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Consequences of the eikonal formula

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We analyze the implications of a model for high-energy hadron-hadron scattering in which the S matrix is given by an eikonal formula and is therefore explicitly unitary. The eikonal χ is a Hermitian operator which represents the lowest-order amplitudes for all reactions, elastic and inelastic. (Consequently, the matrix elements of $S = e^{i\chi}$ are not simply related to those of χ.) We derive a physical picture of high-energy scattering as a stochastic process in which quanta associated with harmonic oscillators at each point in a subspace of three-dimensional space are created or annihilated randomly. As the energy increases, the length of this subspace in the longitudinal direction expands and more harmonic oscillators become excited. The expectation values of $\chi^2, \chi^3, \chi^4, \ldots$ become very large, but the S matrix remains unitary. We derive a partial differential equation for a generating functional for the S matrix, through which we show that the target particle becomes completely absorptive as the energy of the projectile goes to infinity.

I. INTRODUCTION

The eikonal formula, originally known in optics, has been applied to many problems, whether relativistic or nonrelativistic, quantum mechanical or field theoretic, in the theory of scattering of high-energy particles. Over thirty years ago, Moliére[1] first extended the eikonal formula from optics to high-energy potential scattering in quantum mechanics. This formula can also be derived from the Dirac and Klein-Gordon equations; it has become one of the cornerstones of various optical models[2,3] for high-energy scattering of strongly interacting particles, and it describes multimeson exchange in quantum electrodynamics (QED) and other field theories.[4]

Consider the case of a fermion or a boson scattered by a potential $V(\vec{x})$. The eikonal formula is valid in the limit in which the energy is large and the scattering angle is small. In this limit the three-dimensional momentum transfer $\vec{\Delta}$ reduces to a two-dimensional vector perpendicular to the direction of the incident momentum, which we take to be along the z axis. Then the eikonal formula for the S matrix is

$$S(\vec{\Delta}) = \int d^2\vec{b}\, e^{-i\vec{\Delta}\cdot\vec{b}} S(\vec{b}) \qquad (1.1)$$

with

$$S(\vec{b}) = e^{i\chi(\vec{b})}, \qquad (1.2)$$

where

$$\chi(\vec{b}) = -\frac{1}{v}\int_{-\infty}^{\infty} dz\, V(\vec{b}+z\vec{e}_z), \qquad (1.3)$$

and where v is the particle's velocity. The magnitude of the two-dimensional vector $\vec{b}$ is the impact distance. Note that the S matrix, as given by (1.1) in momentum-transfer space, or by (1.2) in impact-distance space, is explicitly unitary. The eikonal $\chi(\vec{b})$ in (1.3) is simply the lowest order (Born) term of the scattering amplitude.

For the purpose of illustration, we will derive the eikonal formula for the potential scattering of a particle whose wave function satisfies the Klein-Gordon equation. The time-independent Klein-Gordon equation is

$$[E - V(\vec{x})]^2\psi(\vec{x}) + (\nabla^2 - m^2)\psi(\vec{x}) = 0. \qquad (1.4)$$

The boundary condition is specified by the incident wave

$$\psi_{\text{inc}}(\vec{x}) = e^{ipz}. \qquad (1.5)$$

In (1.4) and (1.5), E is the energy, $V(\vec{x})$ the potential, m the mass, and

$$p = (E^2 - m^2)^{1/2}.$$

We shall study the solution of (1.4) in the limit

$$E \to \infty.$$

Let us set

$$\psi(\vec{x}) = e^{ipz}\phi(\vec{b}, z), \qquad (1.6)$$

where, for reasons that will become obvious, we have explicitly separated the components of $\vec{x}$ into z and $\vec{b}$, where

$$\vec{b} = x\vec{e}_x + y\vec{e}_y. \qquad (1.7)$$

Substituting (1.6) into (1.4), we find that the resulting equation consists of terms of the order of $E^2\phi$, $E\phi$, and ϕ. The $E^2\phi$ terms cancel and neglecting terms of the order of ϕ, we obtain

$$\frac{\partial\phi(\vec{b},z)}{\partial z} \sim -iV(\vec{b},z)\phi(\vec{b},z), \qquad (1.8)$$

where $V(\vec{b},z)$ is just another notation for $V(\vec{x})$. From (1.8), we easily obtain

$$\phi(\vec{b},z)\sim \exp\left[-i\int_{-\infty}^{z}V(\vec{b},z')dz'\right]F(\vec{b}), \quad (1.9)$$

where F depends on $\vec{b}$ only. Substituting (1.9) into (1.6), we get

$$\psi(\vec{b},z)\sim e^{ipz}\exp\left[-i\int_{-\infty}^{z}V(\vec{b},z')dz'\right]F(\vec{b}). \quad (1.10)$$

