Brief Reports

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Relation between the magnetic moments of the states on a Regge trajectory

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(Received 6 October 1980)

For a classical model, which describes a single Regge trajectory, we calculate a relation which exists between the magnetic moments of the various states on the trajectory. The calculation is analogous to a similar calculation of the electron *g* factor.

I. INTRODUCTION AND SUMMARY

Recently we introduced[1] a classical Lagrangian model for a spinning relativistic particle. The states of the quantized form of this model form a Regge sequence with each value of spin, integer as well as half integer, occurring once. The model allows a minimal interaction with the electromagnetic field also in its quantized form. In analogy to what happens for the Dirac equation, this minimal interaction generates magnetic moments (g factor $\neq 0$) for the various states of the Regge trajectory. It is the purpose of the present paper to calculate these magnetic moments for the classical version of our model. Our model differs from somewhat similar models proposed by Hanson and Regge.[2] One of the differences is that minimal electromagnetic coupling cannot generate magnetic moments in the models of Ref. 2.

The basic variables of our classical model are a four-vector $x(s)$, which describes events along the world line of the particle parametrized by s, and an internal (translation invariant) spinor $Q(s)$ consisting of a pair of harmonic oscillators. The Lagrangian, including minimal interaction with the electromagnetic field $A_\mu(x)$, is

$$L=\tfrac{1}{2}\dot{Q}^T\gamma^0 Q+(-\dot{x}^2)^{1/2}f\left(\frac{\dot{x}\cdot V}{(-\dot{x}^2)^{1/2}}\right)+eA_\mu(x)\dot{x}^\mu\,, \qquad (1)$$

where the dots denote differentiation with respect to s and where

$$\gamma^0=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{bmatrix},\quad Q=\begin{pmatrix}q_1\\q_2\\q_3\\q_4\end{pmatrix}=\begin{pmatrix}q_1\\q_2\\p_1\\p_2\end{pmatrix}, \qquad (2)$$

with the Poisson brackets $\{q_\alpha,q_\beta\}=\gamma^0{}_{\alpha\beta}$. Symmetric bilinears in Q are defined by

$$V^\mu=\tfrac{1}{2}Q^T\gamma^0\gamma^\mu Q\,,\quad V^0=\tfrac{1}{2}(p_1{}^2+p_2{}^2+q_1{}^2+q_2{}^2)\,,$$
$$S^{\mu\nu}=\tfrac{1}{8}Q^T\gamma^0(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)Q\,; \qquad (3)$$

here the γ^μ are a real (Majorana) representation of the Dirac matrices with γ^0 given in (2). The spinor structure (2), (3) is a classical version[3] of a quantum-mechanical structure associated with the Majorana equation and used by Dirac.[4] One also has the Poisson brackets

$$\{V^\mu,Q\}=-\tfrac{1}{2}\gamma^\mu Q\,,$$
$$\{S^{\mu\nu},Q\}=-\tfrac{1}{4}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)Q\,, \qquad (4)$$

and the classical identities

$$V_\mu V^\mu=S_{\mu\nu}S^{\mu\nu}=S_{\mu\nu}V^\nu=\epsilon^{\mu\nu\rho\lambda}S_{\mu\nu}S_{\rho\lambda}=\epsilon^{\mu\nu\rho\lambda}S_{\mu\nu}V_\rho=0\,. \qquad (5)$$

The Lagrangian (1) is invariant for chronometric transformations that replace s by a function of itself. (More exactly, the action is unchanged.) This chronometric invariance implies a constraint

$$\phi=\Pi^2-\alpha(\Pi\cdot V)\simeq 0\,, \qquad (6)$$

where V is given by (3), $\Pi_\mu = P_\mu - eA_\mu$, and $P_\mu = \partial L/\partial \dot{x}^\mu$. The function α in (6) is uniquely determined by the arbitrary function f in the Lagrangian (1). For the free system, the constraint (6) is a Regge relation as $P \cdot V$ is proportional to the spin, whereas $-P^2$ is the square of the mass. For details see Ref. 1.

For quantization it is important that there is just one constraint in the theory, which makes it first class; thus the Dirac brackets are identical with the Poisson brackets, unlike what happens in the model of Ref. 2. For the classical model, by imposing conditions on the function α in (6), equivalently on f in (1), one can ensure that both P and $\dot{x}$ are timelike.

