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Soliton mass corrections and explicit models in two dimensions

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We show how to evaluate the first-order soliton mass correction in two-dimensional scalar theories and show it to be finite. We give a formula for the correction which generalizes those previously obtained in special cases. We also investigate two classes of models for which an exact stability analysis is possible. One of these classes has the feature that it permits solitons of arbitrarily small classical mass, even for weak coupling.

I. INTRODUCTION

For Lagrangians of the form

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)\,, \tag{1}$$

where ϕ is a scalar field, conditions necessary for the existence of solitary wave solutions (solitons) are well known.[1,2] $V(\phi)$, which is non-negative, must have at least two zeros; the time-independent soliton will interpolate smoothly and monotonically between adjacent zeros of V. Each of these zeros will support meson sectors which, unlike the sectors of the ϕ^4 and sine-Gordon models, need not be related, i.e., there need not be an internal symmetry transforming one vacuum to the adjacent vacuum. In general such distinct meson sectors will be characterized by mesons of different masses. The simplest such example is perhaps a ϕ^6 model[3] in which V possesses three minima, although the phenomenon of unequal masses is a general one. Previous considerations[1,2] have applied only to the situation of equal meson masses and must be extended to apply in general, particularly for the calculation of the soliton mass correction ΔM.

In this paper we address ourselves to the problem of carrying out the necessary extensions for the general case, and display a general formula for ΔM. We then apply this analysis to two classes of models which are distinguished by their amenability to analytic techniques, to the extent that one can investigate exactly their stability properties. In the quantum theory this corresponds to a precise description to first order of soliton-meson interactions. Together with the sine-Gordon theory, the two classes which we describe exhaust the scalar models for which such an exact description is possible. We feel it is important in the understanding of solitons to have a range of such models available for study; for example, the model of Sec. V shows that even in weakly coupled theories it is possible to have solitons of low mass.

The quantum theory of the models under consideration displays the feature that the classical solitary wave does not necessarily correspond to a quantum particle. The existence of the quantum soliton depends on the quantum vacuum states being degenerate, and although this is guaranteed by symmetry for the ϕ^4 and sine-Gordon models there is no corresponding guarantee for models lacking such symmetries. In general, quantum effects will break the classical degeneracy of the lowest-energy solutions and the classical solitary wave will not have a stable quantum soliton counterpart; rather, the solution describes the decay of the false vacuum, rendered unstable by barrier penetration, into the ground state of lowest energy as Coleman[4] has described. In this case there is no reason to expect a well-defined mass for the soliton at the quantum level and indeed we find there exists an arbitrariness in the first-order mass correction ΔM. However, one can restore the vacuum degeneracy at the quantum level by the addition of suitable finite counterterms to the Lagrangian. This corresponds to selecting a particular value for the mass μ which is used to normal order $V(\phi)$, and we show that precisely for this choice of μ the ambiguity in ΔM disappears. The classical solution can therefore again be interpreted as representing a quantum particle.

In the calculation of ΔM one expects that after subtraction of the vacuum energy and renormalization ΔM will be finite and indeed this occurs for models previously studied. The vacuum-energy subtraction requires one in general to take account of the two distinct vacuums and, briefly, this can be done in the following way.[1-3] We perturb about the classical solution $\phi_c(x)$ in the form

$$\phi(x,t)=\phi_c(x)+\eta(x,t)\,,$$

where η can be written

$$\eta(x,t)=e^{i\omega_k t}\eta_k(x)\,.$$

$\eta_k(x)$ are eigenfunctions of the Schrödinger operator K:

$$K\eta_k(x)=\omega_k^2\eta_k(x)\,,\qquad K\equiv -\frac{d^2}{dx^2}+U(x)\,,\tag{2}$$

where

$$U(x)=V''(\phi_c(x))\,.$$

The potential $U(x)$ determines meson-soliton scattering properties, with reflection and transmission allowed in general as well as the possibility of bound states. The discussion given for ϕ^6 applies generally.[3]

If the classical energy (mass) E_c is $O(\lambda^{-1})$, where λ is the coupling constant, then the first-order mass correction will be $O(\lambda^0)$ and is given by $\Delta M = \frac{1}{2}\sum \omega_k$, i.e., half the sum of zero-point oscillations about the soliton (infrared and ultraviolet cutoffs are assumed). This correction diverges quadratically but subtraction of the vacuum energy should render this divergence logarithmic, and the addition of renormalization counterterms should then produce a finite answer. The vacuum energy is calculated by considering the situation in which the soliton is removed, i.e., $\phi_c(x)$ and in particular $U(x)$ take only their asymptotic values. Let $U_\pm$ be the asymptotic values of $U(x)$:

$$U_\pm = \lim_{x\to\pm\infty} U(x)$$

and let

$$U_0(x)=U_+,\quad x\geqslant 0$$
$$\qquad =U_-,\quad x<0\,.$$

These asymptotic values are in fact the squared meson masses of the two adjacent meson sectors and in general will be different. The vacuum energy will be $\frac{1}{2}\sum\omega_k^0$, where $(\omega_k^0)^2$ is an eigenvalue of the Schrödinger operator K_0:

$$K_0\equiv -\frac{d^2}{dx^2}+U_0(x)\,.\tag{3}$$

The mass correction becomes $\frac{1}{2}\sum_k(\omega_k-\omega_k^0)$ which we can write, even when the infrared cutoff is removed, as

$$\tfrac{1}{2}\mathrm{Tr}(\sqrt{K}-\sqrt{K_0})\,.\tag{4}$$

(There still remains an ultraviolet cutoff in order to handle the expected logarithmic divergence.) This expression, in which K and K_0 are given by Eqs. (2) and (3), is completely general for scalar field theories with the Lagrangian (1).

