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Probabilistic ideas in the theory of Fermi fields: Stochastic quantization of the Fermi oscillator

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We give a complete description of the Fermi oscillator in terms of ordinary, numerical-valued, Markov processes. This description includes a path-integral formulation for the Hamiltonian semigroup and for the configurational Schwinger functions and the explicit formulation and solution of the stochastic differential equations describing the system in the sense of Nelson's stochastic mechanics.

I. INTRODUCTION

Probabilistic ideas and methods have proven to be extremely fruitful in the quantum theory of relativistic Bose fields. The best proof of this statement is, of course, the success of the approach to Euclidean quantum field theory as classical statistical mechanics in the rigorous construction of models of interacting fields.[1]

The formulation of the quantum theory of Bose fields in terms of local (Markov) probabilistic fields has also offered the possibility of quite a new approach to the study of physical properties of realistic models by means of Monte Carlo numerical experiments.[2]

The deep reasons for the success of the probabilistic language have been studied at the foundational level[3] and traced back to the possibility of a probabilistic formulation of quantum mechanics itself.[4,5]

In this paper, using as a concrete example the explicitly soluble Fermi oscillator, we show the possibility of performing on a Fermi system the three steps, by now familiar in the theory of Bose fields, towards the complete translation of the quantum-mechanical problem into probabilistic language:

(a) Feynman-Kac representation for the Hamiltonian semigroup;

(b) path-integral representation of the configurational Schwinger functions;

(c) stochastic quantization in the sense of Nelson.

We wish to stress that these three steps can be performed in terms of ordinary, numerical-valued, Markov processes, and simple ones, indeed, with values in $Z_2=\{-1,1\}$.

As in the Bose case, therefore, we realize the connection of the quantum problem with a simple classical statistical-mechanics problem which, among other things, is simply implementable in numerical experiments.

In Sec. II we construct the Euclidean theory of the Fermi oscillator, while in Sec. III we treat the same system according to Nelson's stochastic approach to quantum mechanics. Section IV contains conclusions and outlook.

II. THE FERMI OSCILLATOR: EUCLIDEAN THEORY

We define the Fermi oscillator by the equations of motion

$$\dot{Q}=P\,,\quad \dot{P}=-Q \tag{2.1}$$

and by the canonical anticommutation relations

$$\{Q,P\}=0\,,\quad Q^2=1\,,\quad P^2=1$$

for the Hermitian operators Q,P.

This general two-level system can be seen from several different points of view (as a Majorana-Dirac field in one space-time dimension, or as one normal mode of a Dirac field in any dimension, or alternatively as a spin-$\frac{1}{2}$ system), each of which points to a distinct development of the present approach.

From the point of view of the general framework of Nelson's stochastic mechanics, it is most stimulating to adopt the perspective of having one and the same *classical* system, defined by the equations of motion (2.1), for which in a space-time of dimension 1 both Bose stochastic quantization[3] and Fermi stochastic quantization (the one to be developed here) are consistent.

In this sense we are conducting a preliminary exploration of the approach to stochastic quantization that a spin-statistics theorem will force for classical fields of half-integral spin in higher dimension.

We adopt here the "Q-space" representation of the time-zero field algebra on the Hilbert space

$$\mathcal{H} = L^2(Z_2, d\sigma)$$

(where $Z_2 = \{-1, 1\}$; $\int d\sigma = \frac{1}{2}\sum_{\sigma=\pm 1}$) defined by the action on any $\psi \in \mathcal{H}$:

$$(Q\psi)(\sigma) = \sigma\psi(\sigma), \quad (P\psi)(\sigma) = i\sigma\psi(-\sigma).$$

The Hamiltonian of the system is, correspondingly, given by

$$(H_0\psi)(\sigma) = \tfrac{1}{2}[\psi(\sigma) - \psi(-\sigma)].$$

For future reference we observe that the most general normalized solution of the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t,\sigma) = \tfrac{1}{2}[\psi(t,\sigma) - \psi(t,-\sigma)] \tag{2.2}$$

is given, within an inessential overall phase factor, by

$$\psi(t,\sigma) = \cos\alpha + \sigma\sin\alpha\,\exp[-i(t-t_0)],$$

where α and t_0 are integration constants.

