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Duality for heavy-quark systems. II. Coupled channels

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We derive the duality relation $\langle\sigma\rangle\simeq\langle\sigma_{\text{bound}}\rangle$ which relates a suitable energy average of the physical coupled-channel cross section $\sigma=\sigma(e^+e^-\to\text{hadrons})$ to the same average of the cross section σ_{bound} for the production of bound $q\bar{q}$ states in a single-channel confining potential. The average $\langle\sigma_{\text{bound}}\rangle$ is equated by our previous work to the average cross section $\langle\sigma_{\text{free}}\rangle$ for production of a $q\bar{q}$ pair moving freely in the nonconfining color Coulomb potential. Thus, $\langle\sigma\rangle\simeq\langle\sigma_{\text{free}}\rangle$. The corrections to these duality relations are calculable. We give an exactly solvable coupled-two-channel model and use it to verify duality for both weak and strong coupling.

I. INTRODUCTION

Duality equates an appropriate energy average of the observed cross section for $e^+e^-\to$ hadrons to the same average of the calculated quantum-chromodynamics (QCD) cross section.[1-3] It is used in the ψ and Υ regions to predict leptonic widths $\Gamma_{e^+e^-}$ (Refs. 4 and 5) to determine quark masses,[6,7] and to test QCD.[3,8-10] In a previous paper[11] we gave a proof of duality, under suitable circumstances, for confining potentials which may include short-range (e.g., color Coulomb) components. We found that the duality relation holds up to correction terms which can be calculated from data, and illustrated this relation for three exactly solvable potentials (linear, harmonic oscillator, and Hulthén, which mocks Coulomb at short range).

However, our proof and other studies of duality for potential models[12-16] have been restricted to the single-channel process $e^+e^-\to\psi_n$, where ψ_n is the nth $^3S_1$ $q\bar{q}$ bound state. It has been assumed that the subsequent decay of the resonant states to the observed hadrons does not affect the duality relation. The real process is, of course, a multichannel process with complicated couplings to the allowed final states, e.g., $D\bar{D}$, $D^*\bar{D}$, ... The coupling to these channels results in significant mass shifts for bound states and resonances, distortion of resonance shapes, and changes in decay probabilities.[17] It would not be surprising if these effects seriously damaged duality. However, we show in this paper that duality survives the coupled-channel effects.

In Sec. II we construct a very naive, but exactly solvable, coupled-channel model which illustrates both the effects noted above and our approach to the coupled-channel problem.

In Sec. III, we derive a new form of the duality relation, applicable to coupled channels, which equates an energy average of the multichannel cross section to the same energy average of the single-channel bound-state cross section. In our previous paper,[11] hereafter referred to as paper I, we showed that the single-channel bound-state cross section could in turn be related to the free $q\bar{q}$ cross section, either with or without a short-range color Coulomb interaction. The combination of these two results, each with calculable correction terms, is duality for multichannel potential models.

In Sec. IV we show numerically that duality holds for the model of Sec. II for both weak and strong coupling. The accuracy of the duality relations (~1% error) is remarkable considering the displacements and extreme distortions of resonances caused by the coupling. We conclude that duality, with corrections, gives a very reliable method for comparing QCD cross sections to experimental data.

II. A COUPLED-CHANNEL MODEL

The problem we are considering is $e^+e^-\to\gamma\to$ hadrons, where there are many hadron channels open. The open channels do not include a free $q\bar{q}$ pair, since quarks are confined, but do include such states as $D\bar{D}$, $D^*\bar{D}$, ... However, the state $q\bar{q}$ does appear as a closed channel, and is the only channel to which the photon couples directly.[18] The final hadrons result from the coupling of the $q\bar{q}$ to the physical states.

We must define the wave functions for this problem carefully. We will assume that the quarks and mesons are nonrelativistic, and that their interactions can be described by a multichannel potential model. The complete Hamiltonian H is then a matrix operator which acts on the space of open channels plus the $q\bar{q}$ channel.[18a] The exact eigenstates of H are denoted by ψ_β, where the physical channel β corresponds for $r\to\infty$ to an observed state ($D\bar{D}$, $D^*\bar{D}$, ...). For $r\to 0$, within the range of the strong interaction, H induces mixing between channels and the ψ_β have several components. We denote the component of ψ_β in channel i for small r by $\psi_{\beta i}(r,E)$. The closed-channel (confined) $q\bar{q}$ state is labeled by $i = 1$.

