Stationary world lines and the vacuum excitation of noninertial detectors

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The stationary world lines, on which quantized field detectors in a vacuum have time-independent excitation spectra, are discussed. They are characterized by the requirement that the geodetic interval between two points depends only on the proper time interval. To construct these world lines a generalization of the Frenet equations to Minkowski space is developed. The curvature invariants are found to be the proper acceleration and angular velocity of the world line. The equations are solved for constant invariants and the solutions are shown to be the stationary world lines. A classification into six types is made. The equivalence of the timelike Killing vector field orbits and the stationary world lines is demonstrated. The classification scheme therefore extends to Killing orbits and stationary coordinate systems in flat spacetime. Finally, the vacuum excitation spectra of detectors on a representative sample of the stationary world lines are calculated.

I. INTRODUCTION

Several years ago Unruh[1] showed that a scalar-particle detector moving with constant linear acceleration in the vacuum of flat spacetime will be excited. The detector will behave as if in contact with a bath of scalar particles with energies in a Planck spectrum of temperature: acceleration/2π. Similar results have been described for detectors of electromagnetic radiation.[2] Recently[3] it has been shown that a detector in uniform circular motion will also be excited. The spectrum differs from a Planck spectrum in a manner dependent on the detector's angular velocity. It is noteworthy that the spectrum is time independent in all these cases; in general, this will not be true. For example, if a detector was moving with a very slowly increasing linear acceleration one would expect the spectrum to be Planckian, though with a slowly increasing temperature.

The probability for a detector moving along a world line $x^\mu(\tau)$ to be found in an excited state of energy E at $\tau=\tau_0$ is[4]

$$P(E)=C(E)\int_{-\infty}^{\tau_0}d\tau\int_{-\infty}^{\tau_0}d\tau'\, e^{-iE(\tau-\tau')}\times\langle 0|\phi(x(\tau))\phi(x(\tau'))|0\rangle\,, \quad (1)$$

where $C(E)$ is a function characterizing the detector's sensitivity. The Wightman function for a scalar field is[5]

$$\langle 0|\phi(x(\tau))\phi(x(\tau'))|0\rangle=[2\pi^2 W(\tau,\tau')]^{-1}\,, \quad (2)$$

where the geodetic interval is

$$W(\tau,\tau')=[x_\mu(\tau)-x_\mu(\tau')][x^\mu(\tau)-x^\mu(\tau')]\,. \quad (3)$$

In terms of the proper time interval $s=\tau-\tau'$, the rate of excitation to the state with energy E is

$$\frac{dP(E)}{d\tau_0}=2C(E)\int_{-\infty}^{0}ds\,\cos(Es)\langle 0|\phi(x(\tau_0))\phi(x(\tau_0+s))|0\rangle\,. \quad (4)$$

The detector is therefore effectively immersed in a bath of scalar particles with energy spectrum

$$S(E,\tau)=2\pi\rho(E)\int_{-\infty}^{0}ds\,\cos(Es)\times\langle 0|\phi(x(\tau))\phi(x(\tau+s))|0\rangle\,, \quad (5)$$

where $\rho(E)$ is the density of states.

The condition that (5) be time independent is just the time independence of the Wightman function (2). This will be assured if, and only if,

$$W(\tau,\tau+s)=W(0,s)\,. \quad (6)$$

The spectrum is therefore time independent when the geodetic interval between two points on the detector's world line depends only on the proper time interval between them.

The world lines which satisfy the requirements above are called stationary. The geometric properties of these lines are independent of proper time. In the next section the Frenet equations of classical differential geometry are extended to Minkowski space. The curvature invariants are shown to be the proper acceleration and angular velocity of a world line. The stationary world lines are therefore solutions of the Frenet equations when the curvature invariants are constant.

In Sec. III the differential equation defining the stationary world lines is solved. We are therefore able to exhibit the explicit form of these world lines as functions of proper time. It is shown that there are six classes of stationary world lines. Within each class the world lines are qualitatively the same. This classification is shown to extend to the timelike Killing vector orbits and hence the stationary coordinate systems in flat spacetime. Finally, the excitation spectra of detectors on several of the stationary world lines are calculated. These spectra have a bearing on the question of particle definition in noninertial coordinates.