We concentrate here first of all on the Hamiltonian semigroup $\exp(-tH_0)$, $t \geq 0$ and observe that it admits the Feynman-Kac representation

$$(\exp(-tH_0)\psi)(\sigma) = \int_\Omega d\mu_\sigma(\epsilon)\psi(\epsilon(t)),$$

where μ_σ is the probability measure on the set Ω of paths $\epsilon: t \in [0, +\infty) \to \epsilon(t) \in Z_2$ determined by the initial measure which assigns probability one to $\epsilon(0) = \sigma$ and by the transition probability density from the configuration x at time t to the configuration x' at time $t' \geq t$ given by

$$P(t, x\,|\,t', x') = 1 + xx' e^{-(t'-t)}.$$

The very simple proof of this statement just requires an explicit calculation showing that

$$(\exp(-tH_0)\psi)(\sigma) = \tfrac{1}{2}\sum_{\sigma'=\pm 1} P(0,\sigma\,|\,t,\sigma')\psi(\sigma')$$

and the observation that the P's satisfy the characteristic properties of the transition probabilities for a Markov process with values in $\{-1, 1\}$.

Similar straightforward considerations show that if $\Omega_0 = 1$ is the normalized ground-state wave function, for $G_0, G_1, \ldots, G_n$ arbitrary functions of the time-zero configuration operator Q, and for $t_1, t_2, \ldots, t_n$ arbitrary positive numbers, the following Feynman-Kac formula holds:

$$(\Omega_0, G_0(Q)e^{-t_1H_0}G_1(Q)e^{-t_2H_0}\cdots e^{-t_nH_0}G_n(Q)\Omega_0) = \int_\Omega d\mu(\epsilon)G_0(\epsilon(0))G_1(\epsilon(t_1))\cdots G_n(\epsilon(t_1+\cdots+t_n)),$$

where μ is the measure on Ω determined by the transition probabilities already considered and by the initial measure which assigns equal probability $\frac{1}{2}$ to $\epsilon(0) = \pm 1$.

In particular, the configurational Schwinger functions of the Fermi oscillator, initially defined for $t_1 \leq t_2 \leq \cdots \leq t_n$ as

$$S(t_1, t_2, \ldots, t_n) = (\Omega_0, Q\,e^{-(t_2-t_1)H_0}\,Q\,e^{-(t_3-t_2)H_0}Q\cdots e^{-(t_n-t_{n-1})H_0}Q\,\Omega_0)$$

admit for every $t_1, t_2, \ldots, t_n$ the following representation as moments of the measure μ:

$$S(t_1,\ldots,t_n) = \int_\Omega d\mu(\epsilon)\epsilon(t_1)\cdots\epsilon(t_n) = \begin{cases} 0 \text{ for } n \text{ odd} \\ e^{-(t_{i_2}-t_{i_1})}e^{-(t_{i_4}-t_{i_3})}\cdots e^{-(t_{i_{2k}}-t_{i_{2k-1}})} \text{ for } n = 2k. \end{cases}$$

Here the permutation

$$\begin{pmatrix} 1 \cdots 2k \\ i_1 \cdots i_{2k} \end{pmatrix}$$

is such that

$$t_{i_1} \leq t_{i_2} \leq \cdots \leq t_{i_{2k}}.$$

It is interesting to observe that these correlation functions are equal to those of a one-dimensional Ising model, which thus turns out to be the classical statistical-mechanics system underlying the Euclidean Fermi oscillator.

We conclude this section with the observation that, according to the analysis carried through for the Dirac field in Ref. 6, we have considered only *configurational* Schwinger functions.

We wish to point out that, as in the theory of Bose fields, the knowledge of just the configurational Schwinger functions already gives a complete specification of the quantum system.

One can indeed reconstruct $\mathcal{H}$ as the L^2 space on the configurations of the time-zero Euclidean field, and recover the Hamiltonian by requiring that for ψ, $\psi' \in \mathcal{H}$

$$(\psi', e^{-tH_0}\psi) = \int_\Omega d\mu(\epsilon)(\overline{j_0\psi'})(j_t\psi)$$

with $j_t\psi = \psi(\epsilon(t))$; after which

$$Q(t) = e^{itH_0}Q(0)\,e^{-itH_0},$$

$$P(t) = \dot{Q}(t) = i[H, Q(t)].$$

III. STOCHASTIC QUANTIZATION OF THE FERMI OSCILLATOR

According to Nelson's quantization procedure, the classical equations of motion, viewed as stochastic differential equations, to be solved in a class of stochastic processes whose specification amounts to the additional quantum hypothesis to be *added* to the classical framework, leads to a scheme mathematically equivalent to the one based on the Schrödinger equation.