The coupling of the asymptotic (observed) channel β to the photon depends in the nonrelativistic treatment on $\psi_{\beta 1}(0,E)$. The partial cross section is

$$\sigma_\beta(E) = (24\pi^3\alpha^3 e_q^2/m_q^2 W^2)\,\rho_\beta(E)\,|\psi_{\beta 1}(0,E)|^2, \quad (1)$$

where E is the nonrelativistic energy, $W=E+2m_q$, m_q and e_q are the mass and the charge of the quark, α is the fine-structure constant, and $\rho_\beta(E)$ is the density of states in channel β at energy E. The total cross section is

$$\sigma_{e^+e^-\to\text{ hadrons}}(E) = \sum_\beta \sigma_\beta(E). \quad (2)$$

The Hamiltonian used to discuss the ψ system in Ref. 17 is restricted to a set of coupled two-body channels,

$$H = \begin{pmatrix} -\frac{1}{m_q}\nabla^2 + V_1 & V_{12} & \cdots \\ V_{12} & -\frac{1}{m_2}\nabla^2 + m_2 - m_q + V_2 & \cdots \\ \cdots & & \end{pmatrix}, \quad (3)$$

where $\hbar = c = 1$ and $\frac{1}{2}m_i$ is the reduced mass of the ith two-body system. This Hamiltonian neglects the (apparently small) coupling to three-body final states. In this section we consider a very simple, exactly solvable two-channel model with a confining infinite-square well in channel 1.[19] H is defined by setting $V_1=0$ and $V_2=0$ for $r\le R$, $V_1=\infty$ and $V_2=0$ for $r>R$, and $V_{12}=V$ for $r\le R$. This naive model could be generalized but is sufficient to illustrate the coupled-channel effects which appear in Ref. 17, and to give a simple test of duality in Sec. IV.

The elementary radial solutions to the Schrödinger equation for $r<R$ are $\psi_i(r) = A_i \sin kr/r$, where

$$\begin{bmatrix} \frac{k^2}{m_1} - E_1 & V \\ V & \frac{k^2}{m_2} - E_2 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0. \quad (4)$$

Here $m_1 = m_q$, $E_1 = E$, and $E_2 = E - (m_2 - m_1)$. We assume that $m_2 > m_1$. There are two solutions for k_i^2:

$$\begin{aligned} 2k_1^2 &= m_2E_2 + m_1E_1 - [(m_2E_2 - m_1E_1)^2 + 4m_1m_2V^2]^{1/2} \\ &\to 2m_1E_1, \quad V\to 0, \\ 2k_2^2 &= m_2E_2 + m_1E_1 + [(m_2E_2 - m_1E_1)^2 + 4m_1m_2V^2]^{1/2} \\ &\to 2m_2E_2, \quad V\to 0, \end{aligned} \quad (5)$$

with

$$\frac{A_{2i}}{A_{1i}} = \frac{m_1E_1 - k_i^2}{m_1V}. \quad (6)$$

The general solutions for $\psi_1(r)$ and $\psi_2(r)$ for $r<R$ are of the form

$$\begin{aligned} \psi_1(r) &= (A_{11}\sin k_1 r + A_{12}\sin k_2 r)/r, \\ \psi_2(r) &= (A_{21}\sin k_1 r + A_{22}\sin k_2 r)/r. \end{aligned} \quad (7)$$

For $r>R$, $\psi_1(r)$ vanishes identically and $\psi_2(r)$ is given by

$$\begin{aligned} \psi_2(r) &= \sin(k_2' r + \delta)/k_2', \quad k_2'^2 = m_2E_2 > 0, \\ \psi_2(r) &= N e^{-|k_2'|r}/r, \quad E_2 < 0. \end{aligned} \quad (8)$$

We use the usual plane-wave normalization for $E_2 > 0$.

The boundary conditions require that $\psi_1(r)$ vanish at $r=R$ and that $\psi_2(r)$ and its first derivative be continuous. The general solutions for $E_2>0$ are

$$r\psi_1(r) = A(\sin k_2R \sin k_1 r - \sin k_1 R \sin k_2 r) \quad (9)$$

and

$$\begin{aligned} r\psi_2(r) &= (A/m_1V)[(m_1E_1 - k_1^2)\sin k_2R \sin k_1 r - (m_1E_1 - k_2^2)\sin k_1R \sin k_2 r], \quad r<R \\ &= k_2'^{-1}\sin(k_2' r + \delta), \quad r>R \end{aligned} \quad (10)$$

with

$$\begin{aligned} e^{2i\delta} &= S \\ &= e^{-2ik_2'R}\,\frac{k_1(m_1E_1 - k_1^2)\cos k_1R \sin k_2R - k_2(m_1E_1 - k_2^2)\sin k_1R\cos k_2R + ik_2'(k_2^2 - k_1^2)\sin k_1R \sin k_2R}{k_1(m_1E_1 - k_1^2)\cos k_1R \sin k_2R - k_2(m_1E_1 - k_2^2)\sin k_1R\cos k_2R - ik_2'(k_2^2 - k_1^2)\sin k_1R \sin k_2R}. \end{aligned} \quad (11)$$

The normalization constant A is

$$A = \frac{m_1V\sin(k_2'R + \delta)}{k_2'(k_2^2 - k_1^2)\sin k_1R \sin k_2R}. \quad (12)$$