For the field-free case one can explicitly solve the equations of motion generated by ϕ.[1] Both P_μ and $P \cdot V$ are conserved. In space-time the general solution is a helix, P giving the direction of the screw and the (timelike) world line following its thread. The total angular momentum is constant, it consists of the orbital part $L_{\mu\nu}$ and the spin $S_{\mu\nu}$ given in (3); neither of these is separately conserved. Writing $M = \sqrt{-P^2}$ and choosing s in a natural way, we present from Ref. 1 that part of the solution we need in the following:

$$x^\mu(s+\Delta s) = x^\mu(s) + \left[\left(\frac{2}{\alpha'(z)} + \frac{z}{M^2}\right)\Delta s - \frac{z}{M^2}\sin\Delta s\right]\frac{P^\mu}{M} - \sin\Delta s\,\frac{V^\mu(s)}{M} - \frac{(1-\cos\Delta s)}{M^2}S^{\mu\rho}(s)P_\rho\,,$$

$$V^\mu(s+\Delta s) = V^\mu(s)\cos\Delta s - \frac{z}{M^2}(1-\cos\Delta s)P^\mu + \sin\Delta s\, S^{\mu\rho}(s)\frac{P_\rho}{M}\,, \tag{7}$$

$$S^{\mu\nu}(s+\Delta s) = S^{\mu\nu}(s) - \frac{1}{M}[P^\mu V^\nu(s) - P^\nu V^\mu(s)]\sin\Delta s - \frac{1}{M^2}[P^\mu S^\nu{}_\rho(s) - P^\nu S^\mu{}_\rho(s)]P^\rho(1-\cos\Delta s)\,.$$

(Δs need not be infinitesimal or small.) Measured in units of proper time, one finds the frequency of the helical motion to be MS^{-1} and the radius of the helix to be $M^{-1}S$, where S is the average spin in units of $\hbar$. One may say that the internal spinor variables keep the particle away from its average position and force the helical motion. Such motion generates a magnetic moment, the calculation of which is the subject of Sec. III. Although this calculation is completely classical, there is a close analogy with what happens for the Dirac equation. To exhibit this, we present in Sec. II a calculation of the g factor for the electron in a somewhat unusual way, the role of the helical motion being played by the *Zitterbewegung*. Our result for the present model is that the g factor is determined by the function f in (1). Let the Regge relation implied by this same f be $M = M_0\beta(S)$, where S is again average spin/$\hbar$, then our final result is

$$g(S) = \frac{d\ln[\beta(S)]}{d\ln S}\,. \tag{8}$$

Hence if we assume $\beta(S) = S^n$, then $g(S) = n$, i.e., for the usual assumption $n = \frac{1}{2}$, $g(S) = \frac{1}{2}$ for all states of the Regge trajectory. This is in striking contrast to the well-known result $g(S) = 1/S$ obtained from finite-component wave equations describing one spin value at a time.[5,6]

II. ALTERNATIVE CALCULATION OF ELECTRON g-FACTOR

Dirac's original derivation[7] of the electron g factor involves squaring the Hamiltonian, and cannot be adapted to our model. We give here an alternative approach as preparation for our work in Sec. III.

We use the Heisenberg picture and set $\hbar = c = 1$. In the presence of a homogeneous external magnetic field $\vec{B}$, the operator equations of motion are generated by the Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm int}\,, \quad \mathcal{H}_0 = \vec{\alpha}\cdot\vec{p} + \beta m\,,$$
$$\mathcal{H}_{\rm int} = -e\vec{\alpha}\cdot\vec{A}(\vec{x})\,, \quad \vec{A}(x) = \tfrac{1}{2}\vec{B}\times\vec{x}\,. \tag{9}$$

The operators $\vec{\alpha}$, $\vec{x}$, ... appearing here are of course time dependent. In its present form $\mathcal{H}_{\rm int}$ does not yet involve terms such as $\vec{L}\cdot\vec{B}$ and $\vec{S}\cdot\vec{B}$, where L is the orbital angular momentum and the spin $\vec{S}$ is

$$\vec{S} = \tfrac{1}{2}\vec{\alpha} = -\tfrac{1}{4}i\vec{\alpha}\times\vec{\alpha}\,. \tag{10}$$

We shall now show that if one averages over the oscillatory motion (*Zitterbewegung*) of the electron, and takes the low-momentum limit, $\mathcal{H}_{\rm int}$ is actually of the physically expected form.

Working to lowest order in eB, we may assume that in

$$\mathcal{H}_{\rm int} = \tfrac{1}{2}e\vec{B}\cdot\vec{\alpha}(t)\times\vec{x}(t) \tag{11}$$

the time dependences in $\vec{\alpha}$ and $\vec{x}$ correspond to free motion due to $\mathcal{H}_0$. Following Schrödinger and Dirac,[6] this motion is explicitly known: $\vec{p}$ and $\mathcal{H}_0$ are constant, while $\vec{\alpha}$ and $\vec{x}$, at any two times t, $t+\Delta t$ are connected by