In this section we consider the classical equations of motion (2.1), defining the classical harmonic oscillator, as stochastic differential equations in a sense to be specified below. We add the information that we are considering the two-level Fermi oscillator instead of the ordinary Bose quantum oscillator by suitably specifying the class of stochastic processes in which solutions of (2.1) are to be found. At this stage, as compared to previous attempts to describe Fermi fields in terms of c-number path integrals,[7] the constraint $Q^2=1$ will play a major role.

In order to get oriented, we consider first of all the stochastic process $\epsilon(t)$ associated to the ground state in the sense, discussed in Sec. II, that its moments reproduce the configurational Schwinger functions.

We observe that for this process the functions

$$p^{\pm}(t,\sigma)=\lim_{\Delta t\to 0^+} \pm\Delta t^{-1}E(\epsilon(t\pm\Delta t)-\epsilon(t)\,|\,\epsilon(t)=\sigma),$$

where $E(\ \ |\epsilon(t)=\sigma)$ stands for conditional expectation given that $\epsilon(t)=\sigma$ are well defined and are explicitly given by

$$p^{\pm}(t,\sigma)=\mp\sigma .$$

Defining mean forward and backward derivatives of the process $\epsilon(t)$ by

$$(D^{\pm}\epsilon)(t)=p^{\pm}(t,\epsilon(t))$$

we observe that

$$\tfrac{1}{2}(D^+p^-+D^-p^+)=-\epsilon .$$

Namely, the stochastic process $\epsilon(t)$ associated to the ground state in the sense of Sec. II satisfies the classical equations of motion in the sense of the following *problem A*:

Find the Markov processes $q(t)$ with values in Z_2 for which the mean forward and backward derivatives

$$D^{\pm}q\equiv p^{\pm}$$

are defined and satisfy

$$\tfrac{1}{2}(D^+p^-+D^-p^+)=-q .$$

Most of this section will be devoted to the study of this problem and of its relation to the quantum mechanics of the Fermi oscillator.

We start our discussion by defining a few notational conventions which will be useful in the following.

Every function F on Z_2 is a linear combination of the characters $\chi_0(\sigma)=1$, $\chi_1(\sigma)=\sigma$. By F_0 and F_1 we indicate the corresponding coefficients, namely, we set

$$F(\sigma)=F_0+\sigma F_1 .$$

We define the operator ∇ by the position

$$\nabla F=F_1$$

and observe that the following integration-by-parts formula holds:

$$\int d\sigma F\,\nabla G=\int d\sigma\,\nabla^+ F G\,,\quad \nabla^+=(\sigma-\nabla).$$

Next we study what conditions the hypothesis of existence of the mean forward and backward derivatives imposes on a Markov process $q(t)$ with values in Z_2.

We first of all define the density

$$\rho(t,\sigma)=\rho_0(t)+\sigma\rho_1(t)$$

associated to the process $q(t)$ by the prescription that for every $F(t,\sigma)$ the expectation of $F(t,q(t))$ be given by

$$E(F(t,q(t)))=\int d\sigma\,\rho(t,\sigma)F(t,\sigma).$$

The conditions of normalization and positivity of ρ require

$$\rho_0=1, \tag{3.1}$$
$$-1\leqslant\rho_1(t)\leqslant 1. \tag{3.2}$$

The observation that

$$p^+(t,\sigma)=\lim_{\Delta t\to 0^+} -\sigma\frac{P(t,\sigma|t+\Delta t,-\sigma)}{\Delta t}$$

and the positivity of the transition probabilities require that

$$p_1^+(t)\leqslant 0. \tag{3.3}$$

Similarly

$$p_1^-(t)\geqslant 0. \tag{3.4}$$

For every function $F(t,\sigma)$ differentiable in t the mean stochastic derivatives with respect to t are easily shown to be

$$\begin{aligned}(D^{\pm}F)(t,\sigma)\equiv\lim_{\Delta t\to 0^+} &\pm\Delta t^{-1}E(F(t\pm\Delta t,q(t\pm\Delta t))\\ &-F(t,q(t))\,|\,q(t)=\sigma)\\ &=\frac{\partial}{\partial t}F(t,\sigma)+p^{\pm}(t,\sigma)(\nabla F)(t).\end{aligned}$$