We recall from Eq. (1) that $|\psi_1(r=0)|^2$ determines the cross section for $e^+e^-\to$ hadrons. This quantity is determined by Eqs. (9), (11), and (12), and is of course a function of energy,

$$|\psi_1(0,E)|^2=\frac{m_1{}^2V^2(k_1\sin k_2R-k_2\sin k_1R)^2}{|k_1(m_1E_1-k_1{}^2)\cos k_1R\sin k_2R-k_2(m_1E_1-k_2{}^2)\sin k_1R\cos k_2R-ik_2'(k_2{}^2-k_1{}^2)\sin k_1R\sin k_2R|^2}\,. \tag{13}$$

The exact result for σ for $E_2>0$ obtained by substituting Eq. (13) in Eq. (1) is rather opaque. We will therefore illustrate the coupled-channel effects initially by taking the coupling V to be small and calculating the positions and widths of resonances to order V^2 (Born approximation), and will afterwards demonstrate the effects of strong coupling numerically.

The resonances in the coupled-channel problem are at the poles of S in Eq. (11), and for V small we expect them to be near the bound states in the single-channel square-well problem,

$$m_1E_n^0=k_n^{0\,2}=n^2\pi^2/R^2\,. \tag{14}$$

Thus we let $k_1R=n\pi+z$, where z is small, expand the denominator of Eq. (11) in powers of z and find its zeros. The resultant poles in the E plane are at the values (correct to order V^2)

$$E_n=E_n^0+\Delta E_n-i\Gamma_n/2\,, \tag{15}$$

where

$$\Delta E_n=-\frac{2m_2V^2k_n^{0\,2}}{(k_2'^2-k_n^{0\,2})^2}\,\frac{\sin k_2'R\cos k_2'R}{k_2'R} \tag{16}$$

and

$$\frac{\Gamma_n}{2}=\frac{2m_2V^2k_n^{0\,2}}{(k_2'^2-k_n^{0\,2})^2}\,\frac{\sin^2k_2'R}{k_2'R}\,, \tag{17}$$

with E_n^0 and k_n^0 given by Eq. (14).

The features displayed by this simple model are qualitatively the same as in Ref. 17. The positions of resonances are shifted from their single-channel values by amounts comparable to the half-widths of the resonances. The Γ's acquire potentially nontrivial structure through the factor $\sin^2k_2'R$ and decrease rapidly for large energies. Equation (17) is exactly the Born-approximation result for Γ_n if the process is considered to proceed through the production and decay of bound states.

The exact expression for the cross section is given in Eqs. (1) and (2). For our example, $m_1=m_q$, β takes on only one value, and $\rho_\beta(E)=m_2k_2'/4\pi^2$. It can easily be seen that $|\psi_1(0,E)|^2$ will show resonance structure: the denominator of $|\psi_1(0,E)|^2$ is just the absolute square of the denominator of $S(E)$. Thus $|\psi_1(0,E)|^2$ has poles at the same locations as $|S(E)|^2$. Using the same small-z expansion for $|\psi_1(0,E)|^2$ as for $S(E)$, we find for V small that

$$|\psi(0,E)|^2\rho(E)=\frac{k_n^{0\,2}}{2\pi R}\,\frac{\Gamma_n/2}{(E-E_n-\Delta E_n)^2+(\Gamma_n/2)^2}\,,\quad E\sim E_n\,. \tag{18}$$

The factor $k_n^{0\,2}/2\pi R$ is just $|\psi_1^0(0,E_n)|^2$, the value of the normalized bound-state wave function for the (uncoupled) square-well potential at $r=0$. We can therefore write σ approximately as

$$\sigma(E)\simeq(24\pi^3\alpha^2e_q{}^2/m_q{}^2W^2)\times\sum_n|\psi_1^0(0,E_n)|^2\,\frac{\Gamma_n/2\pi}{(E-E_n-\Delta E_n)^2+(\Gamma_n/2)^2}\,. \tag{19}$$

For the Γ's sufficiently small, the Breit-Wigner factors can be replaced by δ functions, and $\sigma\simeq\sum_n\sigma_{n,\text{bound}}$ where $\sigma_{n,\text{bound}}$ is the usual cross section for the production of the nth bound state in e^+e^- annihilation. This, of course, is the expected result. For V large, we must use the exact expression for $|\psi_1(0,E)|^2$ in Eq. (13).

For $E_2<0$, the expression for the cross section in Eq. (1) includes a sum over discrete bound states, plus the continuum contribution

$$\sigma(E)=\frac{24\pi^3\alpha^2e_q{}^2}{m_q{}^2W^2}\left[\sum_n|\psi_1(0,E_n)|^2\delta(E-E_n)+\frac{m_2k_2'}{4\pi^2}\,|\psi_1(0,E)|^2\theta(E_2)\right]. \tag{20}$$

The energies of bound states are determined by a small-z expansion of $S(E)$ as before, $E_n=E_n^0+\Delta E_n$, with

$$\Delta E_n=-\frac{2m_2V^2k_n^{0\,2}}{(k_n^{0\,2}+\kappa_n{}^2)}\,\frac{e^{-\kappa_nR}\sinh\kappa_nR}{\kappa_nR}\,, \tag{21}$$

where $\kappa_n{}^2=m_2(m_2-m_1-E_n^0)$, $E_n^0=n^2\pi^2/m_1R^2$, and V is still assumed small. Note that the energies of the bound states are lowered by the coupling to the second (meson) channel.

