Number of Feynman diagrams in arbitrary order of perturbation theory

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Recurrence relations are established to determine the number of Feynman diagrams in arbitrary order of perturbation theory for four expansions: (i) the Green's function $\mathcal{G}$ expanded in the noninteracting Green's function $\mathcal{G}^{(0)}$ and the bare interaction V, (ii) the proper self-energy Σ expanded in $\mathcal{G}^{(0)}$ and V, (iii) Σ expanded in $\mathcal{G}$ and V, and (iv) Σ expanded in $\mathcal{G}$ and the particle-particle (hole-hole) ladder sum Γ. In each case, the number of diagrams has the asymptotic behavior const$\times(2n+1)!!$ for large n.

In a series of papers, Trainor and co-workers have used the methods of group theory to investigate the expansions (in the interaction and the bare propagator) of the one-particle[1,2] and many-particle[3] Green's functions; they found not only the number of diagrams in each order but also an effective enumeration procedure for them. Their motivation was to investigate the analytic problem of obtaining an upper bound on the nth-order contribution to the above expansions.

The present article deals only with the counting of the diagrams; the results of Refs. 1 and 2 are extended to more sophisticated expansions. The method used, however, is that of functional derivatives,[4] and the results, where they overlap with those of Refs. 1 and 2, are simpler. A very simple recurrence relation is found for the number of nth-order diagrams in the expansion of the proper self-energy in the full Green's function and the bare interaction. The relation is then used to generate recurrence relations for the three other expansions considered here. Finally, the recurrence relations are used to determine asymptotic expressions for the number of diagrams in each expansion. The treatment is not self-contained, however, for appeal is made to a result of Ref. 2 to evaluate a constant appearing in one of these asymptotic expressions.

The Hamiltonian under consideration is the nonrelativistic

$$\hat{H}_0=\sum_{s_1}\int d^3r_1\hat{\psi}^\dagger(\vec{r}_1,s_1)\left(\frac{-\hbar^2}{2m}\nabla_1^{\,2}\right)\hat{\psi}(\vec{r}_1,s_1)$$
$$+\tfrac{1}{2}\sum_{s_1}\sum_{s_2}\int d^3r_1\int d^3r_2\hat{\psi}^\dagger(\vec{r}_1,s_1)\hat{\psi}^\dagger(\vec{r}_2,s_2)V(\vec{r}_1,s_1;\vec{r}_2,s_2)\hat{\psi}(\vec{r}_2,s_2)\hat{\psi}(\vec{r}_1,s_1)\,. \tag{1}$$

$\hat{\psi}^\dagger(\vec{r},s)$ and $\hat{\psi}(\vec{r},s)$ are the field operators, in the Schrödinger representation, of a particle of spin projection s at point $\vec{r}$. The interaction potential V is assumed to be symmetric $[V(\vec{r}_1,s_1;\vec{r}_2,s_2)=V(\vec{r}_2,s_2;\vec{r}_1,s_1)]$.

The expansion of the proper self-energy Σ in the Green's function $\mathcal{G}$ and the bare interaction V is considered in the Appendix; there it is shown that M_n, the number of nth-order (in V) diagrams, satisfies the recurrence relation

$$M_n=(2n-1)M_{n-1}+(n-1)\sum_{m=2}^{n-2}M_mM_{n-m},\quad n\geqslant 4 \tag{2}$$

with $M_1=2$, $M_2=2$, and $M_3=10$. From this result, it follows that the ratio M_n/M_{n-1} has the asymptotic expansion

$$M_n/M_{n-1}\sim 2n+1+4n^{-1}+21n^{-2}+145n^{-3}+O(n^{-4})\,. \tag{3}$$

In turn, the latter result gives an asymptotic expansion for M_n:

$$M_n\sim\mu(2n+1)!!\,[1-2n^{-1}-\tfrac{7}{4}n^{-2}-\tfrac{34}{3}n^{-3}+O(n^{-4})]\,, \tag{4}$$

where μ is independent of n. An analytical result for μ cannot be obtained by the above method; a numerical calculation gives $\mu=0.1353$, in agreement with the analytical result $\mu=e^{-2}$ obtained below.