As for any such F and G, it must be

$$\frac{d}{dt}E(F(t,q(t))G(t,q(t)))=E(F(t,q(t))(D^+G)(t,q(t)))+E((D^-F)(t,q(t))G(t,q(t)))$$

we draw the conclusion that for any $G(t,\sigma)$ the following relation must hold between ρ and $p^{\pm}$:

$$G\frac{\partial\rho}{\partial t}=p^+\rho\nabla G+(\sigma-\nabla)p^-G\rho\,.$$

In particular, for $G=1$

$$\frac{\partial\rho}{\partial t}=(\sigma-\nabla)(p^-\rho)$$

or, equivalently,

$$\dot\rho_1=p_0^-+\rho_1p_1^-\,.$$

For $G=\sigma$ we obtain, instead,

$$\sigma\frac{\partial\rho}{\partial t}=\rho p^++(\sigma-\nabla)(\sigma p^-\rho)$$

or, equivalently,

$$\dot\rho_1=p_0^++\rho_1p_1^++\sigma[(p_1^++p_1^-)+\rho_1(p_0^++p_1^-)]\,.$$

These statements are easily summarized in terms of the functions

$$p(t,\sigma)=\frac{p^++p^-}{2}\,,\quad \delta p(t,\sigma)=\frac{p^+-p^-}{2}$$

into the conclusion that the continuity equation

$$\dot\rho_1=p_0+\rho_1p_1 \tag{3.5}$$

and the constraint equations

$$p_1+\rho_1p_0=0\,, \tag{3.6}$$
$$\delta p_0+\rho_1\delta p_1=0 \tag{3.7}$$

must hold.

The other condition appearing in problem A, namely, that the equation

$$\tfrac{1}{2}(D^+p^-+D^-p^+)=-q$$

must hold, leads to the differential equation

$$\frac{\partial p}{\partial t}+p\;\nabla p-\delta p\;\nabla\delta p=-\sigma$$

or, equivalently,

$$\delta p_0\,\delta p_1=\dot p_0+p_0p_1\,, \tag{3.8}$$
$$(\delta p_1)^2=1+p_1{}^2+\dot p_1\,. \tag{3.9}$$

Having reformulated problem A into conditions (3.1)–(3.9), we observe now that conditions (3.5), (3.7), and (3.8) imply

$$\ddot\rho_1(t)=-\rho_1(t)$$

and therefore completely determine ρ_1 within two integration constants as

$$\rho_1(t)=C\cos(t-\tau_0)\,,$$

where $C\in[-1,1]$ because of the inequalities (3.2).

Equations (3.5) and (3.6) in turn determine p_0 and p_1 in terms of ρ_1 as

$$p_0(t)=\frac{\dot\rho_1(t)}{1-\rho_1(t)^2}\,,$$

$$p_1(t)=\frac{-\rho_1(t)\,\dot\rho_1(t)}{1-\rho_1(t)^2}\,.$$

Equation (3.9) then determines δp_1 as

$$\delta p_1(t)=\xi(t)\,\frac{[1-\rho_1(t)^2-\dot\rho_1(t)^2]^{1/2}}{1-\rho_1(t)^2}$$

while Eq. (3.7) gives

$$\delta p_0(t)=-\xi(t)\,\frac{\rho_1(t)[1-\rho_1(t)^2-\dot\rho_1(t)^2]^{1/2}}{1-\rho_1(t)^2}\,,$$

where $\xi(t)=\pm1$ is to be determined in such a way that (3.3) and (3.4) hold.

A particular solution of problem A is thus determined for each choice of the integration constants C and τ_0.