In summary, for V small, we find shifts in the positions of both resonances [Eq. (16)] and bound states [Eq. (21)]. The magnitude of these shifts ΔE_n is on the order of $\Gamma_n/2$. We further find momentum-dependent structure in Γ_n [Eq. (17)], which can distort the resonance shapes and alter the decay probabilities.

For large V, we must treat the problem numerically. We choose parameters to mock the ψ region, $m_1 = m_q = 1.5$ GeV, $m_2 = 1.8$ GeV, and $R = 5.736$ GeV^{-1}. These parameters give a spectrum with one bound state (the "ψ") and a spacing to the first resonance of 600 MeV, with the "$D\bar{D}$" threshold halfway between. The function $|\psi_1(0,E)|^2$ calculated from Eq. (13) is shown in Fig. 1 for several values of the coupling potential V. We also show on Fig. 1 the location of the bound state. The value $V = 0.1$ GeV gives a fairly weak coupling and for this value the resonances are approximately Breit-Wigner curves except near threshold. As V increases, the resonances are strikingly distorted and their positions shift. The structure near $E = 0.6$ GeV results from a zero in the numerator of $|\psi_1(0,E)|^2$ caused by unequal mass kinematics.

In Table I we give the energy of the bound state and $|\psi_1(0,E_1)|^2$ for several values of V. The bound-state ("ψ") energies are shifted downward for stronger coupling by amounts which would significantly alter single-channel fits to the spectrum. Stronger coupling also leads to a marked decrease in $|\psi_1(0,E_1)|^2$, which is proportional to the cross section for $e^+e^- \to$ "ψ". These two effects result physically from leakage of the $q\bar{q}$ wave function into the *closed* ($E_2 < 0$) "$D\bar{D}$" channel.

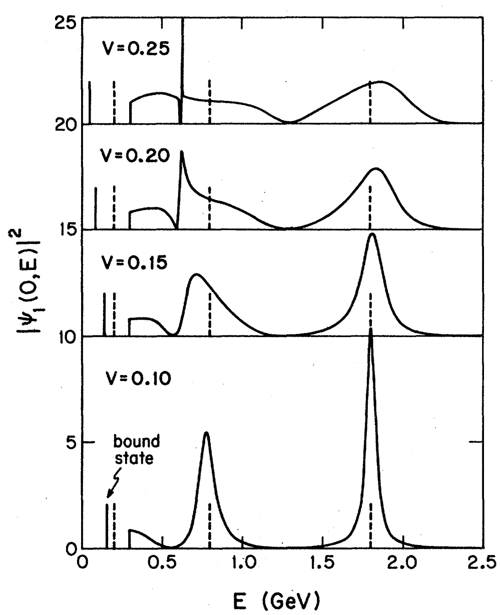

FIG. 1. The dependence of $|\psi_1(0,E)|^2$ *versus* energy on the magnitude of the coupling V for the two-channel problem of Sec. II. $\psi_1(0,E)$ is the quark-antiquark component of the total wave function at the origin. $|\psi_1(0,E)|^2$ is given in Eq. (13). We also note the location of the bound state for this solution. The cross section $\sigma_{e^+e^-\to \text{hadrons}}$ is related to $|\psi_1(0,E)|^2$ by Eq. (20). $E = 0$ corresponds to the free $q\bar{q}$ threshold. The parameters chosen are $m_1 = m_q = 1.5$ GeV, $m_2 = 1.8$ GeV, and $R = 5.736$ GeV^{-1} corresponding to bound-state energies $E_n^0 = (0.2 \text{ GeV})\, n^2$ in the single-channel problem. The single-channel bound-state energies E_n^0 are indicated by dashed lines.

TABLE I. Variation of the bound-state energy E_1 and of $|\psi_1(0,E_1)|^2$ with the coupling V for the two-channel model of Sec. II. Here $\psi_1(0,E_1)$ is the value of the quark-antiquark component of the total wave function at the origin. $\psi_1^0(0,E_1) = (\pi/2R^3)$ is the value of the lowest bound-state wave function at the origin for the corresponding single-channel problem, $V = 0$. The parameters used were $m_1 = m_q = 1.5$ GeV, $m_2 = 1.8$ GeV, $R = 5.736$ GeV^{-1}, and $E_1^0 = (\pi^2/m_1R^2) = 0.2$ GeV.