II. MASS CORRECTION

We can write down a formula for the expression (4) in terms of the asymptotic scattering data.[5] Such an expression has been worked out[6] for the case in which $U(x)$ is reflectionless and $U_+=U_-$, using a method of box regularization which unfortunately is not appropriate in general.[3] The expression (4) has also been computed[2] with only the restriction $U_+=U_-$, but the general formula below for $U_+\neq U_-$ involves an additional term. Proofs of the assertions that follow are given separately.[5]

We can assume that $U_+\geqslant U_-$. Let λ be an eigenvalue of K and define the wave numbers

$$k_\pm=(\lambda-U_\pm)^{1/2}\,.\tag{5}$$

K will have a finite number of discrete eigenvalues $\lambda_1,\ldots,\lambda_n$ and the continuum begins at $\lambda=U_-$, i.e., $k_-=0$. We can define reflection and transmission coefficients $r_\pm, t_\pm$, which are functions of λ, from the asymptotic solutions of the Schrödinger equation $K\eta=\lambda\eta$:

$$\eta(x)\to\begin{cases} t_-e^{-ik_-x} & \text{as } x\to-\infty\,,\\ e^{-ik_+x}-r_+e^{ik_+x} & \text{as } x\to+\infty\end{cases}\tag{6}$$

and

$$\eta(x)\to\begin{cases} e^{ik_-x}-r_-e^{-ik_-x} & \text{as } x\to-\infty\,,\\ t_+e^{ik_+x} & \text{as } x\to+\infty\,.\end{cases}\tag{7}$$

The properties of $r_\pm$ and $t_\pm$ are summarized in the following lemma[5]:

(i) $r_\pm$ and $t_\pm$ are holomorphic functions of λ in a strip containing the real axis, cut along $(-\infty, U_+]$, except possibly for poles in the lower half plane. For λ in this region the following identities hold:

$$\begin{aligned} &r_-(\lambda)\bar{r}_-(\bar\lambda)+\frac{k_-(\lambda)}{k_+(\lambda)}\,t_-(\lambda)\bar{t}_-(\bar\lambda)=1\,,\\ &r_+(\lambda)\bar{r}_+(\bar\lambda)+\frac{k_+(\lambda)}{k_-(\lambda)}\,t_+(\lambda)\bar{t}_+(\bar\lambda)=1\,,\\ &t_+(\lambda)k_+(\lambda)=t_-(\lambda)k_-(\lambda)\,,\\ &k_+(\lambda)r_+(\lambda)\bar{t}_+(\bar\lambda)=-k_-(\lambda)\bar{r}_-(\bar\lambda)t_-(\lambda)\,.\end{aligned}\tag{8}$$

(ii) $r_\pm$ and $t_\pm$ are infinitely differentiable on (U_-, U_+) and

$$|r_-(\lambda)|=1,\quad U_-<\lambda<U_+\,.\tag{9}$$

(iii) If $r_\pm$ and $t_\pm$ are regarded as functions of $k_\pm$, then they are holomorphic in a neighborhood of $k_\pm=0$. Furthermore,

$$\lim_{k_\pm\to 0} r_\pm(k_\pm)=1,\quad \lim_{k_\pm\to 0} t_\mp(k_\pm)=0\,.\tag{10}$$

For technical reasons we need to assume that $U(x)$ falls off exponentially to the values $U_\pm$ at large $|x|$; this will indeed be the case for meson masses which are nonzero. Let G be a function of compact support, then a trace formula can be given[5] for $\mathrm{Tr}[G(K)-G(K_0)]$. We apply this formula to the square-root function (with cutoff) to obtain

$$\tfrac{1}{2}\mathrm{Tr}(\sqrt{K}-\sqrt{K_0})=\tfrac{1}{2}\sum_{i=1}^{n}\sqrt{\lambda_i}+\frac{1}{4\pi i}\int_{U_-}^{U_+}d\lambda\sqrt{\lambda}\,\frac{d}{d\lambda}\ln\left(\frac{r_-}{r_-^0}\right)+\frac{1}{4\pi i}\int_{U_+}^{\infty}d\lambda\sqrt{\lambda}\,\frac{d}{d\lambda}\ln\left(\frac{t_-}{\bar{t}_-}\right). \quad (11)$$

Here r_-^0 is the reflection coefficient appropriate to the potential U_0 and is given by

$$r_-^0=\frac{i\kappa_+-k_-}{i\kappa_++k_-}, \quad (12)$$

where $\kappa_+=-ik_+=(U_+-\lambda)^{1/2}$ is real for $U_-\le\lambda\le U_+$. For $U_+=U_-$ one recovers the formula due to Faddeev and Korepin.[2] We see that the arguments of both logarithms are unitary and so the mass correction depends only on the phase change undergone during meson-soliton scattering, as determined by $K\eta=\lambda\eta$.

We can reach some understanding of the new term in Eq. (11) in the following way. Although the box regularization method of Dashen, Hasslacher, and Neveu[6] is not appropriate in general, it does work for potentials which have either no transmission or no reflection (the latter occurs for sine-Gordon and ϕ^4 theory). We place the soliton in a large box of length L with periodic boundary conditions and consider the sum

$$\tfrac{1}{2}\sum_k(\omega_k-\omega_k^0) \quad (13)$$

in which $\omega_k^2=k_-^2+U_-$ satisfies $U_-\le\omega_k^2\le U_+$. As the soliton is removed from the box we can follow the countable continuum modes, which are uniformly shifted relative to the vacuum values, to obtain a relation of the form

$$k_-^0=k_-+\frac{\gamma(k_-)}{L}, \quad (14)$$

where k_-^0 refers to the vacuum wave number. The mass correction is then[6]

$$-\frac{1}{4\pi}\int_0^{(U_+-U_-)^{1/2}}dk_-\gamma(k_-)\frac{d}{dk_-}(k_-^2+U_-)^{1/2}. \quad (15)$$

Incoming plane waves with energies in the range $U_-\le\omega_k^2\le U_+$ are reflected completely from the barrier $U(x)\sim U_+$ at large x, that is, the wave function dies off exponentially and can be neglected for the box length L sufficiently large, for large x. Periodic boundary conditions then imply [from Eq. (7)]

$$e^{-ik_-L/2}-r_-(k_-)e^{ik_-L/2}=0 \quad (16)$$

or

$$k_-L+\delta(k_-)=2\pi n,$$

where $\delta=-i\ln r_-$ and n is an integer. Similarly for the vacuum,

$$k_-^0L+\delta^0(k_-)=2\pi n_0,$$

and by identifying the modes $n=n_0$, we obtain for γ in Eq. (14)

$$\gamma=\delta-\delta^0=-i\ln\left(\frac{r_-}{r_-^0}\right).$$

After an integration by parts in Eq. (15) we arrive at the term

$$\frac{1}{4\pi i}\int_{U_-}^{U_+}d\lambda\sqrt{\lambda}\,\frac{d}{d\lambda}\ln\left(\frac{r_-}{r_-^0}\right)$$

which appears in Eq. (11). This is the contribution to ΔM of the reflected plane waves of energy $0\le k_-\le(U_+-U_-)^{1/2}$. The remaining terms in Eq. (11) are identical to those derived by Faddeev and Korepin.[2]