To determine $F(\vec{b})$, we note that, as $z\to-\infty$, $\psi(\vec{b},z)$ should be equal to the incident wave e^{ipz}. Thus,

$$F(\vec{b})=1$$

and we have

$$\psi(\vec{b},z)\sim e^{ipz}\exp\left[-i\int_{-\infty}^{z}V(\vec{b},z')dz'\right]. \quad (1.11)$$

Equation (1.11) states that, as the incident waves passes through the potential, the wave function $\psi(\vec{b},z)$ derives an accumulated phase shift equal to

$$-\int_{-\infty}^{z}V(\vec{b},z')dz'.$$

When the wave arrives at $z=\infty$, the total phase shift accumulated is therefore equal to

$$\chi(\vec{b})=-\int_{-\infty}^{\infty}V(\vec{b},z')dz'. \quad (1.12)$$

This is consistent with (1.3) since the particle's velocity is essentially c, which in our units equals one. The S matrix in impact-distance space is equal to

$$S(\vec{b})=e^{i\chi(\vec{b})}. \quad (1.13)$$

Once the potential $V(\vec{x})$ is given, all physical quantities such as the differential or total cross sections are easily determined.

II. HIGH-ENERGY SCATTERING IN FIELD THEORIES

The above considerations apply only to potential scattering or to the generalization of potential scattering to field theory, where an arbitrary number of mesons are exchanged between the two scattered particles. These processes do not describe the very important phenomenon of particle creation and annihilation. The value of the eikonal formulas (1.1)–(1.3) for hadron-hadron scattering, where particle creation and annihilation play a vital role, is therefore at best phenomenological.

It is generally believed that hadron-hadron scattering is described by gauge field theories, in particular quantum chromodynamics (QCD). We have therefore investigated high-energy collisions in QED (Ref. 5) and a SU(2) Yang-Mills theory[6] and have found a generalization of the eikonal formula. In our studies, the S-matrix operator at high energies was found to be[5,6]

$$S(\vec{b},T)=e^{i\chi(\vec{b},T)}. \quad (2.1)$$

In (2.1), $\vec{b}$ again is a vector in the transverse plane, and T is the rapidity defined by

$$T=\frac{1}{2\pi}\ln s\gg 1, \quad (2.2)$$

where s is the square of the energy in the center-of-mass system, and where χ is the eikonal representing the lowest-order amplitudes in all elastic and inelastic reactions, with the propagators in the Yang-Mills case Reggeized.

Unlike the eikonal formula in potential scattering, to which it bears a formal resemblance, (2.1) is not yet in a form from which physical consequences can be readily extracted. Since it describes the creation and annihilation of particles occurring in hadron scattering, the eikonal χ is an operator. Thus (2.1) is an operator in the Fock space. As a result, the matrix elements of $e^{i\chi}$ are not simply related to those of χ, and the physical implications of the eikonal formula (2.1) remain to be deduced. In this paper, we turn our attention to this problem.

The expressions for the eikonal χ in QED and the Yang-Mills theory are fairly complicated. In the QED case, the complication is mostly due to the fact that a created electron must be accompanied by a created positron, while in the Yang-Mills case, the complication is due to the complexity of the vertex factors and the Reggeization of the propagators. As a first attempt to understand the consequences of eikonal formula we shall consider a simplified model in which a particle can be created singly with the vertex factor equal to the coupling constant g, and the propagator in the momentum space is simply $(\vec{q}_\perp^{\,2}+\lambda^2)^{-1}$, where $\vec{q}_\perp$ and λ are, respectively, the transverse momentum and the mass of the virtual particle. For this model, we have

$$\chi(\vec{b},T)=g^2K(b)+g^3\int d^2\vec{b}_1\int_0^T dT_1K(|\vec{b}-\vec{b}_1|)x(\vec{b}_1,T_1)K(|\vec{b}_1|)$$
$$+g^4\int d^2\vec{b}_1d^2\vec{b}_2\int_0^T dT_1\int_0^{T_1}dT_2K(|\vec{b}-\vec{b}_1|)x(\vec{b}_1,T_1)K(|\vec{b}_1-\vec{b}_2|)x(\vec{b}_2,T_2)K(|\vec{b}_2|)+\cdots. \quad (2.3)$$

