1}}\right)\right) = P_n. \tag{2.13}$$

This entire procedure generates graphically a "staircase" as shown in Fig. 1. This figure is drawn for an asymptotically free theory with $g_0(0) = 0$.

In Fig. 2 we sketch a situation where the functions F and G cross each other at a nonvanishing g_0. Here the staircase asymptotically approaches this crossing point. We have a renormalization-group fixed point g_F defined by

$$F(g_F) = G(g_F). \tag{2.14}$$

Note that g_F can be approached either from stronger or weaker coupling. As the bare charge at some very small cutoff passes through g_F, the corresponding initial value P_0 drastically changes as we go from a staircase on one side of g_F to the other. This is the signal of a phase transition in the equivalent statistical mechanical system. The critical exponents of the transition are related to the relative slopes of F and G near the critical point. The absolute slopes depend on the value of a/r chosen to define these functions.

The above example represents a conventional ultraviolet attractive fixed point. One could also imagine a fixed point where again $F(g_F) = G(g_F)$ but

$$\left(\left|\frac{d}{dg}F(g)\right| - \left|\frac{d}{dg}G(g)\right|\right)\Bigg|_{g_F} > 0. \tag{2.15}$$

In this situation the staircase construction will always lead away from g_F. A continuum limit at such an ultraviolet repulsive fixed point is at best possible if g_0 is exactly equal to g_F. It is also conceivable that at some point in the construction Eq. (2.13) may have no solution. Several authors

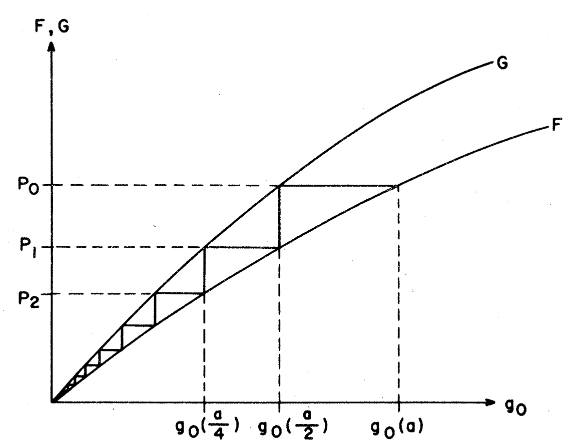

FIG. 1. The "staircase" construction of Sec. II for an asymptotically free theory.

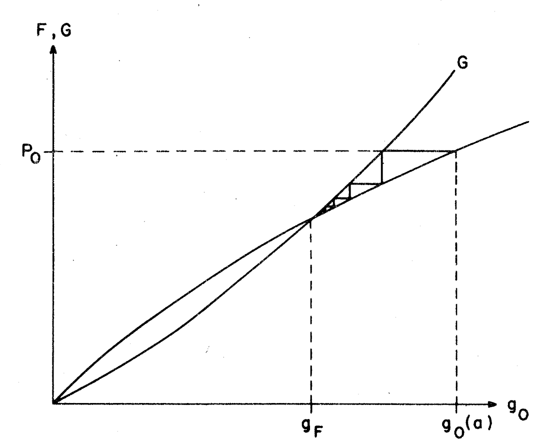

FIG. 2. An example of a nontrivial fixed point.

have argued that this may be the case for four-dimensional ϕ^4 theory, which thereby may not have a nontrivial continuum limit.[9]

Non-Abelian gauge theories are asymptotically free. This means that they have an attractive fixed point at $g_F = 0$. Perturbative expansion about this point yields

$$a\frac{d}{da}g_0(a) \equiv \beta(g_0) = \beta_0 g_0^3 + \beta_1 g_0^5 + O(g_0^7)\,. \quad (2.16)$$

Although in general $\beta(g_0)$ depends on renormalization scheme, the first two coefficients β_0 and β_1 are determined if g_0 becomes the conventionally normalized Yang-Mills charge in the classical limit. For SU(2) gauge theory these coefficients are[10]

$$\beta_0 = \frac{11}{3}\left(\frac{1}{8\pi^2}\right), \qquad \beta_1 = \frac{34}{3}\left(\frac{1}{8\pi^2}\right)^2. \quad (2.17)$$