II. CURVATURE INVARIANTS AND THE FRENET EQUATIONS

An arbitrary timelike world line in flat space is generally described by four functions, $x^\mu(s)$, specifying the coordinates of each point s on the curve. This parameter may be taken to be the arc length or proper time on the world line. The parametric representation is unsatisfactory in two respects: (1) A world line is a geometric object and should not require a coordinate-dependent entity for its definition and (2) there is an inherent redundancy in the parametric representation since three functions suffice to determine the world line. The curvature invariants as described below provide an intrinsic definition of the world line not subject to these criticisms.

To begin, an orthonormal tetrad $V_a^\mu(s)$ is constructed at every point on the world line $x^\mu(s)$. The Latin index everywhere is a tetrad index. The tetrad is formed from the derivatives of $x^\mu(s)$ with respect to proper time (represented by one or more dots). It is assumed that the first four derivatives are linearly independent, the results being practically unchanged when they are not. Members of the tetrad must satisfy the orthonormality condition

$$V_{a\mu} V_b^\mu = \eta_{ab}, \tag{7}$$

where the metric has diagonal components $(1, -1, -1, -1)$ only.

By Gram-Schmidt orthogonalization of the derivatives working upwards from the first, the following expressions for the tetrad members are found:

$$V_0^\mu = \dot{x}^\mu, \tag{8}$$

$$V_1^\mu = \frac{\ddot{x}^\alpha}{(-\ddot{x}_\alpha \ddot{x}^\alpha)^{1/2}}, \tag{9}$$

$$V_2^\mu = \frac{(\ddot{x}_\gamma \ddot{x}^\gamma)\dddot{x}^\mu - (\ddot{x}_\gamma \dddot{x}^\gamma)\ddot{x}^\mu + (\ddot{x}_\gamma \ddot{x}^\gamma)^2 \dot{x}^\mu}{[(\ddot{x}_\alpha \ddot{x}^\alpha)^4 + (\ddot{x}_\alpha \ddot{x}^\alpha)(\ddot{x}_\beta \dddot{x}^\beta)^2 - (\ddot{x}_\alpha \ddot{x}^\alpha)^2(\dddot{x}_\beta \dddot{x}^\beta)]^{1/2}}, \tag{10}$$

$$V_3^\mu = \frac{1}{\sqrt{6}} \epsilon^{\alpha\beta\gamma\mu} V_{0\alpha} V_{1\beta} V_{2\gamma}. \tag{11}$$

Overall signs on these vectors are fixed by the orientation of the tetrad.

The tetrad V_a^μ is a basis for the vector space at a point on the world line. Derivatives of the basis vectors may therefore be expanded in terms of them:

$$\dot{V}_a^\mu = K_a{}^b V_b^\mu, \tag{12}$$

These are the generalized Frenet equations. K_{ab} is a coordinate-independent matrix whose structure must be determined.

Differentiation of the orthonormality condition (7) yields

$$\dot{V}_{a\mu} V_b^\mu + V_{a\mu} \dot{V}_b^\mu = 0, \tag{13}$$

and, in view of (12),

$$K_{ab} = -K_{ba}. \tag{14}$$

A basis vector V_a^μ is defined in terms of the first $a+1$ derivatives of x^μ; therefore, $\dot{V}_a^\mu$ will be a linear combination of the first $a+2$ derivatives. These $a+2$ derivatives are dependent only on the basis vectors V_b^μ where $b \leq a+1$. It follows that K_{ab} is zero if $b > a+1$. This and (14) limit the matrix to the form

$$K_{ab} = \begin{bmatrix} 0 & -\kappa(s) & 0 & 0 \\ \kappa(s) & 0 & -\tau(s) & 0 \\ 0 & \tau(s) & 0 & -\upsilon(s) \\ 0 & 0 & \upsilon(s) & 0 \end{bmatrix}. \tag{15}$$

The three functions of proper time are the invariants

$$\kappa = V_{0\mu} \dot{V}_1^\mu = -\dot{V}_{0\mu} V_1^\mu, \tag{16}$$

$$\tau = V_{1\mu} \dot{V}_2^\mu = -\dot{V}_{1\mu} V_2^\mu, \tag{17}$$

$$\upsilon = V_{2\mu} \dot{V}_3^\mu = -\dot{V}_{2\mu} V_3^\mu. \tag{18}$$

They are, respectively, the curvature, torsion, and hypertorsion of the world line. Sign choices are made for reasons brought out below.