In (2.3),

$$K(b)=K_0(\lambda b)/2\pi$$

is the Fourier transform of $(\vec{q}_\perp{}^2+\lambda^2)^{-1}$, and

$$x(\vec{b},T)=\frac{1}{\sqrt{2}}[a^\dagger(\vec{b},T)+a(\vec{b},T)]\,, \qquad (2.4)$$

where $a^\dagger(\vec{b},T)$ and $a(\vec{b},T)$ are, respectively, the creation and annihilation operators for a particle at transverse coordinate $\vec{b}$ with rapidity $2\pi T$ in the laboratory system. They satisfy the commutation relation

$$[a(\vec{b},T),a^\dagger(\vec{b}',T')]=\delta(T-T')\delta^2(\vec{b}-\vec{b}')\,. \qquad (2.5)$$

Diagrammatically, the first term in (2.3) is represented by the diagrams 1(a), 1(a′), 1(a″),..., etc., where the scattering is elastic an no particle is created or destroyed, while the second term in (2.5) is represented by the diagrams 1(b), 1(c), 1(b′), 1(c′),...,etc., where one particle is created or destroyed. Similarly, the nth term in (2.3) is represented by diagrams in which a total of n particles are either created or destroyed. Among these n particles, no two of them have the same rapidity. Indeed, the rapidities of the particles are ordered successively according to their vertical positions, with the particle at the top having the largest rapidity. This restriction on the rapidities is exhibited more precisely by formula (3.3) in Sec. III. We also note that these diagrams are of the lowest perturbative orders for the corresponding processes.

We shall, in this paper, restrict ourselves to the study of the elastic scattering amplitude. Thus we shall concentrate on the matrix element

$$S(\vec{b},T)=\langle 0|e^{i\chi(\vec{b},T)}|0\rangle\,, \qquad (2.6)$$

where $|0\rangle$ denotes a state containing only the two colliding particles, i.e., there are no created particles. Mathematically, $|0\rangle$ is defined by

$$a(\vec{b},T)|0\rangle=0\,,\quad \text{all } \vec{b} \text{ and all } T\,. \qquad (2.7)$$

We shall study the behavior of the matrix element (2.6) with χ given by (2.3).

III. INTEGRAL REPRESENTATION FOR THE S MATRIX

As an artifice to facilitate the calculation of $S(\vec{b},T)$, we shall divide the $\vec{b}$ and T spaces into small regions and approximate the integrals in (2.3) by sums. Thus we replace the b space by a two-dimensional lattice with the lattice constant (distance between two neighboring lattice points) d, and replace the T space by a one-dimensional lattice with the lattice constant ϵ. We shall use the indices $j,k,1,\ldots$ (two components implied) to denote the $\vec{b}$-space coordinate of a lattice point, and the indices $n_1,n_2,\ldots$ to denote its T-space coordinate. We shall also replace the creation and annihilation operators in (2.4) by $a_j^\dagger(n)$ and $a_j(n)$ which satisfy the commutation rule

$$[a_j(n),a_{j'}^\dagger(n')]=\delta_{jj'}\delta_{nn'}\,. \qquad (3.1)$$

Since we have the correspondence between Dirac δ functions and Kronecker δ functions,

$$\delta(T_n-T_{n'})\leftrightarrow\frac{1}{\epsilon}\delta_{nn'}\,,$$

$$\delta^{(2)}(\vec{b}_j-\vec{b}_{j'})\leftrightarrow\frac{1}{d^2}\delta_{jj'}\,,$$

thus (2.5) and (3.1) give

$$a(b_j,T_n)\leftrightarrow\frac{1}{(\epsilon d^2)^{1/2}}a_j(n)\,,$$
$$a^\dagger(b_j,T_n)\leftrightarrow\frac{1}{(\epsilon d^2)^{1/2}}a_j^\dagger(n)\,. \qquad (3.2)$$

Eventually, we shall let $\epsilon\to 0$ and $d\to 0$ and recover the problem we started out with.