The expansion of the previous paragraph certainly fails when the bare interaction is "large" [for example, when $V(\vec{r})$ has a hard core]. In such cases, it is necessary to expand in the effective interaction Γ which is the sum of the particle-particle (hole-hole) ladder diagrams as in Fig. 1. Let N_n be the number of diagrams in this expansion of Σ (in $\mathcal{G}$ and Γ); note that Γ is expanded in $\mathcal{G}$ rather than $\mathcal{G}^{(0)}$. Define generating functions by

$$\sigma\sim\sum_{n=1}^{\infty}M_nw^n\,, \tag{5}$$

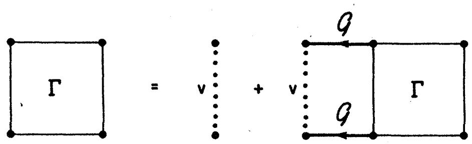

FIG. 1. Integral equation for the function Γ which sums the particle-particle (hole-hole) ladder diagrams.

$$\sigma \sim \sum_{n=1}^{\infty} N_n \gamma^n , \tag{6}$$

where $\gamma = w/(1-w)$; the expansions are asymptotic because of the factorial growth of M_n and N_n, and converge only for $w=0$ and $\gamma=0$. Equating the generating functions gives the following relation between M_n and N_n:

$$N_n = \sum_{m=1}^{n} (-1)^{n-m} \begin{Bmatrix} n-1 \\ m-1 \end{Bmatrix} M_m \tag{7}$$

or, equivalently,

$$M_n = \sum_{m=1}^{n} \begin{Bmatrix} n-1 \\ m-1 \end{Bmatrix} N_m . \tag{8}$$

These results give the asymptotic expression

$$N_n \sim e^{-1/2} M_n [1 + \tfrac{3}{4} n^{-1} + \tfrac{23}{32} n^{-2} + O(n^{-3})] . \tag{9}$$

An alternative procedure is to work from the recurrence relation, valid for $n \geq 4$,

$$N_n = (2n-1)N_{n-1} + 2(n-1)N_{n-2} + \sum_{m=2}^{n-2} (-1)^{n-m-1} N_m + (n-1) \sum_{m=2}^{n-2} N_m (N_{n-m} + N_{n-m-1}) , \tag{10}$$

which follows from Eqs. (2), (7), and (8).

The recurrence relation of Eq. (2) can also be used to determine the number of diagrams in the expansions of the Green's function $\mathcal{G}$ and the proper self-energy Σ in the bare interaction V and the bare propagator $\mathcal{G}^{(0)}$. Let K_n and L_n denote the number of diagrams in these expansions for $\mathcal{G}$ and Σ, respectively; define the generating functions g and σ by

$$g \sim 1 + \sum_{n=1}^{\infty} K_n u^n , \tag{11}$$

$$\sigma \sim \sum_{n=1}^{\infty} L_n u^n . \tag{12}$$

Then, from Dyson's equation, one has

$$g = 1/(1-\sigma) \tag{13}$$

and therefore, with $K_0 = 1$,

$$K_n = \sum_{m=0}^{n-1} K_m L_{n-m} , \quad \text{for } n \geq 1 . \tag{14}$$

To relate the coefficients L_n to the coefficients M_n, one uses the expression

$$\sigma \sim \sum_{n=1}^{\infty} M_n u^n g^{2n-1} ; \tag{15}$$

recall that each nth-order diagram in Σ contains $2n-1$ propagators. With the definition

$$g^n \sim \sum_{m=0}^{\infty} I_m^{(n)} u^m \tag{16}$$

(that is, $I_m^{(n)}$ is the coefficient of u^m in the expansion of the nth power of g), one finds

$$L_n = \sum_{m=1}^{n} M_m I_{n-m}^{(2m-1)} \quad \text{for } n \geq 1 . \tag{17}$$

From the definition of $I_m^{(n)}$, one has

$$I_m^{(1)} = K_m \quad \text{for } m \geq 0 . \tag{18}$$

The following recurrence relation is easily established:

$$I_m^{(n)} = K_m + \sum_{k=0}^{m-1} K_k I_{m-k}^{(n-1)} \quad \text{for } n \geq 2, \ m \geq 1 . \tag{19}$$

Table I provides numerical values of K_n, L_n, M_n, and N_n for $n=1$ to 9; the values of K_n agree with those for the same quantity $N_G^c(n)$ of Ihrig, Rosensteel, Chow, and Trainor.[2] These values were calculated as follows: Eq. (2) was used to determine M_4 to M_9, starting from $M_1=2, M_2=2, M_3=10$; N_n was then obtained from Eq. (7); finally, L_n and K_n were obtained from Eqs. (14), (17), (18), and (19).