To explore the physical significance of the invariants we examine the infinitesimal Lorentz transformations of the tetrad at a point on the world line. The transformations leave the metric invariant

$$\eta_{ab} = L_a{}^c L_b{}^d \eta_{cd}. \tag{19}$$

An infinitesimal transformation may be written

$$L_a{}^c = \delta_a{}^c + d\epsilon_a{}^c, \tag{20}$$

where the elements of $d\epsilon_a{}^c$ are small and must satisfy

$$d\epsilon_{ab} = -d\epsilon_{ba}. \tag{21}$$

The transformations are taken to be active; that is, the transformed tetrad moves $+v$ and is rotated $+\theta$ relative to the untransformed tetrad. Thus the infinitesimal generator is

$$d\epsilon_{ab}=\begin{bmatrix} 0 & -dv_1 & -dv_2 & -dv_3 \\ dv_1 & 0 & -d\theta_{12} & d\theta_{31} \\ dv_2 & d\theta_{12} & 0 & -d\theta_{23} \\ dv_3 & -d\theta_{31} & d\theta_{23} & 0 \end{bmatrix}. \tag{22}$$

The change in the tetrad resulting from this transformation is

$$\dot{V}_a^\mu = (d\epsilon_a{}^b/ds)V_b^\mu. \tag{23}$$

Equations (23) are identical to the Frenet equations (12); therefore, the physical content of the curvature invariants is found by comparison of (15) and (22). The curvature is the proper acceleration of the world line which is always parallel to V_1^μ. The torsion and hypertorsion are the components of proper angular velocity in the planes spanned by V_1^μ and V_2^μ, and V_2^μ and V_3^μ, respectively. The total proper angular velocity is the vector sum of these two invariants.

III. STATIONARY MOTIONS

In this section the general expression for a world line whose curvature invariants are constant is found. These world lines will be called stationary because their geometric properties are independent of proper time. One also finds that only observers on these world lines may establish a coordinate system in which they are at rest and the metric is stationary. Clearly, the geodetic interval between two points on a stationary world line can depend only on the proper time interval, therefore they are the world lines on which a detector's excitation is time independent.

The Frenet equation (12) may be reduced to a fourth-order linear equation in V_0^μ when the curvature invariants are constant:

$$\overset{....}{V_0^\mu} - 2a\ddot{V}_0^\mu - b^2V_0^\mu = 0\,, \tag{24}$$

where $a=\frac{1}{2}(\kappa^2-\tau^2-\upsilon^2)$ and $b=|\kappa\upsilon|$. The other basis vectors are determined from V_0^μ by the equations

$$V_1^\mu = \dot{V}_0^\mu/\kappa\,, \tag{25}$$

$$V_2^\mu = (\ddot{V}_0^\mu - \kappa^2 V_0^\mu)/\kappa\tau\,, \tag{26}$$

$$V_3^\mu = [\dddot{V}_0^\mu - (\kappa^2-\tau^2)\dot{V}_0^\mu]/\kappa\tau\upsilon\,. \tag{27}$$

Equation (24) is homogeneous with constant coefficients. The four roots of the characteristic equation are $\pm R_+$ and $\pm iR_-$, where

$$R_\pm = [(a^2+b^2)^{1/2} \pm a]^{1/2}. \tag{28}$$

The solution is therefore

$$V_0^\mu = A^\mu \cosh(R_+s) + B^\mu \sinh(R_+s) + C^\mu \cos(R_-s) + D^\mu \sin(R_-s)\,, \tag{29}$$

and the coefficients may be fixed by the initial conditions

$$(V_a^\mu)_{s=0} = \delta_a^\mu\,. \tag{30}$$

Solving (29) and (25)–(27) for A^μ, B^μ, C^μ, and D^μ at $s=0$ subject to the conditions (30) yields

$$A^\mu = R^{-2}(R_-^2+\kappa^2, 0, \kappa\tau, 0)\,, \tag{31}$$

$$B^\mu = R^{-2}(0, \kappa(R_-^2+\kappa^2-\tau^2)/R_+, 0, \kappa\tau\upsilon/R_+)\,, \tag{32}$$

$$C^\mu = R^{-2}(R_+^2-\kappa^2, 0, -\kappa\tau, 0)\,, \tag{33}$$

$$D^\mu = R^{-2}(0, \kappa(R_+^2-\kappa^2+\tau^2)/R_-, 0, -\kappa\tau\upsilon/R_-)\,, \tag{34}$$

with $R^2=R_+^2+R_-^2$.