$$\vec{\alpha}(t+\Delta t) = \frac{\vec{p}}{\mathcal{H}_0} + \left[\vec{\alpha}(t) - \frac{\vec{p}}{\mathcal{H}_0}\right]e^{-2i\mathcal{H}_0\Delta t} = \frac{\vec{p}}{\mathcal{H}_0} + e^{2i\mathcal{H}_0\Delta t}\left[\vec{\alpha}(t) - \frac{\vec{p}}{\mathcal{H}_0}\right],$$

$$\vec{x}(t+\Delta t)=\vec{x}(t)+\frac{\vec{p}}{\mathcal{H}_0}\Delta t + \frac{e^{2i\mathcal{H}_0\Delta t}-1}{2i\mathcal{H}_0}\left[\vec{\alpha}(t)-\frac{\vec{p}}{\mathcal{H}_0}\right] = \vec{x}(t)+\frac{\vec{p}}{\mathcal{H}_0}\Delta t + \left[\vec{\alpha}(t)-\frac{\vec{p}}{\mathcal{H}_0}\right]\frac{1-e^{-2i\mathcal{H}_0\Delta t}}{2i\mathcal{H}_0}\,. \quad (12)$$

Superposed on the uniform motion is the high-frequency *Zitterbewegung*. This makes $\mathcal{H}_{\text{int}}$ in (11) a rapidly fluctuating function of time. We define a short-term average of $\vec{\alpha}(t)\times\vec{x}(t)$ as the average of $\vec{\alpha}(t+\Delta t)\times\vec{x}(t+\Delta t)$ with respect to Δt over many cycles of the *Zitterbewegung* centered about t: In this process, terms $e^{\pm 2i\mathcal{H}_0\Delta t}$ and $\Delta t\, e^{\pm 2i\mathcal{H}_0\Delta t}$ are discarded. The result will of course depend on t, but the high-frequency terms will have been removed. Denoting this average by angular brackets, we have after a simple calculation,

$$\langle\mathcal{H}_{\text{int}}\rangle=\frac{e}{2}\vec{B}\cdot\langle\vec{\alpha}(t)\times\vec{x}(t)\rangle = -\frac{e}{2}\vec{B}\cdot\left(\frac{1}{\mathcal{H}_0}[\vec{L}(t)+2\vec{S}(t)]+\frac{i}{2\mathcal{H}_0}\vec{\alpha}(t)\times\vec{p}\right). \quad (13)$$

For a slowly moving electron, the matrix elements of $\vec{\alpha}(t)$ between positive-energy states are proportional to $\vec{p}$, as is clear from

$$\vec{\alpha}\mathcal{H}_0+\mathcal{H}_0\vec{\alpha}=2\vec{p}\,. \quad (14)$$

In that limit, we may drop the term $\vec{\alpha}\times\vec{p}$ as being quadratic in $\vec{p}$, and also replace $\mathcal{H}_0$ by m. This yields

$$\langle\mathcal{H}_{\text{int}}\rangle=-\frac{e}{2m}\vec{B}\cdot(\vec{L}+2\vec{S})\,, \quad (15)$$

which shows that the g factor for the Dirac electron is two, up to the approximation made.

III. MAGNETIC MOMENT CALCULATION

The calculation for the model of Sec. I, although classical and not quantum mechanical, is closely analogous to the calculation of the g factor for the electron as given in Sec. II.

As described in Sec. I, for the free system we have a relation between mass and spin: $P^2=\alpha(P\cdot V)$. This gives an implicit equation for the classical Hamiltonian $\mathcal{H}=P^0$, which is conjugate to the time x^0. Introducing $u=\vec{P}\cdot\vec{P}$, $v=\vec{P}\cdot\vec{V}$, $w=V_0$, (6) reduces to the implicit equation

$$u-\mathcal{H}^2=\alpha(v+\mathcal{H}w)\,; \quad (16)$$

we write the solution as

$$\mathcal{H}=\mathcal{H}_0(u,v,w)\,. \quad (17)$$

An external electromagnetic field interacts with our system via minimal interaction,[1] i.e., $\vec{P}\to\vec{\Pi}=\vec{P}-e\vec{A}$, $P_0\to\Pi_0=P_0-eA_0$. This changes the Hamiltonian (17); assuming the external field to be weak it will suffice to restrict oneself to the first-order (in e) correction to the right-hand side of (17). To find the gyromagnetic factors it will suffice to consider, in analogy to Sec. II, a constant magnetic field: $\vec{A}=-\frac{1}{2}\vec{x}\times\vec{B}$, $A_0=0$, i.e., in the implicit equation (16) only u and v change. Form $d\mathcal{H}_0=(du-\alpha' dv-\mathcal{H}_0\alpha' dw)(2\mathcal{H}_0+w\alpha')^{-1}$, i.e.,

$$\begin{pmatrix}\frac{\partial\mathcal{H}_0}{\partial u}\\ \frac{\partial\mathcal{H}_0}{\partial v}\\ \frac{\partial\mathcal{H}_0}{\partial w}\end{pmatrix}=\frac{1}{2\mathcal{H}_0+w\alpha'}\begin{pmatrix}1\\ -\alpha'\\ -\mathcal{H}_0\alpha'\end{pmatrix}. \quad (18)$$