Integrating Eq. (2.16) gives

$$\frac{1}{g_0^2(a)} = \beta_0 \ln\left(\frac{1}{\Lambda_0^2 a^2}\right) + \frac{\beta_1}{\beta_0}\ln\left[\ln\left(\frac{1}{\Lambda_0^2 a^2}\right)\right] + O(g_0^2)\,. \quad (2.18)$$

Here Λ_0 is an integration constant which depends on the cutoff scheme. In Ref. 2 we used Monte Carlo methods for pure SU(2) and SU(3) gauge theories with Wilson's lattice cutoff to calculate Λ_0 in units of the square root of the string tension K. For SU(2) we found

$$\Lambda_0 = (1.3 \pm 0.2) \times 10^{-2}\sqrt{K}\,. \quad (2.19)$$

Equation (2.18) implies that for a factor of 2 change in cutoff

$$\frac{1}{g_0^2(a/2)} = \frac{1}{g_0^2(a)} + 2\beta_0 \ln 2 + O(g_0^2)\,. \quad (2.20)$$

This gives an asymptotic freedom prediction for the functions F and G used in the staircase construction. For small g_0 we expect for SU(2) gauge theory

$$F(g_0) = G\left(\left(\frac{1}{g_0^2} - \frac{11}{12\pi^2}\ln 2\right)^{-1/2}\right). \quad (2.21)$$

Verification of this behavior is a primary check on our procedure.

When the physical scale r is small enough in an asymptotically free theory, one expects the validity of an asymptotic perturbation series

$$\lim_{a\to 0} P(r,a,g(a)) \equiv P(r) = \sum_{n=0}^{N} p_n g^{2n}(r) + O(g^{2N+2}(r))\,. \quad (2.22)$$

The quantity $g(r)$ represents an "effective" coupling at the scale r. There is considerable freedom in the definition of $g(r)$; the only requirement is that as g_0 is varied with the cutoff in place we have

$$g(r,a,g_0) = g_0 + O(g_0^3)\,. \quad (2.23)$$

In particular, any change in definition satisfying

$$g(r) \to g(r) + O(g^3(r)) \quad (2.24)$$

is a perfectly acceptable new renormalized charge. Indeed, we are free to define

$$g^2(r) = \frac{P(r) - p_0}{p_1}\,. \quad (2.25)$$

With this definition the perturbation series in Eq. (2.22) for $P(r)$ becomes trivial. Thus, with an appropriate shifting and rescaling, we can interpret our SU(2) results in terms of any effective coupling at a variable scale r.

III. RENORMALIZATION AND WILSON LOOPS

We use the standard Wilson formulation of gauge fields on a hypercubical lattice.[4] An element U_{ij} of a unitary gauge group is associated with each nearest-neighbor pair of lattice sites i and j. We restrict ourselves in this paper to the groups SU(2) and U(1). The action of the theory is a sum over all elementary squares $\Box$ of the lattice

$$S = \frac{2}{g_0^2}\sum_{\Box} S_{\Box}\,, \quad (3.1)$$

where

$$S_{\Box} = \mathrm{Tr}(U_{ij}U_{jk}U_{kl}U_{li}) \quad (3.2)$$

for SU(2) or

$$S_{\Box} = \tfrac{1}{2}\,\mathrm{Re}(U_{ij}U_{jk}U_{kl}U_{li}) \quad (3.3)$$

for U(1). Here the indices i, j, k, and l circulate about the "plaquette" $\Box$. The quantum theory is defined via the path integral

$$Z = \int\left(\sum_{\{i,j\}} dU_{ij}\right) e^{S(U)}\,, \quad (3.4)$$

where every link variable is integrated over with the invariant Haar measure for the group. The U(1) theory is known[1,6] to exhibit a phase transition at $g_0 = g_F \approx 1.0$.