The stationary world lines separate naturally into six classes according to the values of the curvature invariants:

(i) $\kappa=\tau=\upsilon=0$,
$$V_0^\mu = (1, 0, 0, 0)\,; \tag{35}$$

(ii) $\kappa=\tau=0$,
$$V_0^\mu = (\cosh\kappa s, \sinh\kappa s, 0, 0)\,; \tag{36}$$

(iii) $|\kappa|<|\tau|$, $\upsilon=0$, $\rho^2=\tau^2-\kappa^2$,
$$V_0^\mu = \rho^{-2}(\tau^2-\kappa^2\cos\rho s, \kappa\rho\sin\rho s, \kappa\tau-\kappa\tau\cos\rho s, 0)\,; \tag{37}$$

(iv) $|\kappa|=|\tau|$, $\upsilon=0$,
$$V_0^\mu = (1+\tfrac{1}{2}\kappa^2s^2, \kappa s, \tfrac{1}{2}\kappa^2s^2, 0)\,; \tag{38}$$

(v) $|\kappa|>|\tau|$, $\upsilon=0$, $\sigma^2=\kappa^2-\tau^2$,
$$V_0^\mu = \sigma^{-2}(\kappa^2\cosh\sigma s-\tau^2, \kappa\sigma\sinh\sigma s, \kappa\tau\cosh\sigma s-\kappa\tau, 0)\,; \tag{39}$$

(vi) $\upsilon\neq0$ [Eqs. (29) and (31)–(34)].

The stationary world lines are the integral curves, or orbits, of the timelike Killing vector fields in Minkowski space. That each of the world lines is a Killing orbit may be seen explicitly by comparison with its tangent vector. Because only stationary world lines are invariant under proper time translations, the converse can be proved by showing that all Killing trajectories have this property. A world line is invariant under proper time translations if the tangent vectors at all points s are related to the tangent vectors at $s+ds$ by an infinitesimal Lorentz transformation which is independent of s. Comparison of Eqs. (23) and (12) shows that world lines are invariant under proper time translations if, and only if, they are stationary. A Killing vector field ξ^α, is defined by

$$\xi_{\alpha,\beta}+\xi_{\beta,\alpha}=0\,. \tag{40}$$

The proper time derivative of ξ^α along its trajectories is the infinitesimal Lorentz transformation

$$\dot{\xi}^\alpha=\xi^\alpha{}_{,\beta}\,\xi^\beta\,. \tag{41}$$

Applying (40) repeatedly one can show that the proper time derivative of $\xi^\alpha{}_{,\beta}$ is zero along any trajectory.

IV. SPECTRA

Combining Eqs. (2), (3), (5), and (6) and noting that

$$\rho(E)=E^2/2\pi^2\,, \tag{42}$$

the excitation spectrum of a detector on a stationary world line is

$$S(E)=\frac{E^2}{4\pi^3}\int_{-\infty}^{\infty}e^{-iEs}\{[x_\mu(s)-x_\mu(0)]\times[x^\mu(s)-x^\mu(0)]\}^{-1}ds\,. \tag{43}$$

In this section the results of Sec. III and (43) are used to calculate excitation spectra for detectors on world lines of types (i) through (v). The world lines are exhibited and described in a convenient Lorentz frame.

(i) The inertial detector, in its rest frame, follows the world line

$$x^\mu(s)=(s,0,0,0)\,. \tag{44}$$

Its excitation spectrum is

$$S(E)=E^3/4\pi^2\,, \tag{45}$$

which corresponds to a ground-state energy per mode of $E/2$. This term appears in the other cases with another spectrum superimposed. It will be subtracted out in the results below.