With such replacements the eikonal of (2.3) becomes

$$\chi_j(N)=g^2K_j+g^3(d^2\epsilon)^{1/2}\sum_k\sum_{N>n_1>0}K_{j-k}x_k(n_1)K_k + g^4d^2\epsilon\sum_{k,l}\sum_{N>n_1>n_2>0}K_{j-k}x_k(n_1)K_{k-l}x_l(n_2)K_l+\cdots\,, \qquad (3.3)$$

where

$$N\epsilon=T\,, \qquad (3.4)$$

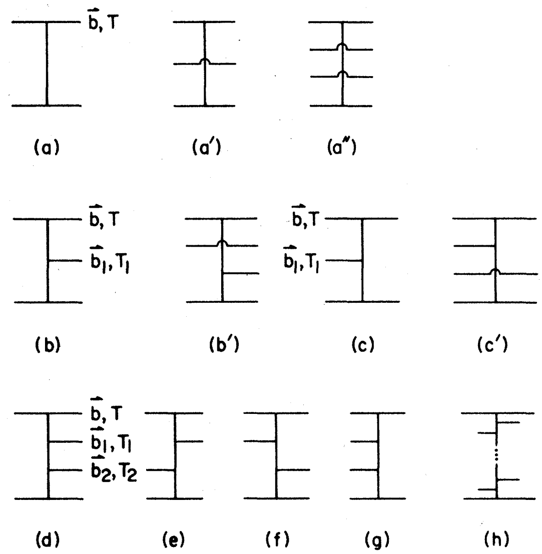

FIG. 1. The matrix elements of the eikonal operator. The impact distance and rapidity of each particle are indicated.

$$K_{j-k}=K(|\vec{b}_j-\vec{b}_k|)\,, \tag{3.5}$$

and

$$x_j(n)=\frac{1}{\sqrt{2}}[a_j^\dagger(n)+a_j(n)]\,. \tag{3.6}$$

Also, the S matrix in (2.6) becomes

$$S_j(N)=\langle 0|e^{i\chi_j(N)}|0\rangle\,. \tag{3.7}$$

It is possible to cast (3.3) into a more compact form. This is done by abbreviating the summation over $j,k,l,\ldots$ with the use of matrix notation. To do this, we define Λ and $x(n)$ as matrices in the $\vec{b}$ lattice with the elements

$$\Lambda_{jk}=\Lambda_{j-k}=\tfrac{1}{2}gd\,K_{j-k} \tag{3.8}$$

and

$$(x(n))_{jk}=x_j(n)\delta_{jk}\,. \tag{3.9}$$

We also define $\vec{\chi}(0)$ as the vector which has a component g^2K_j associated with each $\vec{b}$-lattice point:

$$(\chi(0))_j=g^2K_j\,. \tag{3.10}$$

Then (3.3) can be written as

$$\chi_j(N)=\Big[\chi(0)+2\sqrt{\epsilon}\sum_{N>n_1>0}\Lambda x(n_1)\chi(0)+4\epsilon\sum_{N>n_1>n_2>0}\Lambda x(n_1)\Lambda x(n_2)\chi(0)+\cdots\Big]_j\,. \tag{3.11}$$

Let us denote by $\vec{\chi}(N)$ the vector whose jth component is $\chi_j(N)$. Then we recognize from (3.11) that $\vec{\chi}(N)$ can be expressed by a product of matrices times $\vec{\chi}(0)$:

$$\vec{\chi}(N)=[I+2\sqrt{\epsilon}\,\Lambda x(N-1)][I+2\sqrt{\epsilon}\,\Lambda x(N-2)]\cdots[I+2\sqrt{\epsilon}\,\Lambda x(1)]\vec{\chi}(0), \tag{3.12}$$

where I denotes the identity matrix. If we expand the right-hand side of (3.12) in a power series in $\sqrt{\epsilon}$, we recover Eq. (3.11) precisely.

The operators $x_j(n)$ commute with each other. Thus an eigenstate of $\chi_j(N)$ is a product of the eigenstates of the operators $x_j(n)$ involved in (3.12). In other words, each eigenstate of $\chi_j(N)$ is specified by a designation of the quantum numbers of $x_j(n)$ for all j and for all positive n less than N.