V (GeV)	E_1 (GeV)	$\lvert\psi_1(0,E_1)\rvert^2$ (GeV^3)	$\lvert\psi_1(0,E_1)/\psi_1^0(0,E_1)\rvert^2$
0	0.200	8.32×10^{-3}	1.000
0.1	0.161	6.96	0.836
0.15	0.124	6.27	0.754
0.2	0.083	5.80	0.696
0.25	0.039	5.46	0.656

After seeing the effects of coupling on the hadron cross section, we were pleasantly surprised to find that we could derive a duality relation for the general coupled-channel problem and that this relation is very accurate for the model just considered.

III. PROOF OF DUALITY FOR COUPLED CHANNELS

The duality relation we wish to derive is[11]

$$\langle \sigma(E) \rangle \simeq \langle \sigma_{\text{free}}(E) \rangle\,, \tag{22}$$

where $\langle\ \rangle$ denotes a suitable energy average, $\sigma(E)$ is the total cross section for $e^+e^- \to$ hadrons from Eqs. (1) and (2),

$$\sigma(E) = \frac{24\pi^3\alpha^2 e_q^{\,2}}{m_q^{\,2}W^2} \sum_{\substack{\text{open}\\ \text{channels}\\ \beta}} \rho_\beta(E)\,|\psi_{\beta 1}(0,E)|^2\,, \tag{23}$$

and

$$\sigma_{\text{free}}(E) = (6\pi\alpha^2 e_q^{\,2} m_q^{\,1/2} E^{1/2}/W^2)\,|\psi_f(0,E)|^2\,. \tag{24}$$

Here σ_{free} is the nonrelativistic cross section for the production of a free $q\bar{q}$ pair calculated in Born approximation, $e^+e^- \to \gamma \to q\bar{q}$, and $\psi_f(0,E)$ is the free wave function at the orign. As emphasized in paper I, $\psi_f(0,E)$ should include the effects of any short-range nonconfining interaction between the quarks, e.g., the color Coulomb interaction.

Of course, production of free $q\bar{q}$ pairs is not allowed physically because of color confinement, but if it were, the $q\bar{q}$ channel would be the open

"$\beta=1$" channel. The physical picture is that the $q\bar{q}$ pair produced at the origin in channel 1 acts free for a short time, until it encounters the confining potential. We showed in paper I that the energy average of σ_{tree} over a sufficiently broad energy range picks out this short-time behavior. Since the quarks are confined, the physical cross section σ describes the production of hadrons in the open channels 2, 3, ... The duality relation, therefore, states that the short-time behavior in all open channels taken together is the same as that in the closed channel.[20] This is certainly plausible for weak coupling of channel 1 to the open channels, since the decay of the $q\bar{q}$ system to hadrons will be slow; but we will show it in general.

Our method is a generalization of that used to derive duality for single-channel processes in paper I. We define a multichannel Feynman propagator $K_{ij}(\vec{r}\,',\vec{r},t)$ as

$$K_{ij}(\vec{r}\,',\vec{r},t)=\sum_\beta \psi_{\beta i}(\vec{r}\,')e^{-iE_\beta t}\,\psi^*_{\beta j}(\vec{r})\,. \tag{25}$$

This is a solution of the matrix Schrödinger equation

$$H(\vec{r}\,')K(\vec{r}\,',\vec{r},t)=i\frac{\partial}{\partial t}K(\vec{r}\,',\vec{r},t)\,, \tag{26}$$

with H given, e.g., by Eq. (3), $\hbar=c=1$, and with the boundary condition $K_{ij}(\vec{r}\,',\vec{r},0)=\delta_{ij}\delta(\vec{r}\,'-\vec{r})$. The cross section can be written in terms of the Fourier transform of K_{11},

$$\sigma(E)=(12\pi^2\alpha^2 e_q{}^2/m_q{}^2W^2)\tilde{K}_{11}(0,0,E)\,, \tag{27}$$

where

$$\tilde{K}_{11}(0,0,E)=\int_{-\infty}^{\infty}dt\,e^{iEt}K_{11}(0,0,t)$$
$$=2\pi\sum_\beta \rho_\beta(E)\,|\psi_{\beta 1}(0,E)|^2\,. \tag{28}$$

We average $\tilde{K}$ over energy by convoluting with a smooth function $f(E'-E)$ and define

$$\langle\tilde{K}_{11}(E)\rangle=\int_{-\infty}^{\infty}dE'f(E'-E)\tilde{K}_{11}(0,0,E')\,. \tag{29}$$

From the convolution theorem for Fourier transforms, this becomes

$$\langle\tilde{K}_{11}(E)\rangle=\int_{-\infty}^{\infty}dt\,e^{iEt}K_{11}(0,0,t)\tilde{f}(t)\,. \tag{30}$$

When $\tilde{f}(t)$ is sharply peaked around $t=0$, $\langle\tilde{K}_{11}(E)\rangle$ depends only on the short-time behavior of the propagator [this corresponds to a choice of $f(E'-E)$ which is broad and smooth]. We will relate the short-time propagator $K_{11}(0,0,t)$ to the short-time single-channel propagator $K(0,0,t)$ discussed in paper I. The duality relation then follows from the results of paper I.