Now let us examine the behavior of the integral in Eq. (11) for large λ. This can be done with the help of the asymptotic development[5]

$$t_-\sim1-\frac{i}{2\sqrt{\lambda}}\int_{-\infty}^{\infty}[U(x)-U_0(x)]dx+O(\lambda^{-1}) \quad (17)$$

for large λ. This asymptotic form may be derived by converting the Schrödinger equation to an integral equation, which is then solved by iteration. This provides an asymptotic development of the wave function η in inverse powers of $\lambda^{1/2}$, leading to Eq. (17). Now substitute $\lambda=k^2$ so that for large λ, k is the wave number and the integrand in Eq. (11) becomes, for large k,

$$\frac{k}{4\pi i}\frac{d}{dk}\left(1+\frac{I}{2ik}\right)^2\sim\frac{I}{4\pi k}, \quad (18)$$

where

$$I=\int_{-\infty}^{\infty}[U(x)-U_0(x)]dx. \quad (19)$$

The mass correction therefore diverges logarithmically, but we have yet to add renormalization counterterms. The Lagrangian (1) will be renormalized by a normal ordering of the self-interaction $V(\phi)$. We use the formula

$$V(\phi)=:\exp\left(\frac{l}{2}\frac{d^2}{d\phi^2}\right)V(\phi):, \quad (20)$$

where l is the loop integral (with suitable cutoff Λ)

$$l=\frac{1}{2\pi}\int_0^{\Lambda}\frac{dk}{(k^2+\mu^2)^{1/2}} \quad (21)$$

and μ is a mass parameter. We find that the counterterm of $O(\lambda)$, to be added to $\mathcal{L}$, is $\frac{1}{2}lV''(\phi)$. This contributes to the soliton mass

$$-\frac{l}{2}\int dx\, V''(\phi_c(x)) = -\frac{l}{2}\int dx\, U(x),$$

and from this we must subtract the counterterm which contributes to the vacuum energy, namely

$$-\frac{l}{2}\int dx\, U_0(x).$$

The total correction of $O(\lambda^0)$ by the counterterms to the soliton mass is therefore

$$-\frac{l}{2}\int dx[U(x)-U_0(x)] = -\frac{lI}{2}, \tag{22}$$

which we see is infrared convergent. At high energies (large k) the integrand of this contribution is

$$-\frac{I}{4\pi k}, \tag{23}$$

which cancels the logarithmic divergence of Eq. (11), as shown in Eq. (18). We have demonstrated therefore that under very general conditions the first-order correction to the soliton mass is finite, and is given explicitly by

$$\Delta M = \frac{1}{2}\sum_{i=1}^{n}\sqrt{\lambda_i} + \frac{1}{4\pi i}\int_{U_-}^{U_+} d\lambda\sqrt{\lambda}\,\frac{d}{d\lambda}\ln\left(\frac{r_-}{r_-^0}\right) + \frac{1}{4\pi i}\int_{U_+}^{\infty} d\lambda\sqrt{\lambda}\,\frac{d}{d\lambda}\ln\left(\frac{t_-}{\bar{t}_-}\right) - \frac{l}{2}\int_{-\infty}^{\infty} dx(U-U_0)(x), \tag{24}$$

where r_-^0, l are given in Eqs. (12) and (21).

Two further comments must be made about this formula. Firstly, the divergent loop integral l in Eq. (21) depends on a mass parameter μ, with respect to which $V(\phi)$ is normal ordered, and ΔM then also depends on μ. We can always vary μ by adding finite amounts to l, which will not affect the high-energy behavior and therefore the renormalizability of the theory. The relation between loop integrals with different μ is given by the re-normal-ordering formula

$$\frac{1}{2\pi}\int_0^{\infty}\frac{dk}{(k^2+\mu^2)^{1/2}} - \frac{1}{2\pi}\int_0^{\infty}\frac{dk}{(k^2+m^2)^{1/2}} = \frac{1}{4\pi}\ln\frac{m^2}{\mu^2}.$$

For $U_+\neq U_-$ there is no natural single choice for a normal-ordering mass μ. We will deduce a convenient form for l, i.e., a value of μ, below [Eq. (25)]. Secondly, in our choice of reference potential $U_0(x)$ we have placed the step discontinuity at $x=0$, the "center" of the soliton. In fact, ΔM does depend on the point $x=a$ at which we position the step. Let $U_a(x)=U_0(x-a)$, then the mass correction ΔM_a calculated with this reference potential differs from ΔM by the finite amount

$$\Delta M - \Delta M_a = \frac{1}{2}\mathrm{Tr}(\sqrt{K_a}-\sqrt{K_0}) - \frac{l}{2}\int_{-\infty}^{\infty} dx(U_a-U_0)(x).$$

Here K_a is the Schrödinger operator with potential $U_a(x)$ and has transmission and reflection coefficients $t_\pm^a$, $r_\pm^a$ with the properties

$$\frac{r_-^a}{r_-^0} = e^{2ik_-a}, \quad \frac{t_-^a}{\bar{t}_-^a} = e^{-2i(k_+-k_-)a}.$$

Using Eq. (11) we find

$$\Delta M - \Delta M_a = \frac{a}{8\pi}(U_+-U_-) - \frac{aU_+}{4\pi}\int_0 \frac{dk_+}{(k_+^2+U_+)^{1/2}} + \frac{aU_-}{4\pi}\int_0 \frac{dk_-}{(k_-^2+U_-)^{1/2}} + \frac{la}{2}(U_+-U_-),$$

where the integrations are to be cut off at $\lambda=k_\pm^2+U_\pm=\Lambda$. There is no ambiguity when $U_+=U_-$, for then $\Delta M-\Delta M_a=0$. However, in general there seems to be no unique way to choose $x=a$ as the center of the soliton, but we can in fact remove the ambiguity by choosing l such that $\Delta M-\Delta M_a=0$. We have then

$$l = -\frac{1}{4\pi} + \frac{U_+}{2\pi(U_+-U_-)}\int_0 \frac{dk_+}{(k_+^2+U_+)^{1/2}} - \frac{U_-}{2\pi(U_+-U_-)}\int_0 \frac{dk_-}{(k_-^2+U_-)^{1/2}}. \tag{25}$$