The most studied "order parameter" of lattice gauge theory is the Wilson loop. Given a closed contour C of links in the lattice, one constructs the expectation value of the product of link variables about that contour:

$$W(C) = \left\langle \mathrm{Tr}\left(P \prod_C U_{ij}\right)\right\rangle. \quad (3.5)$$

Here P denotes "path ordering"; the group elements are ordered as they are encountered in a circuit of the contour. The expectation value is taken in the sense of the measure in Eq. (3.4); for an arbitrary function A of the link variables we define

$$\langle A\rangle=\frac{1}{Z}\int\prod_{\{i,j\}} dU_{ij}e^{S(U)}A(U). \tag{3.6}$$

The Wilson loops are often used as a signal for confinement. A linearly rising potential energy between two widely separated sources in the fundamental representation of the gauge group corresponds to large loops having an exponential falloff of $W(C)$ with the minimal surface area enclosed by C. In contrast, a nonconfining theory such as the weakly coupled U(1) model should give a decay of the Wilson loop no faster than exponentially with the perimeter.

We would like to use the Wilson loops to construct a physical function for the analysis of Sec. II. Unfortunately, the bare Wilson loop by itself cannot be used because of ultraviolet divergences. These divergences are of a rather trivial nature, arising from the infinitely thin contour. They represent the self-energy of the pointlike external sources circumnavigating the loop. The problem already appears in the exactly solvable continuum theory of free photons. There the expectation value of the operator

$$\exp\left(ie\int_C A_\mu dx_\mu\right) \tag{3.7}$$

is easily shown to vanish. Inserting an ultraviolet cutoff, one sees that this vanishing is exponential in the perimeter of the loop measured in units of the cutoff. If the contour has sharp corners, as inevitable in the lattice theory, then powers of the cutoff also occur in $W(C)$.

We now assume that removing these perimeter and corner divergences as well as appropriately renormalizing the bare charge is all that is necessary to render the Wilson loop finite in the continuum limit. This immediately implies that the ratio of two Wilson loops of equal perimeter and number of similar corners but with different shapes will remain finite upon cutoff removal. Such ratios, therefore, can serve as the physical quantities for the analysis of Sec. II.

Denote by $W(I,J)$ a rectangular Wilson loop of dimension I and J in lattice units. The above discussion suggests comparing ratios such as

$$\frac{W(2,2)}{W(1,3)} \tag{3.8}$$

with the doubled scale

$$\frac{W(4,4)}{W(2,6)}\,. \tag{3.9}$$

Note, however, that in this simplest case we are already in Eq. (3.9) asking for loops with 6 units on a side. As our largest lattice is only 10 sites long, we prefer to use a more compact ratio consisting of four loops; thus we define

$$F(g_0)=1-\frac{W(2,2)W(1,1)}{[W(2,1)]^2}\,, \tag{3.10}$$

$$G(g_0)=1-\frac{W(4,4)W(2,2)}{[W(4,2)]^2}\,. \tag{3.11}$$

The constant is added so that these quantities vanish as g_0 goes to zero. In Sec. IV we present Monte Carlo measurements of the quantities in Eqs. (3.10) and (3.11).

In Eq. (3.10) we have effectively taken $r=2a$ in the more general physical ratio

$$P(r,a,g_0)=1-\frac{W\left(\frac{r}{a},\frac{r}{a}\right)W\left(\frac{r}{2a},\frac{r}{2a}\right)}{\left[W\left(\frac{r}{a},\frac{r}{2a}\right)\right]^2}\,. \tag{3.12}$$

For small g_0 this quantity can be expanded perturbatively. We have not carried this out for arbitrary cutoff, but for a^2/r^2 small a straightforward calculation gives

$$P(r,a,g_0)=p_1 g_0{}^2+O(g_0{}^4)+O\left(\frac{a^2}{r^2}g_0{}^2\right), \tag{3.13}$$

where

$$p_1=\frac{1}{4\pi^2}[8\arctan 2+2\arctan\tfrac{1}{2}-2\pi-4\ln(\tfrac{5}{4})]$$
$$=0.066\,079\,788\ldots \tag{3.14}$$

for U(1) gauge theory, and

$$p_1=\frac{3}{16\pi^2}[8\arctan 2+2\arctan\tfrac{1}{2}-2\pi-4\ln(\tfrac{5}{4})]$$
$$=0.049\,559\,841\ldots \tag{3.15}$$

for the SU(2) model.