We first divide the matrix H into its diagonal part H^d and the off-diagonal coupling matrix V^c,

$$H=H^d+V^c,\quad (V^c)^T=V^c\,. \tag{31}$$

We assume for simplicity that V^c is finite at $r=0$. The complete set of eigenstates ϕ_{jn} for each element H_{jj} of H^d satisfies

$$H_{jj}\phi_{jn}=E_{jn}\phi_{jn}\,, \tag{32}$$

$$\sum_n \phi_{jn}(\vec{r}\,')\phi^*_{jn}(\vec{r})=\delta(\vec{r}\,'-\vec{r})\,. \tag{33}$$

$K(\vec{r}\,',\vec{r},t)$ is given in matrix form by

$$K(\vec{r}\,',\vec{r},t)=e^{-iHt}\begin{pmatrix}\delta(\vec{r}\,'-r) & 0 & 0 & \cdots\\ 0 & \delta(\vec{r}\,'-\vec{r}) & 0 & \cdots\\ \cdots & & & \end{pmatrix} \tag{34}$$

and can be expressed in terms of H^d, V^c, E_{jn}, ϕ_{jn} as

$$K(\vec{r}\,',\vec{r},t)=e^{-iV^ct}(e^{iV^ct}e^{-iHt}e^{iH^dt})\begin{pmatrix}\sum_n\phi_{1n}(\vec{r}\,')e^{-iE_{1n}t}\phi^*_{1n}(\vec{r}) & 0 & \cdots & \\ 0 & \sum_n\phi_{2n}(\vec{r}\,')e^{-iE_{2n}t}\phi^*_{2n}(\vec{r}) & 0 & \cdots\\ \cdots & & & \end{pmatrix}$$
$$=e^{-iV^ct}\{1-\tfrac{1}{2}t^2[H^d,V^c]+\cdots\}K^d(\vec{r}\,',\vec{r},t)\,. \tag{35}$$

We have introduced the notation K^d for the diagonal matrix of propagators for the single-channel problem with Hamiltonians H_{jj}.

The only component of the matrix K which we need for Eq. (29) is $K_{11}(0,0,t)$, which turns out to be strikingly simple. To order t^2,

$$\langle\tilde{K}_{11}(E)\rangle=\int_{-\infty}^{\infty}dt\,e^{iEt}\tilde{f}(t)\Big[K^d_{11}(0,0,t)-\tfrac{1}{2}t^2\sum_{i>1}V^c_{1i}(0)^2K^d_{11}(0,0,t)+\cdots\Big]$$
$$=\langle\tilde{K}^d_{11}(E)\rangle+O(t^2)\,. \tag{36}$$

Thus K_{11} is expressed for small times in terms of the single-channel $q\bar{q}$ bound-state propagator K^d_{11}, hence in terms of the bound-state wave functions $|\phi_{1n}(0)|^2$. By Eq. (27), we know that $\langle\sigma(E)\rangle$ is related directly to $\langle\tilde{K}_{11}(E)\rangle$. The first term in Eq. (36) is related to the cross section for the production of the $q\bar{q}$ bound states in a single-channel model by[11]

$$\langle\sigma_{\text{bound}}(E)\rangle = (12\pi^2\alpha^2 e_q^{\,2}/m_q^{\,2}W^2)\langle\tilde{K}^d_{11}(E)\rangle$$

$$= \frac{24\pi^3\alpha^2 e_q^{\,2}}{m_q^{\,2}W^2}\sum_n |\phi_{1n}(0)|^2 f(E_{1n}-E)\,. \quad (37)$$

We now have that

$$\langle\sigma(E)\rangle = \langle\sigma_{\text{bound}}(E)\rangle + \text{corrections}\,, \quad (38)$$

where the (calculable) corrections can be made small by choosing $\tilde{f}(t)$ to be sharply peaked around $t=0$, that is, by choosing $f(E'-E)$ to be broad and smooth. If $f(E'-E)$ has a characteristic width Δ which is large compared to the spacing between levels, the corrections are of the order of $\sum_i V_{1i}^{\,2}(0)/\Delta^2$.