This corresponds to the loop integral (21) with a mass μ^2 given by

$$\ln\mu^2 = 1 + \frac{1}{U_+-U_-}(U_+\ln U_+ - U_-\ln U_-).$$

We observe that this expression is symmetric in U_+ and U_-, and for $U_+\to U_-$ is equal to $2+\ln U_-$. For the case $U_+\neq U_-$ we adopt this value of μ^2 as the normal-ordering mass, and there is then no ambiguity in the calculation of ΔM, which is given by Eq. (24) with l defined by Eq. (25).

This resolution of the ambiguity in ΔM is discussed in the Introduction and is connected with the degeneracy of the quantum vacuums. In order to investigate this let us calculate the first-order energy difference per unit volume of the adjacent vacuums. We place the system in a box of length L and the vacuum energy is $\frac{1}{2}\sum\omega_k^0$ as before. The difference in energies of the adjacent vacuums is then

$$\tfrac{1}{2}\sum \omega^0_{k_-} - \tfrac{1}{2}\sum \omega^0_{k_+} - \frac{l}{2}\int_{-L/2}^{L/2} dx(U_- - U_+)\,,$$

where we have included counterterms. The energy difference per unit volume is finite and equal to

$$\frac{1}{2\pi}\int_0 dk_-(k_-^2+U_-)^{1/2} - \frac{1}{2\pi}\int_0 dk_+(k_+^2+U_+)^{1/2} + \frac{l}{2}(U_+ - U_-)\,.$$

The value of l for which this energy difference is zero is precisely that given by Eq. (25). We see therefore that by choosing a suitable normal ordering mass, or equivalently, by adding to the Lagrangian the correct finite counterterms proportional to $V''(\phi)$, we can restore vacuum degeneracy and simultaneously remove the ambiguity in ΔM.

For the explicit models described below, the scattering coefficients can be computed in closed form and ΔM displayed as an explicit integral. However, except for a few models such as ϕ^4 or sine-Gordon, these integrals seem to require numerical evaluation.

III. EXPLICIT MODELS

Many soliton models pose computational difficulties in that exact or perturbative solutions cannot be given in closed form. For the scalar theories under consideration we would like to study the models for which the potential $V(\phi)$ and the soliton solution $\phi_c(x)$ can be explicitly displayed, and for which the small disturbance equation (2) can be solved in terms of elementary or special functions. In such models one has a chance of calculating analytically the quantum corrections, including the description of bound states and evaluation of mass corrections mentioned above. In addition, it turns out that one can also give explicitly the Fourier transform $\tilde{\phi}_c(p)$ of $\phi_c(x)$, which is the matrix element of the quantum field between one-soliton states.[1]

Given a Schrödinger operator

$$K = -\frac{d^2}{dx^2} + U(x)\,,$$

a necessary condition that K be appropriate to a soliton model is that $U(x) = V''(\phi_c(x))$ be smooth and finite everywhere, with finite limits $U_\pm$. The spectrum of K must also be bounded below, and the eigenvector of the lowest eigenvalue must correspond to the zero-frequency mode[1] and be equal to $d\phi_c/dx$. From such an eigenvector one obtains $\phi_c(x)$ by integration. Then the potential is given by

$$V(\phi(x)) = \frac{1}{2}\left(\frac{d\phi_c}{dx}\right)^2\,. \tag{26}$$

We wish to display V as an explicit function of ϕ, which requires one to explicitly invert $\phi_c = \phi_c(x)$ to obtain $x = f(\phi_c)$ as a function of ϕ_c. $V(\phi)$ must be at least a continuous function of ϕ, and be bounded below for all ϕ.

The Schrödinger equations soluble in terms of known functions are also those soluble by the factorization method and are listed in Morse and Feshbach (Ref. 7, p. 789). The only suitable potentials are found to be of the form

$$U(x) = a + b\cosh^{-2}x + c\tanh x\,, \tag{27}$$

$$U(x) = a + \cosh^{-2}x(b + c\sinh x) \tag{28}$$

for various constants a, b, c.

The potential (27) is studied in detail in Morse and Feshbach.[7] The eigenvectors ϕ_0 corresponding to the lowest eigenvalues are (up to normalization)

$$\phi_0 = (\cosh x)^\alpha e^{-\beta x} \tag{29}$$

for (27), and for (28)

$$\phi_0 = (\cosh x)^\alpha \exp(\beta \tan^{-1}\sinh x)\,, \tag{30}$$

where α, β are constants related to a, b, c. Setting $\phi_0 = d\phi_c/dx$, we need to integrate Eqs. (29) and (30) and explicitly invert to obtain $x = f(\phi_c)$. We can do this in the following cases.

(i) $\alpha = -1$, $\beta = 0$ in either Eqs. (29) or (30). This will give us the sine-Gordon model.

(ii) $\beta = -\alpha - 2$ in Eq. (29). This gives a model containing an arbitrary dimensionless parameter and is a generalization of ϕ^4. We will write the potential as

$$V(\phi) = \alpha\phi^2(\phi^\beta - v^\beta)^2\,, \tag{31}$$

where α, β, v are new parameters.

(iii) $\alpha = -1$ in Eq. (30). This gives a model also containing a dimensionless parameter, and we can write $V(\phi)$ as

$$V(\phi) = \tfrac{1}{2}\alpha^2\phi^2\cos^2(\beta\ln\lambda\phi^2)\,, \tag{32}$$

where α, β, λ are new parameters.