In the strong-coupling limit we have for U(1)

$$W(I,J)=\left(\frac{1}{2g_0{}^2}\right)^{IJ}\left[1+O\left(\frac{1}{g^4}\right)\right] \tag{3.16}$$

and

$$W(I,J)=\left(\frac{1}{g_0{}^2}\right)^{IJ}\left[1+O\left(\frac{1}{g^4}\right)\right] \tag{3.17}$$

for SU(2). This translates into

$$F(g_0)=1-\frac{1}{2g_0{}^2}+O(g_0{}^{-6})\,, \tag{3.18}$$

$$G(g_0)=1-\frac{1}{16g_0{}^8}+O(g_0{}^{-12}) \tag{3.19}$$

for U(1), or

$$F(g_0)=1-\frac{1}{g_0^{2}}+O(g_0^{-6}), \qquad (3.20)$$

$$G(g_0)=1-\frac{1}{g_0^{8}}+O(g_0^{-12}) \qquad (3.21)$$

for SU(2). All interesting structure should occur in the coupling regime which interpolates between the behaviors of Eq. (3.13) and Eqs. (3.18)–(3.21).

IV. NUMERICAL RESULTS

In previous publications we have discussed the Monte Carlo algorithms used to bring our lattices into equilibrium.[1,2] Here we always work on a hypercubical lattice of 10^4 sites and with periodic boundary conditions. To hasten the approach to equilibrium, the bare coupling is first given a damped oscillation about the values where measurements are finally taken. Once satisfied with the equilibrium of the lattice, we make a sequence of ~5–10 Monte Carlo sweeps through the lattice and measure Wilson loops after each. Any particular shape loop is averaged over all similar loops in the entire lattice, that is, over all translations and rotations. To estimate the statistical errors in some quantity, we first calculate the standard deviation of the mean over the sequence of sweeps. To allow for correlations between successive lattices, these errors are all increased by 40%. This factor is estimated from a measurement of

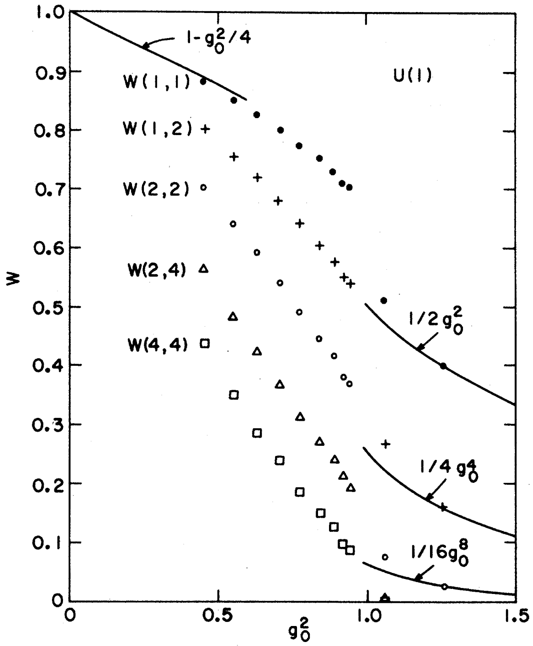

FIG. 4. Wilson loops for U(1) gauge theory.

the successive correlations between 1×1 loops over a sequence of 10 000 iterations on a 2^4 lattice at $g_0^2=1.75$ with the group SU(2). The validity of this error estimate was also confirmed on a measurement of $F(g_0)$ for U(1) over a sequence of 1000 iterations on a 4^4 lattice at $g_0^2=0.82$. For U(1) we keep g_0^2 at least five percent away from