Equation (38) is a new duality relation for the coupled-channel problem. In paper I we showed that $\langle\sigma_{\text{bound}}(E)\rangle$ for the single-channel case can be related by duality to $\langle\sigma_{\text{free}}(E)\rangle$, the average of a free $q\bar{q}$ cross section which includes the effects of any short-range interactions between quarks, e.g., the color Coulomb interaction, but not the effects of the long-range confining potential. The duality relation

$$\langle\sigma_{\text{bound}}(E)\rangle \simeq \langle\sigma_{\text{free}}(E)\rangle \quad (39)$$

again includes calculable correction terms. Combining Eqs. (38) and (39), we conclude that

$$\langle\sigma(E)\rangle \simeq \langle\sigma_{\text{free}}(E)\rangle\,. \quad (40)$$

Equivalently,

$$\langle\tilde{K}_{11}(E)\rangle_{\text{exact}} \simeq \langle\tilde{K}_{11}(E)\rangle_{\text{bound}}$$

$$\simeq \langle\tilde{K}_{11}(E)\rangle_{\text{free}}\,, \quad (41)$$

where $\tilde{K}_{\text{bound}} \equiv \tilde{K}^d$ and

$$\langle\tilde{K}_{11}(E)\rangle_{\text{free}} = \frac{m_1^{\,3/2}}{2\pi}\int_0^\infty dE'f(E'-E)\sqrt{E'}\,. \quad (42)$$

The correction terms for Eq. (41) are given in Eq. (36) and paper I.

Although our derivation of the multichannel duality relation was restricted to coupled two-body channels and nonrelativistic kinematics, the result is clearly quite general and should hold for many-particle channels and relativistic interactions. We intend to discuss these extensions in a future paper.

IV. APPLICATION AND COMMENTS

We now apply the duality relations to the simple two-channel model considered in Sec. II. When the coupling term V is small, the widths of the resonances Γ_n and the energy shifts E_n are small by Eqs. (16) and (17). The Breit-Wigner factors in Eq. (19) then act approximately as δ functions when integrated with a smooth, broad function $f(E'-E)$. In this case duality clearly works well, and $\langle\sigma(E)\rangle \simeq \langle\sigma_{\text{bound}}(E)\rangle$ by inspection.

When the coupling term V is large we must test duality numerically. We choose the Gaussian

$$f(E'-E) = \frac{1}{(2\pi\Delta^2)^{1/2}}\,e^{-(E'-E)^2/2\Delta^2} \quad (43)$$

for our smearing function, so that

$$\tilde{f}(t) = e^{-\Delta^2 t^2/2}\,. \quad (44)$$

The cross section $\sigma(E)$ and $\tilde{K}_{11}(0,0,E)$ are related by Eq. (27). $\tilde{K}_{11}(0,0,E)$ is expressed in terms of the bound-state wave function $|\psi_1(0,E_1)|^2$ of Table I and $|\psi_1(0,E)|^2$ of Fig. 1,

$$\tilde{K}_{11}(0,0,E) = 2\pi|\psi_1(0,E_1)|^2\delta(E-E_1)$$

$$+\frac{m_2^{\,3/2}}{2\pi}|\psi_1(0,E)|^2\sqrt{E_2}\,\theta(E_2)\,, \quad (45)$$

where $E_2 = E-(m_2-m_1)$. We smear $\tilde{K}_{11}(0,0,E)$ to get $\langle\tilde{K}_{11}\rangle_{\text{exact}}$, using the Gaussian of Eqs. (43) and (44) with $\Delta = 0.6$ GeV, or a full width at half maximum of 1.4 GeV (slightly larger than the separation between the first two resonances in Fig. 1 and equal to the separation between the next two resonances).

In Table II we compare $\langle\tilde{K}_{11}\rangle_{\text{exact}}$ to the quantities to which it is related by duality. The first comparison is by Eq. (36) to $\langle\tilde{K}_{11}\rangle_{\text{bound}}$ plus a correction. For $V=0.2$ and small E, this correction gets as large as 3%. Otherwise, it is quite small so we have only displayed the corrected $\langle\tilde{K}_{11}\rangle_{\text{bound}}$ in the table. The corrected $\langle\tilde{K}_{11}\rangle_{\text{free}}$ is obtained by converting the sum in Eq. (37) to an integral. The first term is $\langle\tilde{K}_{11}\rangle_{\text{free}}$ in Eq. (42) and again we have not displayed the (very small) corrections. It can be seen from the table that the exact results oscillate around the smooth free result, which is a consequence of the somewhat narrow Gaussian used for smearing.

The agreement between columns in Table II is within 1.5% which we again emphasize is a striking result, given the complexity of the coupled-channel cross section. The agreement would be even better for potentials more realistic than a square well, e.g., the linear potential, since the density of resonance states would increase with increasing energy and eliminate the ripple in $\langle\tilde{K}\rangle_{\text{exact}}$. On the

TABLE II. Verification of the *corrected* duality $\langle\tilde{K}_{11}\rangle_{\text{exact}} \simeq \langle\tilde{K}_{11}\rangle_{\text{bound}} \simeq \langle\tilde{K}_{11}\rangle_{\text{free}}$ for the two-channel model of Sec. II with three values of the coupling V. The cross section $\sigma(E)$ is related to $K_{11}(E)$ by Eq. (27). The operation $\langle\ \rangle$ was taken to be convolution with the Gaussian smearing function $f(E' - E)$ given by Eq. (43) with $\Delta = 0.6$ GeV. $\tilde{K}_{11\,\text{exact}}$ was calculated using Eqs. (13) and (20). $\tilde{K}_{11\,\text{bound}}$ was calculated from Eqs. (36) and (37), with bound states $|\phi_{1n}(0)|^2 = (\pi n^2/2R^3)$. $\tilde{K}_{11\,\text{free}}$ was calculated from Eq. (37) by converting the sums to integrals. The parameters used were $m_1 = m_q = 1.5$ GeV, $m_2 = 1.8$ GeV, and $R = 5.736$ GeV^{-1}.