Asymptotic formulas for K_n and L_n are easily obtained from the above results and

$$I_m^{(n)} = 2^m \begin{Bmatrix} n \\ m \end{Bmatrix} + \text{higher-order terms.} \tag{20}$$

One finds

$$L_n \sim e^2 \mu (2n+1)!! [1 + O(n^{-1})] , \tag{21}$$

$$K_n \sim L_n [1 + O(n^{-1})] . \tag{22}$$

TABLE I. Values of K_n, L_n, M_n, and N_n for $n=1$ to 9.

n	K_n	L_n	M_n	N_n
1	2	2	2	2
2	10	6	2	0
3	74	42	10	8
4	706	414	82	56
5	8 162	5 058	898	624
6	110 410	72 486	12 018	8 256
7	1 708 394	1 182 762	187 626	127 488
8	29 752 066	21 573 054	3 323 682	2 233 920
9	576 037 442	434 358 018	65 607 682	43 657 280

At this point, one can make use of Theorem 5.4 of Ihrig, Rosensteel, Chow, and Trainor[2]: if $K_n = \beta(n)(2n+1)!!$ then $\beta(n)\to 1$ as $n\to\infty$. The value of μ is thereby determined to be

$$\mu = e^{-2}\,. \tag{23}$$

It is perhaps worth remarking that each of the four expansions has the asymptotic behavior const $\times(2n+1)!!$, despite the considerable renormalization undertaken in going from the expansion of $\mathcal{G}$ in $\mathcal{G}^{(0)}$ and V to the expansion of Σ in $\mathcal{G}$ and Γ. The convergence of these expansions is in doubt, whatever the size of the interaction.

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APPENDIX: DERIVATION OF EQUATION (2)

The approach used here is that of the functional-derivative method of condensed-matter physics as discussed in Chap. 5 of Kadanoff and Baym.[4] The diagrams generated by this method apply, however, to any Hamiltonian of the form of Eq. (1). Imaginary times are used to exploit the formal equivalence between the time-development operator $\exp(-i\hat{H}t/\hbar)$ and the statistical operator $\exp(-\beta\hat{H})$. The Hamiltonian is $\hat{H}(\tau_1)=\hat{H}_0+\hat{H}_1(\tau_1)$, where $\hat{H}_0$ is given by Eq. (1) and

$$\hat{H}_1(\tau_1)=\sum_{s_1}\int d^3r_1\hat{\psi}^\dagger(\vec{r}_1,s_1)W(1)\hat{\psi}(\vec{r}_1,s_1)\,, \tag{A1}$$

where $W(1)=W(\vec{r}_1,s_1,\tau_1)$ is an external potential and $\tau_1=it_1$ is a real variable.

In the grand canonical ensemble, the quantity of interest is $\hat{K}(\tau_1)=\hat{H}(\tau_1)-\mu\hat{N}$ where μ is the chemical potential and $\hat{N}$ is the number operator. The Green's function is defined as[5]

$$\mathcal{G}(1,2)=\frac{-\mathrm{Tr}\{\hat{U}(\beta\hbar,0)T_\tau[\hat{\psi}_K(1)\hat{\psi}_K^\dagger(2)]\}}{\mathrm{Tr}[\hat{U}(\beta\hbar,0)]}\,, \tag{A2}$$

where

$$\hat{U}(\tau,\tau_0)=1+\sum_{n=1}^{\infty}(-\hbar)^{-n}\int_{\tau_0}^{\tau}d\tau_1\cdots\int_{\tau_0}^{\tau_{n-1}}d\tau_n\hat{K}(\tau_1)\cdots\hat{K}(\tau_n) \tag{A3}$$

is a time-development operator and the subscript K means that the operators are in the Heisenberg-type representation,

$$\hat{\psi}_K(1)=\hat{U}^{-1}(\tau_1,0)\hat{\psi}(\vec{r}_1,s_1)\hat{U}(\tau_1,0)\,, \tag{A4}$$

$$\hat{\psi}_K^\dagger(1)=\hat{U}^{-1}(\tau_1,0)\hat{\psi}^\dagger(\vec{r}_1,s_1)\hat{U}(\tau_1,0)\,. \tag{A5}$$

Note that $\hat{U}$ is not unitary and hence that $\hat{\psi}_K^\dagger(1)\neq\hat{\psi}_K^\dagger(1)$; finally, the T_τ operator orders the field operators so that the τ arguments increase to the left, and multiplies the result by $(-1)^P$, where P is the number of interchanges of fermion field operators.

The functional derivative of the Green's function with respect to the external potential W is given by

$$\frac{\hbar\delta\mathcal{G}(1,2)}{\delta W(3)}=\mp[\mathcal{G}(1,2)\mathcal{G}(3,3^+)-\mathcal{G}_2(1,3;2,3^+)]\,, \tag{A6}$$

where the upper (lower) sign is for bosons (fermions) and $\mathcal{G}_2$ is the two-particle Green's function

$$\mathcal{G}_2(1,2;3,4)=\frac{\mathrm{Tr}\{\hat{U}(\beta\hbar,0)T_\tau[\hat{\psi}_K(1)\hat{\psi}_K(2)\hat{\psi}_K^\dagger(4)\hat{\psi}_K^\dagger(3)]\}}{\mathrm{Tr}[\hat{U}(\beta\hbar,0)]}\,. \tag{A7}$$