We consider the potentials (31) and (32) separately below. However, let us first mention the Fourier transform $\tilde{\phi}_c(p)$ of $\phi_c(x)$. The one-soliton matrix element is given by[1]

$$\langle p'|\Phi|p\rangle = \int dx\exp[i(p'-p)x]\phi_c(x)\,. \tag{33}$$

Interesting features of this matrix element are the positions of poles, and any symmetry properties under the interchange of p and p'. For $U_+ = U_-$, i.e., for solitons interpolating between meson sectors of equal mass μ, $\tilde{\phi}_c(p)$ has poles at $p = i\mu n$ for integers n (Ref. 8). For $U_+ \neq U_-$ the position of the poles is not obvious in general, but can be calculated directly for the potential (31) above. We can calculate $\tilde{\phi}_c(p)$ as follows: if ϕ_c interpolates between vacuum values v_1, v_2, i.e., $v_1 \le \phi_c(x) \le v_2$, then

$$\tilde{\phi}_c(p) = \int_{-\infty}^{\infty} e^{ipx}\phi_c(x)dx = (v_1+v_2)\pi\delta(p) + \frac{i}{p}\tilde{\phi}_c'(p), \tag{34}$$

where we have regarded $\phi_c(x)$ as a generalized function. By shifting ϕ_c suitably we could choose $v_1+v_2=0$. Note that since $\phi_c'(x) \to 0$ for $|x| \to \infty$, the Fourier transform of $\phi_c'(x)$ is convergent integral.

IV. $V(\phi) = \alpha\phi^2(\phi^\beta - v^\beta)^2$

For $\beta=1$ this potential is the well-known ϕ^4 potential with the kink solution. For $\beta=2$ we have a ϕ^6 potential with three minima previously investigated.[3] For β an odd integer $2n-1$, V is a polynomial of degree $4n$, with only two minima, at $\phi=0$ and $\phi=v$. For β an even integer $2n$, V is an even polynomial of degree $4n+2$ with three minima, at $\phi=0$ and $\phi=\pm v$. There seems to be no reason why β should not take nonintegral values, although ϕ might have to be replaced by $|\phi|$, e.g., for $\beta=\frac{1}{2}$. In any case we need $\beta \ge -1$, and for $\beta=0$ and $\beta=-1$ the potential degenerates to that for a free massive field.

We study the soliton interpolating between $\phi=0$ and $\phi=v$. Mesons built on these vacuums have mass

$$[V''(0)]^{1/2} = \mu = \sqrt{2\alpha}\,v^\beta, \qquad [V''(v)]^{1/2} = \beta\mu, \tag{35}$$

respectively. Only for $\beta=1$ (ϕ^4) are these masses equal. The one-soliton solution ϕ_c is easily calculated:

$$\phi_c = v(1+e^{-\beta\mu x})^{-1/\beta} = v\left(\frac{1+\tanh(\beta\mu x/2)}{2}\right)^{1/\beta}. \tag{36}$$

The classical mass is

$$E_c = \frac{\sqrt{2\alpha}\,\beta v^{\beta+2}}{2(\beta+2)}. \tag{37}$$

Next, let us examine the small-disturbance equation $-\eta'' + U(x)\eta = \omega^2\eta$. We find

$$\frac{U(x)}{\mu^2} = \frac{\beta^2+1}{2} + \frac{\beta^2-1}{2}\tanh\frac{\mu\beta x}{2} - \frac{(2\beta+1)(\beta+1)}{4\cosh^2(\mu\beta x/2)}. \tag{38}$$

We observe that $U_- = \mu^2 \ne U_+ = \beta^2\mu^2$ in general, so that we have the situation of unequal meson masses alluded to in the Introduction. The Schrödinger equation with the potential (38) is extensively discussed in Morse and Feshbach (Ref. 7, p. 1651). Their notation corresponds with ours as follows:

$$z \to \frac{\mu\beta x}{2}, \quad v \to \frac{3(\beta+1)(\beta+2)}{\beta(2\beta+1)}, \quad \epsilon \to \frac{\beta+5}{2\beta+1} + \frac{4\omega^2}{\mu^2\beta^2}, \quad e^{2|\mu|} \to \frac{3\beta}{\beta+2}. \tag{39}$$

The solutions can be expressed in terms of hypergeometric functions and are listed in Ref. 7. We will be content to note the reflection and transmission coefficients as defined in Eqs. (6) and (7).

For $\omega^2 < \mu^2$ there is always one discrete level $\omega^2=0$, with eigenvector ϕ_c', but for $0<\beta<\sqrt{2}$ there is another discrete level with eigenvalue

$$\omega^2 = \frac{\mu^2\beta^2}{4}(4-\beta^2). \tag{40}$$

This level represents a meson-soliton bound state and possesses excitation levels. The corresponding wave function is

$$\eta = \left[\frac{e^{(1-\beta^2)z}}{\cosh z}\right]^{1/\beta}(\beta-1+\tanh z), \tag{41}$$

where $z = \mu\beta x/2$.

For $\mu^2 \le \omega^2 \le \mu^2\beta^2$ incoming plane waves are reflected with the coefficient of reflection [as defined by Eq. (7)] given by

$$r_- = \frac{-\Gamma\left(\frac{2ik_-}{\mu\beta}\right)\Gamma\left(\frac{\kappa_+}{\mu\beta} - \frac{ik_-}{\mu\beta} - \frac{\beta+1}{\beta}\right)\Gamma\left(\frac{\kappa_+}{\mu\beta} - \frac{ik_-}{\mu\beta} + \frac{2\beta+1}{\beta}\right)}{\Gamma\left(-\frac{2ik_-}{\mu\beta}\right)\Gamma\left(\frac{\kappa_+}{\mu\beta} + \frac{ik_-}{\mu\beta} - \frac{\beta+1}{\beta}\right)\Gamma\left(\frac{\kappa_+}{\mu\beta} + \frac{ik_-}{\mu\beta} + \frac{2\beta+1}{\beta}\right)}, \tag{42}$$

where $k_- = (\omega^2-\mu^2)^{1/2}$, $\kappa_+ = (\mu^2\beta^2-\omega^2)^{1/2}$. Physically this means that mass-μ mesons in this energy range are completely reflected by the soliton.