(ii) This is linear motion with constant proper acceleration κ. In the rest frame of the detector at $s=0$ the world line is hyperbolic

$$x^\mu(s)=\kappa^{-1}(\sinh\kappa s,\cosh\kappa s,0,0)\,. \tag{46}$$

It is convenient to express the spectrum in terms of a dimensionless energy $\epsilon_\kappa=E/\kappa$. The spectrum

$$S(\epsilon_\kappa)=\frac{\epsilon_\kappa{}^3}{2\pi^2(e^{2\pi\epsilon_\kappa}-1)} \tag{47}$$

is Planckian with temperature $\kappa/2\pi$ and is shown in Fig. 1.

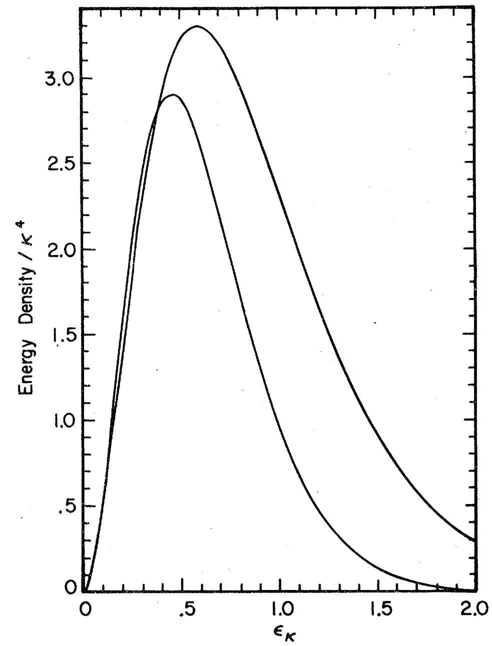

FIG. 1. Spectra for detectors on world lines with $\tau=\upsilon=0$ (lower curve) and $\kappa=\tau$, $\upsilon=0$ (upper curve).

(iii) The detector moves in a circle of radius κ/ρ^2 with constant velocity κ/τ. The world line is a helix

$$x^\mu(s)=\rho^{-2}(\tau\rho s,\kappa\cos\rho s,\kappa\sin\rho s,0)\,. \tag{48}$$

Equation (43) cannot be integrated analytically in this case. The results of a numerical integration are shown in Fig. 2. A dimensionless energy $\epsilon_\rho=E/\rho$ is used. The curves are drawn for various values of κ/ρ. When $\kappa/\rho=0$, the spectrum is flat

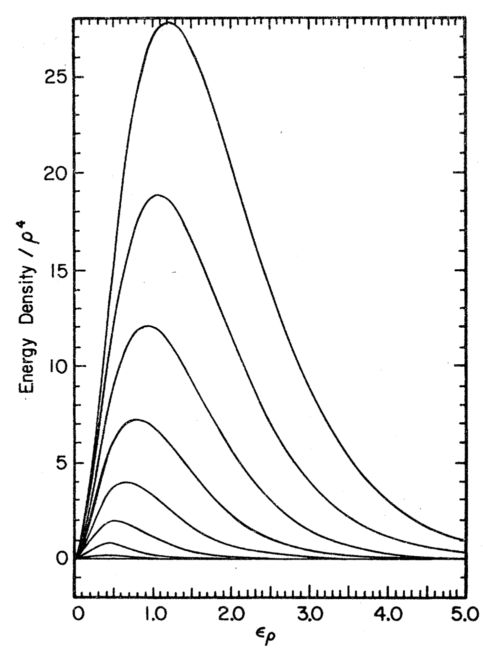

FIG. 2. Spectra for detectors on world lines with $|\tau|>|\kappa|$ and $\upsilon=0$. $\kappa/\rho=0, 0.25, 0.5, \ldots, 1.75, 2.0$ from the lower to the upper curve, respectively.