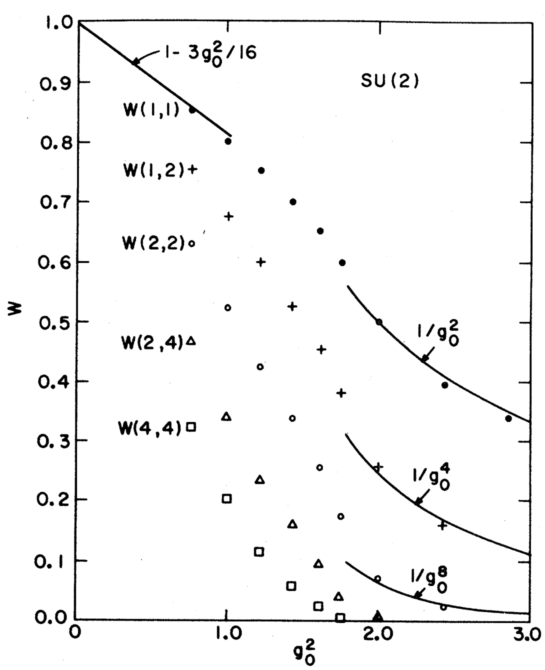

FIG. 3. Wilson loops as a function of g_0^2 for SU(2) gauge theory.

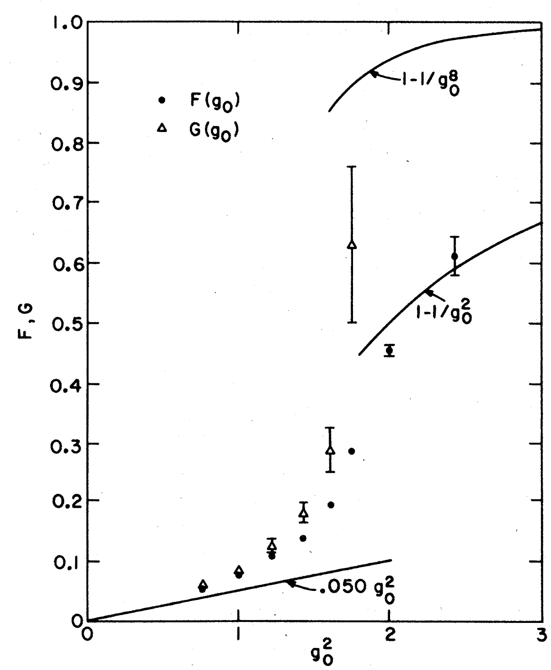

FIG. 5. The quantities F and G for the SU(2) theory.

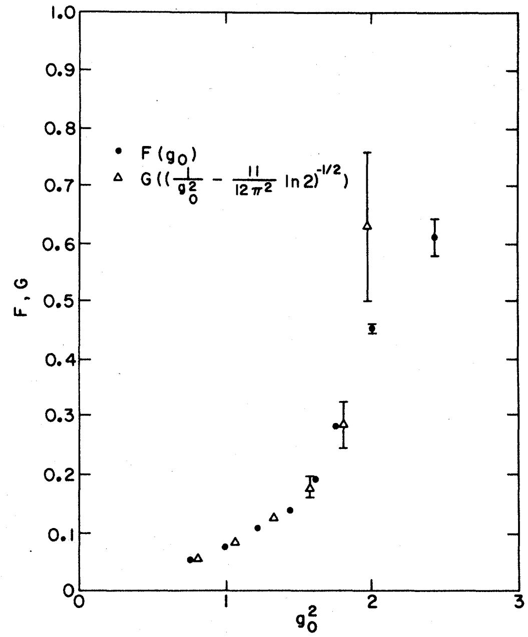

FIG. 6. Testing the prediction of asymptotic freedom.

the transition value to avoid severe critical correlations. Because of this we draw no conclusion on the critical exponents of the transition.