For $\omega^2 \ge \mu^2\beta^2$ we can have both reflection and transmission. The coefficient t_- is given by

$$t_- = \frac{\Gamma\left(-\frac{ik_-}{\mu\beta} - \frac{ik_+}{\mu\beta} + \frac{2\beta+1}{\beta}\right)\Gamma\left(-\frac{ik_-}{\mu\beta} - \frac{ik_+}{\mu\beta} - \frac{\beta+1}{\beta}\right)}{\Gamma\left(-\frac{2ik_+}{\mu\beta}\right)\Gamma\left(1-\frac{2ik_-}{\mu\beta}\right)}, \tag{43}$$

where $k_+ = (\omega^2 - \mu^2\beta^2)^{1/2}$. Evidently incoming mesons can be reflected or transmitted from the soliton; transmitted mesons acquire a different mass, appropriate to the sector they occupy.

We can now calculate the first-order soliton-mass correction by substituting into Eq. (24). The value of I, as defined by Eq. (19), is

$$I = -\frac{\mu(\beta+1)(2\beta+1)}{\beta},$$

and r_-, t_- are given above. We have not attempted to evaluate the integrals in Eq. (24), although this is possible for special values such as $\beta = 1$.

Finally, let us mention the Fourier transform of ϕ_c, the matrix element discussed in Sec. III. From Eq. (34) we obtain

$$\tilde{\phi}_c(p) = \pi v\delta(p) + \frac{i}{p}\tilde{\phi}_c'(p)$$
$$= \pi v\delta(p) + \frac{iv\Gamma(1/\beta + ip/\beta\mu)\Gamma(1-ip/\beta\mu)}{p\Gamma(1/\beta)}. \tag{44}$$

For $\beta = 1$ we recover the ϕ^4 matrix element, proportional to $[\sinh(\pi p/\mu)]^{-1}$. In general, we have poles at

$$p = -i\mu\beta n$$

and (45)

$$p = i\mu\beta n + i\mu, \quad n = 0, 1, \ldots .$$

Unlike ϕ^4, there is no evident symmetry under the exchange $p \to -p$.

V. $V(\phi) = (\alpha^2/2)\phi^2\cos^2(\beta\ln\lambda\phi^2)$

A graph of this potential is shown for a typical value of β in Fig. 1. There are an infinite number of minima at

$$\phi = v_n \equiv \frac{1}{\sqrt{\lambda}}\exp\left[(2n-1)\frac{\pi}{4\beta}\right], \tag{46}$$

where $n = 0, \pm 1, \ldots$. There is also a minimum at $\phi = 0$, i.e., the limit $n \to -\infty$, where V and V' (but not V'') are continuous. The potential can also be written

$$V(\phi) = \frac{\alpha^2}{4}\phi^2\left(1 + \frac{\lambda^{2i\beta}}{2}\phi^{4i\beta} + \frac{\lambda^{-2i\beta}\phi^{-4i\beta}}{2}\right). \tag{47}$$

The parameter β is dimensionless and we take β

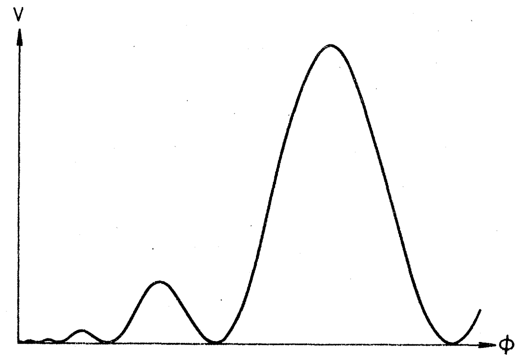

FIG. 1. The potential $V = \phi^2\cos^2(\beta\ln\phi^2)$ plotted for $\beta = 2$. The distance between minima grows exponentially, and the heights of the maxima grow quadratically.

>0. However, the potential makes sense for some imaginary values of β, for example $\beta = i/2$, when we obtain a ϕ^4 potential. α, which we take to be positive, is a mass parameter. The coupling constant λ, also positive, is defined according to $V(\lambda, \phi) = (1/\lambda)V(1, \sqrt{\lambda}\phi)$ and can be scaled out classically. We observe that V is not analytic at $\lambda = 0$.

In the quantum theory we can build meson states on any of the vacuums shown in (46) and the mesons all have mass $\mu = 2\alpha\beta$, independent of n. However, it is not clear whether one can build a meaningful quantum theory on the minimum at $\phi = 0$, where $V''(\phi)$ is discontinuous. A natural approach would be to take the limit $n \to -\infty$ of the theory based on the vacuum v_n in Eq. (46).

There are solitons interpolating between adjacent vacuums and these are obtained by integration of $(\phi')^2 = 2V(\phi)$. We obtain

$$\phi_c = \frac{1}{\sqrt{\lambda}}\exp\left(\frac{\pm 1}{2\beta}\tan^{-1}\sinh\mu x\right)$$
$$= \frac{1}{\sqrt{\lambda}}\left(\frac{i + \sinh\mu x}{i - \sinh\mu x}\right)^{\pm i/4\beta}. \tag{48}$$

The various solutions appear as the different branches of the $\tan^{-1}$ function. The classical mass of the solution interpolating between the vacuums v_n and v_{n+1} is

$$E_n = \frac{\mu}{2\lambda(\beta^2+1)}\cosh\frac{\pi}{2\beta}\,e^{n\pi/\beta}. \tag{49}$$

The exponential dependence on n shows that however weak the coupling λ we can have solitons of arbitrarily small classical mass by taking n large and negative. Of course, quantum effects will contribute corrections of order μ, the meson mass, but evidently it is possible to have solitons of low mass even in weakly coupled theories. In the limit $n \to -\infty$, when the soliton degenerates to be built entirely on the vacuum $\phi = 0$, the classical mass of $O(\lambda^{-1})$ disappears altogether so that the lowest-order mass [of $O(\lambda^0)$] appears as a quantum correction.