For each such solution $q(t)$

$$E(q(t))=\rho_1(t)=C\cos(t-\tau_0)$$

while, defining

$$D=\frac{D^++D^-}{2}\,,$$

$$\begin{aligned}E((Dq)(t))&=\tfrac{1}{2}\sum_{\sigma=\pm1}\rho(t,\sigma)p(t,\sigma)\\&=p_0(t)+\rho_1(t)p_1(t)\\&=\dot\rho_1(t)=-C\sin(t-\tau_0)\,.\end{aligned}$$

These equations relate the integration constants C and τ_0 to the initial values of the "mean position" and the "mean velocity" of the process.

For comparison, observe that for the solution of the Schrödinger equation determined by the integration constants α and t_0

$$\psi(t,\sigma)=\cos\alpha+\sigma\sin\alpha\,\exp[-i(t-t_0)]$$

it is

$$(\psi(t),Q\psi(t))=\sin2\alpha\cos(t-t_0)\,,$$

$$(\psi(t),P\psi(t))=-\sin2\alpha\sin(t-t_0)\,.$$

Therefore, to such a solution ψ of the Schrödinger equation we can unambiguously associate a solution q_ψ of the stochastic equations of motion by requiring that, for every t,

$$E(q_\psi(t))=(\psi(t),Q\psi(t))\,.$$

This condition uniquely determines the integration constants in q_ψ to be

$$C=\sin 2\alpha\,,$$

$$\tau_0=t_0\,.$$

Notice that it is also

$$E((Dq_\psi)(t))=(\psi(t),P\psi(t))\,.$$

In particular we can determine the stochastic process associated in the previous sense to the ground state ($\alpha=0$).

For such a process, we immediately check that

$$\rho(t,\sigma)=1$$

and

$$p^{+}(t,\sigma)=-\sigma\,.$$

In particular the transition probability density per unit time satisfies

$$\lim_{\Delta t\to 0^+}\frac{P(t,\sigma|\,t+\Delta t,-\sigma)}{\Delta t}=1\,.$$

From the last condition the transition probability densities are explicitly obtained by exponentiation as

$$P(t,\sigma|t',\sigma')=1+\sigma\sigma' e^{-(t'-t)}\,,\quad t'\geq t\,.$$

Namely, as already observed for Bose fields, the stochastic process associated to the ground state in the sense of Nelson's stochastic mechanics coincides with the Euclidean process of Sec. II.

IV. CONCLUSION

We wish to comment here on the general conceptual framework emerging from the previous considerations.

Suppose we are given a quantum system whose observables at fixed time form an algebra $\mathcal{B}$. In the Heisenberg picture these observables evolve according to the dynamical group generated by the Hamiltonian H. Under very general hypotheses a complete description of the system will be given by the knowledge of the vacuum expectation values of some observables (fields) at generic times.

We are exploring a possible alternative description of the same quantum system in purely probabilistic language, such as offered by Nelson's stochastic mechanics.

Suppose that a maximal Abelian subalgebra $\mathcal{A}$ of $\mathcal{B}$ and the Hamiltonian H generate $\mathcal{B}$. Represent the Hilbert space of the states of the system as the L^2 space on the spectrum of $\mathcal{A}$, $L^2(\mathcal{A})$. Suppose furthermore that the Hamiltonian semigroup $\exp(-tH)$ is positivity preserving on $L^2(\mathcal{A})$. One can then introduce a Markov process with values in the spectrum of $\mathcal{A}$, whose expectation values at any time reproduce the vacuum expectation values of the elements of $\mathcal{A}$ at that time, and which evolves in such a way that $\exp(-tH)$ is the kernel of the transition probabilities.

A complete description of the quantum system is in this stochastic process, in the sense that from its knowledge one can reconstruct $\mathcal{A}$, H and therefore $\mathcal{B}$.

For systems with a well-defined classical analog Nelson's stochastic mechanics offers a nonambiguous procedure to construct such a stochastic process and therefore the complete quantum-mechanical structure, starting from the classical equations of motion and from a definite probabilistic hypothesis about the sense in which such equations are to be viewed as stochastic differential equations.

What we have shown here is that all the previous steps *can* be performed on the Fermi oscillator and that the constraint $Q^2=1$ on the solutions of the stochastic differential equations of motion (which, unlike the classical equations of motion do not become trivial under this constraint) forces Fermi statistics on the excitations of the corresponding quantum system.

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