Equation (A6) combines with the result

$$\mathcal{G}(1,2)=\mathcal{G}^{(0)}(1,2)\mp\int d3\int d4\,\mathcal{G}^{(0)}(1,4)\hbar^{-1}v(4,3)\mathcal{G}_2(4,3;2,3^+) \tag{A8}$$

[which follows from the equation of motion of $\mathcal{G}(1,2)$] to yield

$$\mathcal{G}(1,2)=\mathcal{G}^{(0)}(1,2)+\int d3\int d4\,\mathcal{G}^{(0)}(1,4)[-\hbar^{-1}v(4,3)]\times\left[\pm\mathcal{G}(4,2)\mathcal{G}(3,3^+)+\frac{\hbar\delta\mathcal{G}(4,2)}{\delta W(3)}\right], \tag{A9}$$

where $v(1,2)=V(\vec{r}_1,s_1;\vec{r}_2,s_2)\delta(\tau_1-\tau_2)$ and

$$\int dn\cdots=\sum_{s_n}\int d^3r_n\int_0^{\beta\hbar}d\tau_n\cdots\,. \tag{A10}$$

Application of the operator $\hbar\delta/\delta W$ to Dyson's equation,

$$\mathcal{G}(1,2)=\mathcal{G}^{(0)}(1,2)+\int d3\int d4\,\mathcal{G}^{(0)}(1,4)\Sigma(4,3)\mathcal{G}(3,2)\,, \tag{A11}$$

yields an integral equation whose solution is

$$\frac{\hbar\delta\mathcal{G}(1,2)}{\delta W(3)}=\mathcal{G}(1,3^+)\mathcal{G}(3,2)+\int d4\int d5\,\mathcal{G}(1,4)\frac{\hbar\delta\Sigma(4,5)}{\delta W(3)}\mathcal{G}(5,2)\,. \tag{A12}$$

Equations (A9), (A11), and (A12) combine to give the final result

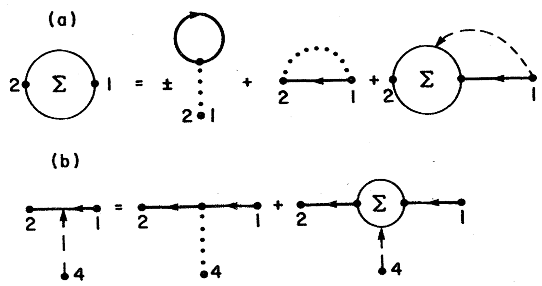

FIG. 2. (a) Diagrammatic representation of Eq. (A13). (b) Diagrammatic representation of Eq. (A12), after multiplication by $[-\hbar^{-1}v(4,3)]$ and integration (summation) over the variables $(\vec{r}_3, \tau_3, s_3)$.

$$\Sigma(1,2) = \pm\, \delta(1,2) \int d3\, \mathcal{G}(3,3^+)[-\hbar^{-1}v(1,3)] + \mathcal{G}(1,2^+)[-\hbar^{-1}v(1,2)] + \int d3 \int d4\, \mathcal{G}(1,3)\frac{\hbar\delta\Sigma(3,2)}{\delta W(4)}[-\hbar^{-1}v(1,4)], \tag{A13}$$

which is Eq. (5-25b) of Kadanoff and Baym.[4]

Equation (A13) is given in diagrammatic form in Fig. 2(a); Eq. (A12), after multiplication by the operator $\int d3[-\hbar^{-1}v(4,3)]$, is shown in Fig. 2(b). Iteration of these equations yields the expansion of Σ in $\mathcal{G}$ and V; note that only connected diagrams appear.

Let Σ_n be the sum of the nth-order diagrams (of which there are M_n) and $\Sigma_n^> = \Sigma_{n+1} + \Sigma_{n+2} + \cdots$; clearly $M_1 = M_2 = 2$. After generation of the diagrams for Σ_{n-1}, there will remain M_{n-1} diagrams containing $(n-2)$ V lines and a dashed line working on Σ, plus M_{n-2} diagrams containing $(n-3)$ V lines and a dashed line working on $\Sigma_1^>$, and so on, down to M_2 diagrams containing one V line and a dashed line working on $\Sigma_{n-3}^>$, plus one diagram with a dashed line working on $\Sigma_{n-2}^>$. Since each nth-order diagram has $2n-1$ $\mathcal{G}$ lines, the number of diagrams in nth order is (for $n \geq 3$)

$$M_n = \sum_{m=1}^{n-2} M_{n-m} M_m (2m-1) + M_{n-1}(2n-3). \tag{A14}$$

Then $M_3 = 10$ and a rearrangement of the sum yields Eq. (2).

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