Next, let us study perturbations around the soliton by solving $-\eta'' + U(x)\eta = \omega^2\eta$. We find

$$\frac{U(x)}{\mu^2} = 1 + \frac{3\sinh\mu x}{2\beta\cosh^2\mu x} + \frac{(1-8\beta^2)}{4\beta^2\cosh^2\mu x} . \qquad (50)$$

This potential is independent of n, i.e., of the particular vacuum or soliton chosen. The shape of $U(x)$ for a typical value of β is shown in Fig. 2. The Schrödinger equation has three singularities, at $z = \sinh\mu x = \pm i$ and $z = \infty$, and these singularities are all regular, showing that the equation can be reduced to the hypergeometric equation. We can calculate the exponents at the singular points and express the solution in terms of Riemann's P function.[9] We find

$$\eta(x) = P\begin{Bmatrix} -i & i & \infty & \\ 1+\frac{i}{4\beta} & 1-\frac{i}{4\beta} & \frac{ik}{\mu} & z \\ -\frac{1}{2}-\frac{i}{4\beta} & -\frac{1}{2}+\frac{i}{4\beta} & -\frac{ik}{\mu} & \end{Bmatrix}, \qquad (51)$$

where $k = (\omega^2 - \mu^2)^{1/2}$, $z = \sinh\mu x$. By standard manipulations[9] this can be converted to

$$\eta = P\begin{Bmatrix} 0 & 1 & \infty & \\ \frac{ik}{\mu} & 1-\frac{i}{4\beta} & 1+\frac{i}{4\beta} & s^{-1} \\ -\frac{ik}{\mu} & -\frac{1}{2}+\frac{i}{4\beta} & -\frac{1}{2}-\frac{i}{4\beta} & \end{Bmatrix}, \qquad (52)$$

where $s = \frac{1}{2}(1 - i\sinh\mu x)$. The solutions we seek are branches of this P function and are hypergeometric functions.

First we study the scattering solutions. Let $\eta_\pm$ denote the branches of (52) with exponents $\pm ik/\mu$ at $s = \infty$. We may take then

$$\begin{aligned} \eta_+ &= (-4is)^{-ik/\mu}(1-s^{-1})^{1-i/4\beta} \\ &\quad\times F(a, a-c+1, a-b+1; s^{-1}), \\ \eta_- &= (-4is)^{ik/\mu}(1-s^{-1})^{1-i/4\beta} \\ &\quad\times F(b, b-c+1, b-a+1; s^{-1}), \end{aligned} \qquad (53)$$

where $a = 2 + ik/\mu$, $b = 2 - ik/\mu$, $c = \frac{5}{2} + i/2\beta$, and we have included phase factors for convenience. For large $|x|$ we have

$$-4is \to \begin{cases} e^{-\mu x} \text{ as } x \to -\infty, \\ -e^{\mu x} \text{ as } x \to +\infty, \end{cases} \qquad (54)$$

and so we obtain the plane-wave solutions

$$\eta_\pm \to \begin{cases} e^{\pm ikx} \text{ as } x \to -\infty, \\ e^{\mp ikx}e^{\pm\pi k/\mu} \text{ as } x \to +\infty. \end{cases} \qquad (55)$$

Here we must take care with the branches of $(-4is)^{\pm ik/\mu}$; we specify that the branch cut lies along the positive real axis in the complex s plane so that for $\mathrm{Im}(s) \gtrless 0$ we set $-s = e^{\pm i\pi}s$.

As x varies from $-\infty$ to $+\infty$, $s = \frac{1}{2}(1 - i\sinh\mu x)$ moves in the complex s plane parallel to the imaginary axis down $\mathrm{Re}(s) = \frac{1}{2}$, i.e., between the two singular points at $s = 0$, $s = 1$. To find the scattering and reflection coefficients we must analytically continue η_+ along this path. During this analytic continuation we will find

$$\eta_\pm \to p_\pm\eta_+ + q_\pm\eta_-$$

for coefficients $p_\pm, q_\pm$. We use the identity

$$\begin{aligned} F(a, a-c+1, a-b+1, s^{-1}) &= \frac{\Gamma(a-b+1)\Gamma(1-c)(-s)^a}{\Gamma(a-c+1)\Gamma(1-b)} F(a,b,c;s) \\ &\quad+ \frac{\Gamma(a-b+1)\Gamma(c-1)}{\Gamma(a)\Gamma(c-b)}(-s)^{a-c+1} \\ &\quad\times F(a-c+1, b-c+1, 2-c; s), \end{aligned} \qquad (56)$$

valid for $|\arg(-s^{-1})| < \pi$. This places the cut as before along the positive real axis. The contour for s, as x varies, can be deformed into a simple clockwise loop about the origin. Analytic continuation around this path leads to the replacement

$$\begin{aligned} (-s)^a &\to (-s)^a e^{-2\pi i a}, \\ (-s)^{a-c+1} &\to (-s)^{a-c+1}e^{-2\pi i(a-c+1)} \end{aligned} \qquad (57)$$

in Eq. (56). Now we reexpress the hypergeometric functions in terms of the branches at infinity, thereby completing the analytic continuation. We obtain

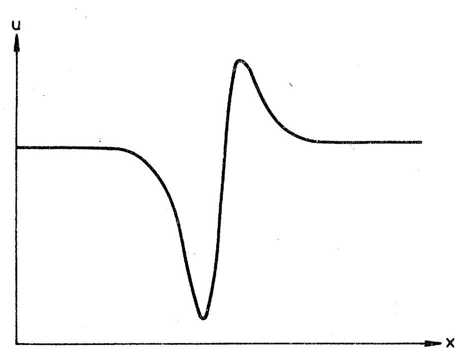

FIG. 2. The Schrödinger potential $U(x) = 1 + (\cosh x)^{-2}[3(\sinh x)/2\beta + (8\beta^2 - 1)/4\beta^2]$ plotted for the value $\beta = \frac{1}{2}$ (with unit mass scale). The potential falls off exponentially to its asymptotic values $U_+ = U_- = 1$.

$$F(a,a-c+1,a-b+1;s^{-1})\to F(a,a-c+1,a-b+1;s^{-1})e^{(\pi k/\mu-\pi/2\beta)}b(k)$$
$$+F(b,b-c+1,b-a+1;s^{-1})(-s)^{a-b}e^{(2\pi k/\mu-\pi/2\beta)}2^{4ik/\mu}a(k)\,, \tag{58}$$

where

$$a(k)=\frac{\left(\frac{ik}{\mu}-1\right)\left[\Gamma\left(\frac{1}{2}+\frac{ik}{\mu}\right)\right]^2}{\left(\frac{ik}{\mu}+1\right)\Gamma\left(\frac{1}{2}+\frac{ik}{\mu}+\frac{i}{2\beta}\right)\Gamma\left(\frac{1}{2}+\frac{ik}{\mu}-\frac{i}{2\beta}\right)}\,, \tag{59}$$

$$b(k)=\frac{\sinh(\pi/2\beta)}{\cosh(\pi k/\mu)}\,.$$

In addition, we need to analytically continue the coefficients of the hypergeometric function in Eq. (53) for η_+ around the same contour. We find

$$(-4is)^{-ik/\mu}(1-s^{-1})^{1-i/4\beta}\to e^{(\pi/2\beta-2\pi k/\mu)}(-4is)^{-ik/\mu}(1-s^{-1})^{1-i/4\beta}. \tag{60}$$