In Figs. 3 and 4 we show as functions of g_0^2 the measured values of those loops needed for the evaluation of Eqs. (3.10) and (3.11). We also plot their strong-coupling limits. The absolute errors, which are too small to show on this graph, are not strongly dependent on loop scale. However, the small numerical values for physically large loops make their errors relatively more important. Indeed, the main source of statistical error in our analysis comes from the 4×4 loops, which are barely measurable for g_0 in the strong-coupling regime. Note the approach to the strong-coupling equations (3.16) and (3.17) for $g_0^2 > 2$ for SU(2) or $g_0^2 > 1$ for U(1).

In Fig. 5 we show for SU(2) the quantities F and G of Eqs. (3.10) and (3.11) as functions of g_0^2. Note that $F(g_0)$ apparently always lies below $G(g_0)$ suggesting a picture like that of Fig. 1. There is no evidence for any fixed point other than at $g_0 = 0$. The figure also displays the strong-coupling forms of Eqs. (3.18) and (3.19). For $g_0^2 > 2$ the numerical results already approximate this limiting behavior; consequently no further structure is expected. Also note the approach to the weak-coupling behavior of Eqs. (3.13)–(3.15) when g_0^2 becomes small.

In Fig. 6 we plot $F(g_0)$ and

$$G\left(\left(\frac{1}{g_0^2}-\frac{11}{12\pi^2}\ln 2\right)^{-1/2}\right) \tag{4.1}$$

versus g_0. Note the excellent agreement with the asymptotic freedom prediction of Eq. (2.21) for $g_0 < 1.8$. This prediction should only apply in the weak-coupling regime. This agreement is rather astonishing in the light of the neglected $O(a^2/r^2)$ terms.

In Fig. 7 we show the results for F and G with the U(1) theory. In the weak-coupling phase of this model the loops are strongly dominated by a perimeter behavior. As this is canceled in F and G the resulting signal is quite small. The data suggest that when g_0^2 is less than g_F^2 is less than g_F^2 the function F lies below G. This effect, however, is close to the error bars and must be regarded with some caution in light of the strong cancellations involved in calculating F and G from the loops. When g_0 is substantially below g_F the functions F and G are equal within errors, indicating a scale-invariant theory. This is evidence for the massless photon dominating the behavior of these loop ratios.

V. DISCUSSION

Comparing physical loop ratios on different length scales, we have Monte Carlo evidence that SU(2) non-Abelian lattice gauge theory does not exhibit a renormalization-group fixed point away from vanishing coupling. This provides further evidence that confinement and asymptotic freedom coexist in the continuum limit of the theory.

In this analysis we have been able to push the lattice spacing to extremely small values. At the easily reached bare coupling $g_0^2 = 1$, the cutoff in

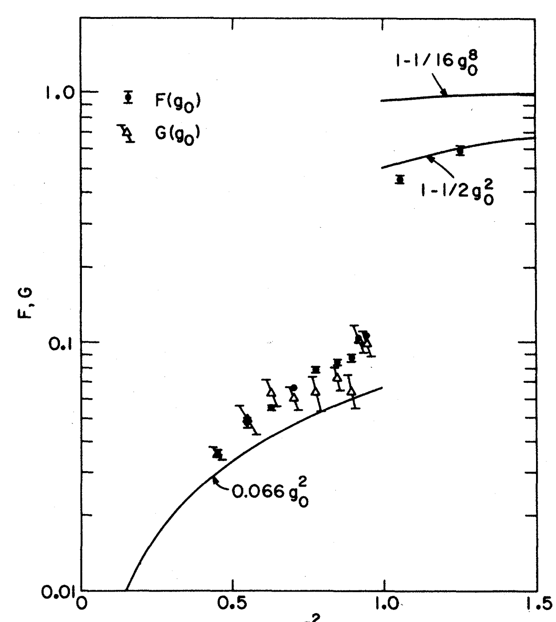

FIG. 7. The quantities F and G for the U(1) theory.

units of the string tension can be calculated from Eqs. (2.18) and (2.19). This gives

$$a=(6\times 10^{-3})K^{-1/2}. \tag{5.1}$$

Thus we have seen asymptotic freedom at cutoff scales much smaller than a typical hadronic size and where direct measurement of the string tension by Monte Carlo techniques would be intractable.