Hence we have in lowest order in e

$$\mathcal{H}_{\text{total}}=\mathcal{H}_0(\vec{\Pi}\cdot\vec{\Pi},\vec{\Pi}\cdot\vec{V},V_0)=\mathcal{H}_0(u,v,w)+\mathcal{H}_{\text{int}}\,, \quad (19)$$

with

$$\mathcal{H}_{\text{int}}=\frac{-e}{2\mathcal{H}_0+w\alpha'}(\vec{L}-\tfrac{1}{2}\alpha'\vec{x}\times\vec{V})\cdot\vec{B}\,. \quad (20)$$

The variables here are functions of the parameter s.

We next follow the method of Sec. II for handling the unfamiliar term $\vec{x}\times\vec{V}$, and replace it by its short-term average. As we are only working to lowest order in e, we may again compute this average on the basis of the free motion, which is described by (7). Thus we form the product $\vec{x}(s+\Delta s)\times\vec{V}(s+\Delta s)$ from (7), and drop all oscillatory terms such as $\sin\Delta s$, $\cos\Delta s$, $\sin\Delta s\cos\Delta s$, $\Delta s\sin\Delta s$, $\Delta s\cos\Delta s$, and obtain a result which of course depends on s:

$$\langle\vec{x}(s)\times\vec{V}(s)\rangle=-\frac{z}{M^2}\vec{x}(s)\times\vec{P}-\frac{2z}{M^4}\vec{P}\times[\vec{S}(s)\cdot P]-\frac{\vec{V}(s)}{M^2}\times[\vec{S}(s)\cdot P]\,. \quad (21)$$

(Here $z=P\cdot V$ and the three-vector $[\vec{S}(s)\cdot P]$ has components $S_{i\mu}P^{\mu}$.) The last term here can be simplified using

$$V_\lambda S_{\mu\nu}+V_\mu S_{\nu\lambda}+V_\nu S_{\lambda\mu}=0$$

from (5). We then drop terms quadratic in $\vec{P}$, as well as terms linear in S_{oj} because in the quantized version in the Majorana representation of $S_{\mu\nu}$, the S_{oj} have vanishing $\Delta j=0$ matrix elements. Then we find

$$\langle \vec{x}(s) \times \vec{V}(s) \rangle \approx -\frac{z}{M^2}[\vec{L}(s) - \vec{S}(s)], \tag{22}$$

and

$$\langle \mathcal{H}_{\text{int}} \rangle \approx \frac{-e}{2\mathcal{H}_0 + w\alpha'}\left[\left(1+\frac{z\alpha'}{2M^2}\right)\vec{L}(s) - \frac{z\alpha'}{2M^2}\vec{S}(s)\right]\cdot\vec{B}. \tag{23}$$

Considering the limit $\vec{P} \to 0$, one has

$$\begin{aligned}(2\mathcal{H}_0+w\alpha')^{-1}\left(1+\frac{z\alpha'}{2M^2}\right) &\to (2\mathcal{H}_0+V_0\alpha')^{-1}\left(1+\alpha'\frac{\mathcal{H}_0V_0}{2\mathcal{H}_0^2}\right) = \frac{1}{2\mathcal{H}_0},\\ -(2\mathcal{H}_0+w\alpha')^{-1}\frac{z\alpha'}{2M^2} &\to -(2\mathcal{H}_0+V_0\alpha')^{-1}\frac{\alpha' V_0}{2\mathcal{H}_0} = \frac{V_0}{2\mathcal{H}_0^2}\frac{\partial\mathcal{H}_0}{\partial V_0}\end{aligned} \tag{24}$$

from (18). For $\langle \mathcal{H}_{\text{int}} \rangle$ one finds the result

$$\langle \mathcal{H}_{\text{int}} \rangle \approx \frac{-e}{2\mathcal{H}_0}\left(\vec{L}(s) + \frac{V_0}{\mathcal{H}_0}\frac{\partial\mathcal{H}_0}{\partial V_0}\vec{S}(s)\right)\cdot\vec{B}, \tag{25}$$

i.e., the gyromagnetic factor is given by

$$g = \frac{\partial \ln\mathcal{H}_0}{\partial \ln V_0}\bigg|_{u=v=0}, \tag{26}$$

which is just (8).

ACKNOWLEDGMENT

This work was supported in part by the National Science Foundation and the Department of Energy.

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