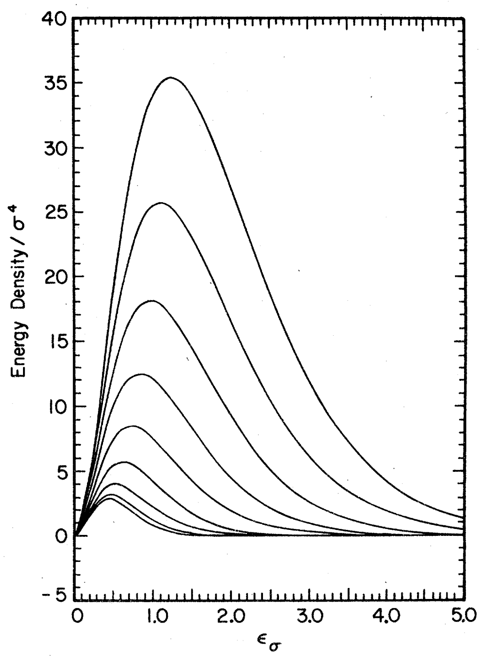

FIG. 3. Spectra for detectors on world lines with $|\kappa|>|\tau|$ and $\upsilon=0$. $\tau/\sigma=0, 0.25, 0.5, \ldots, 1.75, 2.0$ from the lower to the upper curve, respectively.

because the detector is inertial. In the limit $\kappa/\rho \gg 1$ the spectrum tends toward the analytic result in (iv) below.

(iv) This peculiar cusped motion has spatial projection

$$y=\tfrac{1}{3}\sqrt{2}\,\kappa x^{3/2}. \tag{49}$$

The world line is, however, not only smooth but self-similar. In the detector's rest frame at $s=0$, the motion is

$$x^\mu(s)=(s+\tfrac{1}{6}\kappa^2 s^3, \tfrac{1}{2}\kappa s^2, \tfrac{1}{6}\kappa^2 s^3, 0). \tag{50}$$

The spectrum may be calculated exactly and using the dimensionless energy ϵ_κ is

$$S(\epsilon_\kappa)=\frac{\epsilon_\kappa^{\,2}}{8\pi^2\sqrt{3}}\,e^{-\sqrt{12}\,\epsilon_\kappa}. \tag{51}$$

This spectrum is plotted in Fig. 1.

(v) This is an unbounded detector motion along the catenary

$$x=\kappa\cosh(y/\tau). \tag{52}$$

The world line is

$$x^\mu(s)=\sigma^{-2}(\kappa\sinh\sigma s, \kappa\cosh\sigma s, \tau\sigma s, 0). \tag{53}$$

The spectrum cannot be found analytically. Numerical results are shown in Fig. 3 with the spectrum given as a function of $\epsilon_\sigma=E/\sigma$. For $\tau/\sigma=0$, the spectrum becomes Planckian as would be expected. As τ/σ gets large, the spectrum tends toward the spectrum found in (iv).

(vi) In the general case the world line may be written as

$$x^\mu(s)=\left(\frac{\Delta}{RR_+}\sinh(R_+s), \frac{\Delta}{RR_+}\cosh(R_+s), \frac{\kappa\tau}{R\Delta R_-}\cos(R_-s), \frac{\kappa\tau}{R\Delta R_-}\sin(R_-s)\right), \tag{54}$$

where $\Delta^2=\tfrac{1}{2}(R^2+\kappa^2+\tau^2+\upsilon^2)$. This is a superposition of the constant linearly accelerated motion and uniform circular motion. The spatial path of a detector on this world line is helicoid with a pitch that decreases to zero at $s=0$ and increases thereafter. The spectra form a two-parameter set of curves and have not been calculated.

V. CONCLUSION

In this paper the stationary world lines in flat spacetime have been described and the vacuum excitation of detectors on these world lines calculated. These results have immediate application in the study of the coordinate dependence of quantum field theory in flat spacetime.[6] Because the stationary world lines are trajectories of timelike Killing vector fields, a stationary coordinate system adapted to each world line may be constructed. In this system the world line is a coordinate line and proper time is proportional to coordinate time. The spectrum of vacuum fluctuations as measured by observers at rest in the stationary coordinate systems are the spectra calculated in Sec. IV. The existence of the Killing vector field allows a consistent quantum field theory to be developed in these systems. The classification afforded by the curvature invariants shows that there are six essentially different stationary coordinate systems. Only three are well known: Minkowski, Rindler, and rotating coordinates. General statements concerning quantum field theory (particularly particle and vacuum definitions) in the flat-space stationary coordinate systems are made possible by the examination of the remaining three systems.

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