E (GeV)	$\langle\tilde{K}_{11}\rangle_{\text{exact}}$	$\langle\tilde{K}_{11}\rangle^{\text{corr}}_{\text{bound}}$ (GeV^3) $V=0.10$ GeV	$\langle\tilde{K}_{22}\rangle^{\text{corr}}_{\text{free}}$	$\langle\tilde{K}_{11}\rangle_{\text{exact}}$	$\langle\tilde{K}_{11}\rangle^{\text{corr}}_{\text{bound}}$ (GeV^3) $V=0.15$ GeV	$\langle\tilde{K}_{11}\rangle^{\text{corr}}_{\text{free}}$	$\langle\tilde{K}_{11}\rangle_{\text{exact}}$	$\langle\tilde{K}_{11}\rangle^{\text{corr}}_{\text{bound}}$ (GeV^3) $V=0.20$ GeV	$\langle\tilde{K}_{11}\rangle^{\text{corr}}_{\text{free}}$
0	0.0943	0.0942	0.0937	0.0954	0.0949	0.0946	0.0968	0.0959	0.0957
0.4	0.1651	0.1649	0.1656	0.1654	0.1649	0.1657	0.1659	0.1649	0.1659
0.8	0.2384	0.2384	0.2408	0.2382	0.2385	0.2404	0.2381	0.2385	0.2398
1.2	0.3116	0.3117	0.3069	0.3111	0.3112	0.3064	0.3102	0.3106	0.3058
1.6	0.3705	0.3698	0.3617	0.3694	0.3679	0.3614	0.3678	0.3652	0.3609
2.0	0.3909	0.3909	0.4081	0.3912	0.3912	0.4079	0.3915	0.3918	0.4076
2.4	0.4239	0.4255	0.4491	0.4254	0.4290	0.4489	0.4273	0.4340	0.4487
2.8	0.5240	0.5239	0.4862	0.5225	0.5223	0.4861	0.5204	0.5200	0.4860

other hand, as discussed in paper I, the corrections to naive duality can be large for the more realistic potentials and must not be ignored.

We conclude from our derivations and illustrations in this paper and paper I that corrected duality gives a very reliable and useful method for comparing calculated (QCD) cross sections to experimental data. However, the interpretation of Eqs. (39)–(41) involves some subtlety. The observed cross section is σ and the calculated cross section is σ_{free}. To relate $\langle\sigma\rangle$ to $\langle\sigma_{\text{free}}\rangle$ by duality, we introduced relations $\langle\sigma\rangle \simeq \langle\sigma_{\text{bound}}\rangle$ and $\langle\sigma_{\text{bound}}\rangle \simeq \langle\sigma_{\text{free}}\rangle$. The cross section σ_{bound} is to be calculated using the confining potential in the $q\bar{q}$ channel only. It is probably adequate for the purpose of calculating corrections to the duality relation to identify that potential with one of the single-channel potentials used to fit the $q\bar{q}$ data.[21] However, the poles in σ_{bound} do not correspond precisely to the resonances or bound states of the physical cross section, because of the sizable mass shifts caused by coupling to other channels (see our Table I, and Sec. IV and Table VII of the second paper, Ref. 17). These mass shifts call into question results on $q\bar{q}$ potentials derived from single-channel fits to the Ψ and Υ data, and those results should be regarded with caution.

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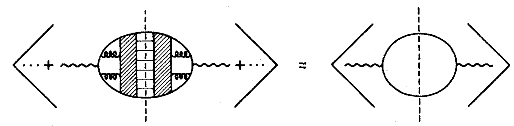
[18]We will consider the coupling to only one quark flavor, and will neglect, e.g., mixing of $c\bar{c}$ and $b\bar{b}$ states. This mixing proceeds through exchange of three gluons, and is expected to be small.

[18a]*Note added in proof.* A rigorous discussion of the scattering problem for an open channel coupled to a closed confining channel was given by R. F. Dashen, J. B. Healy, and I. J. Muzinich, Phys. Rev. D 14, 2773 (1976). We would like to thank Dr. Muzinich for pointing out this reference and its relevance to the present work. See also I. J. Muzinich, in *A Festschrift for Maurice Goldhaber* (New York Academy of Sciences, New York, 1980).

[19]This can be regarded as a simplified nonrelativistic bag model.

[20]For our colleagues who think diagrammatically, we note that this statement of the duality relation relates a sum of loops with arbitrary decorations to a plain loop:



+ SHORT RANGE CORRECTIONS

The correction terms do not depend on long-range effects, in particular, the confining interaction.

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