Recent conjectures based on the analogy of the four-dimensional U(1) gauge model with the two-dimensional XY model of statistical mechanics suggest a continuous line of fixed points for g_0 less than a critical value.[6] Our results suggest that the functions F and G may truly cross each other at g_F. If this effect is real, we conjecture that our analysis is indicating how to take a continuum limit in which the magnetic monopoles[11] of the compactified U(1) lattice theory survive. A continuum limit should also be possible holding g_0 constant below the fixed point, but in this limit the only surviving spectrum will be free photons. The monopole mass will go to infinity with the inverse of the lattice spacing. In contrast, allowing g_0 to go to the fixed point under the construction of Sec. II may keep the monopole mass finite.

In this model a nontrivial continuum limit may be possible from either side of the fixed point. Starting on the weak-coupling side should give monopole electrodynamics (with a possibly vanishing monopole moment). Alternatively, the strong-coupling side should give a Higgs phase where the monopoles have condensed into a "magnetic superconductor." Such a phase exhibits electric confinement as implicit in Wilson's strong-coupling expansion.

Returning now to the SU(2) case, we can calculate from $F(g_0)$ a renormalized charge $g(r=2a)$ using Eqs. (2.25), (3.13), and (3.15). In Fig. 8 we plot the inverse renormalized charge at $2a$ versus the bare charge. For small $1/g_0^2$ we also indicate the strong-coupling limit $1/g^2=p_1(1-g_0^{-2})^{-1}+O(g_0^{-6})$. Note that at weak coupling, i.e., large inverse charges, the graph approaches a straight line. The unit slope of this line demonstrates that the a^2/r^2 corrections to Eq. (3.15) are remarkably small. The intercept measures the ratio of the asymptotic freedom scales for g and g_0,

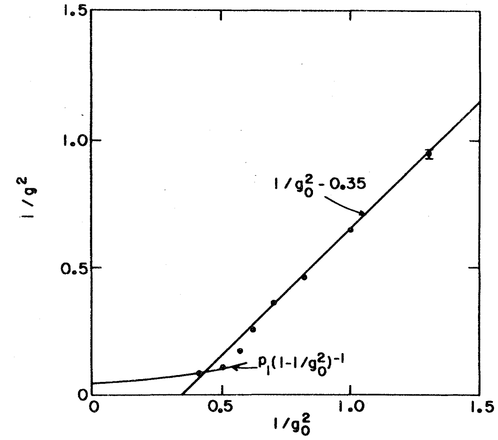

FIG. 8. The inverse renormalized charge squared at $r=2a$ versus the inverse bare charge squared for the SU(2) theory.

$$\frac{1}{g_0^2(a)}-\frac{1}{g^2(2a)}=2\beta_0\ln\left(\frac{2\Lambda}{\Lambda_0}\right)+O(g_0^{-4}). \tag{5.2}$$

Here Λ is defined in analogy to Λ_0:

$$\frac{1}{g^2(r)}=\beta_0\ln\left(\frac{1}{\Lambda^2r^2}\right)+\frac{\beta_1}{\beta_0}\ln\left(\ln\left(\frac{1}{\Lambda^2r^2}\right)\right)+O(g^2(r)). \tag{5.3}$$

Estimating the intercept from Fig. 8 we obtain

$$2\beta_0\ln\left(\frac{2\Lambda}{\Lambda_0}\right)=0.35 \tag{5.4}$$

or

$$\Lambda=22\Lambda_0. \tag{5.5}$$

We do not put an error on this number because of unknown systematic effects due to finite-cutoff corrections. The large factor in Eq. (5.5) is typical of comparisons of the lattice Λ_0 with more physical definitions.[12] Using Eq. (2.19) for Λ_0 we obtain

$$\Lambda=0.3\sqrt{K}. \tag{5.6}$$

Verification of the number in Eq. (5.5) is in principle possible with a one-loop perturbative calculation.

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