Combining all terms we obtain

$$\eta_+\to\eta_+e^{-\pi k/\mu}b(k)+\eta_-e^{\pi k/\mu}a(k)\,. \tag{61}$$

By comparing with the asymptotic solutions in Eq. (55) we deduce that there are scattering solutions η with the behavior

$$\eta(x)\to\begin{cases}e^{ikx} & \text{for } x\to-\infty\\ a(k)e^{ikx}+b(k)e^{-ikx} & \text{for } x\to+\infty\,,\end{cases} \tag{62}$$

where $a(k)$, $b(k)$ are defined in Eqs. (59). Note that $a(k)$ and $b(k)$ satisfy the unitarity relations $\bar{a}(k)=a(-k)$, $\bar{b}(k)=b(-k)$, and $|a|^2-|b|^2=1$. By comparing with the reflection and transmission coefficients defined in Eqs. (6) and (7), together with the relations (8)–(10) we find that

$$a=1/\bar{t}_-,\quad b=r_-/t_-\,, \tag{63}$$

enabling us to obtain in turn $t_-, t_+=t_-, r_-$ and $r_+=-\bar{r}_-t_-/\bar{t}_-$. As a check we can verify the asymptotic expansion for t_- given in Eq. (17) and find as required

$$t_-\sim 1-\frac{iI}{2k}+O(k^{-2})\,,$$

where

$$I=\int_{-\infty}^{\infty}[U(x)-\mu^2]dx=\mu\,\frac{(1-8\beta^2)}{2\beta^2}\,. \tag{64}$$

Bound states could occur for $\omega^2<\mu^2$, i.e., for k imaginary. The solution with asymptotic behavior shown in Eq. (62) will be square integrable for $k=-i\kappa$, κ real and $\kappa\geq 0$, only if $a(\kappa)=0$. Hence

$$\frac{\left(\frac{\kappa}{\mu}-1\right)\left[\Gamma\left(\frac{1}{2}+\frac{\kappa}{\mu}\right)\right]^2}{\left(\frac{\kappa}{\mu}+1\right)\Gamma\left(\frac{1}{2}+\frac{\kappa}{\mu}+\frac{i}{2\beta}\right)\Gamma\left(\frac{1}{2}+\frac{\kappa}{\mu}-\frac{i}{2\beta}\right)}=0\,. \tag{65}$$

There is only one solution $\kappa=\mu$, which of course is the zero mode $\omega^2=0$. The wave function, found by substituting $k=-i\mu$ into the expression for η_+ [Eq. (53)], is

$$s^{-1/2-i/4\beta}(s-1)^{-1/2+i/4\beta}\,, \tag{66}$$

which is in fact $d\phi_c/dx$. We conclude that meson-soliton bound states cannot form for any finite value of β.

This completes our analysis of the meson-soliton interactions. We can now evaluate the first-order soliton mass correction ΔM, by substitution into Eq. (24):

$$\Delta M=\frac{-\mu(1-8\beta^2)}{8\pi\beta^2}+\frac{1}{4\pi}\int_0^\infty\frac{dk}{(k^2+\mu^2)^{1/2}}\left[ik\ln\left(\frac{a}{\bar{a}}\right)-\frac{\mu(1-8\beta^2)}{2\beta^2}\right]\,, \tag{67}$$

where we have carried out an integration by parts and substituted for I from Eq. (64). We have not attempted to evaluate analytically this integral, which we know to be convergent, although it could be done for special values of β such as $\beta=\infty$.

Finally we mention the Fourier transform of ϕ_c. Let us choose the branch of ϕ_c that interpolates between v_n and v_{n+1}. Then

$$\tilde{\phi}_c(p)=(v_n+v_{n+1})\pi\delta(p)+\frac{i}{p}\,\tilde{\phi}_c'(p) \tag{68}$$

and

$$\frac{i}{p}\,\tilde{\phi}_c'(p)=\frac{\pi e^{[(2n-1)/4\beta+p/2\mu]\pi}}{\mu\beta\sqrt{\lambda}\sinh(\pi p/\mu)}\,F\left(1-\frac{i}{2\beta},1+\frac{ip}{\mu},2;2\right). \tag{69}$$

As expected[8] there are poles at $p=i\mu n$ for n any integer. However, there are no evident symmetry properties under the exchange $p\to -p$.

VI. CONCLUSION

We have extended the general theory of solitons in two-dimensional scalar models to include the possibility that adjacent vacuums might be unrelated, i.e., there will be no internal symmetry connecting the two vacuums and the corresponding meson masses might be different. Such an extension is not completely straightforward in the quantum theory, for one must subtract correctly the vacuum energies arising from unrelated vacuums and the soliton mass correction must be modified accordingly. In the absence of multisoliton solutions we have discussed only the single intrinsic soliton property, its mass. However, quantum corrections involve a stability analysis of the soliton, which can be interpreted in terms of soliton-meson interactions. We have listed all the models in which such a stability analysis can be given exactly, enabling one to describe precisely to first order the soliton-meson interactions. It is easy to list all such models because the small-disturbance equation for scalar theories is the one-dimensional Schrödinger equation, for which all exactly soluble potentials are known. It is straightforward to work backwards and derive the corresponding field theoretic potential $V(\phi)$; here we have been interested only in the potentials which can be given explicitly.

Models involving nonlinear structures are usually difficult to study by analytic techniques and so it seems particularly useful to have at hand such a range of models on which to test conjectures. In higher dimensions, where gauge fields are involved, the nonlinearities and consequent difficulties are much more severe and any knowledge gained from two-dimensional models is likely to be of use.

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