Let us calculate $S_j(N)$ in (3.7) by expanding the ground state $|0\rangle$ into a superposition of the eigenstates of $\chi_j(N)$. The ground-state wave function of a harmonic oscillator is $\pi^{-1/4}\exp(-\frac{1}{2}x^2)$. Thus we have from (3.7) and (3.12) that

$$S_j(N)=\int\prod_{0<n<N}\prod_{k}\left\{\frac{dx_k(n)}{\pi^{1/2}}\exp[-x_K{}^2(n)]\right\}\times\exp(i\{[I+2\sqrt{\epsilon}\,\Lambda x(N-1)][I+2\sqrt{\epsilon}\,\Lambda x(N-2)]\cdots[I+2\sqrt{\epsilon}\,\Lambda x(1)]\vec{\chi}(0)\}_j)\,. \tag{3.13}$$

We may think of $x(n)$ in (3.13) as a diagonal matrix whose matric elements $x_j(n)$ are random variables with Gaussian distributions. Thus $S_j(N)$ is equal to the average value of the exponential of the jth component of a random vector $i\vec{\chi}(N)$, by (3.12), is equal to a product of random matrices operating on $\vec{\chi}(0)$. We shall study $S_j(N)$ in the limit $d\to 0$, $\epsilon\to 0$, with $T=\epsilon N\gg 1$.

IV. SCATTERING AS A STOCHASTIC PROCESS

The physical meaning of the eikonal forms (3.13) is very suggestive. Let us imagine a three-dimensional lattice with its lattice points specified by the index (j,n). We may think of j as the index specifying the transverse position on the lattice and n the index specifying the longitudinal position on the lattice. While the transverse dimension of the lattice is infinite, the longitudinal dimension of the lattice, restricted by (3.4), is equal to T. As $s=e^{2\pi T}$ increases, the longitudinal dimension of the lattice also increases.

There is associated with each lattice point (j,n) a harmonic oscillator with the creation operator $a_j^\dagger(n)$ and the annihilation operator $a_j(n)$. When two high-energy particles collide, they can excite any of these harmonic oscillators associated with the three-dimensional lattice in any arbitrary manner. The scattering is therefore a stochastic process in which quanta of the harmonic oscillators are created and annihilated in a random way. It is interesting to observe that the relevant physical entity which directly enters is not the creation operator or the annihilated operator separately, but the combination $x=(1/\sqrt{2})(a+a^\dagger)$. The eigenvalue of x plays the role of a random variable which can take any value between $-\infty$ and ∞, with the probability distribution equal to the Gaussian $\pi^{-1/2}e^{-x^2}$. It is also important to observe that the random variables $x_j(n)$ enter in the form of a power series for the eikonal χ, not for the S matrix. As s becomes larger and larger, the three-dimensional lattice expands in the longitudinal direction and more and more harmonic oscillations are involved. Thus χ receives contributions from an increasing number of random variables as s increases. Consequently, the expectation value of $\langle 0|\chi_j^2(N)|0\rangle$, for example, is very large as $s\to\infty$. Indeed, in the QED case, this expectation value corresponds to the sum of tower diagrams[5] and violates the Froissart bound. It is important that the S matrix $S_j(N)$, being equal to $\langle 0|e^{i\chi_j(N)}|0\rangle$, always satisfies unitarity no matter how large $\chi_j(N)$ becomes. Indeed, let $P_{j,N}(\chi)$ be the probability that the eigenvalue of $\chi_j(N)$ is equal to χ, then we have

$$S_j(N)=\int_{-\infty}^{\infty} d\chi\, P_{j,N}(\chi)e^{i\chi}. \tag{4.1}$$

If $P_{j,N}(\chi)$ is non-negligible only if χ is very large, we expect that the rapid oscillation of the integral makes the integral in (4.1) vanish.

V. DIFFERENTIAL EQUATION FOR THE S MATRIX

Carrying out the infinitely multiple integral for $S_j(N)$ as given by (3.13) is a very difficult task. Instead, it is possible to derive and study a partial differential equation for a generating functional $S(\vec{\xi},T)$ for the S matrix. In the discrete T-space formulation in which we have been processing ($T=N\epsilon$), we define

$$S(\xi,N\epsilon) \equiv\int\prod_{n=1}^{N-1}\prod_{k}\left\{\frac{dx_k(n)}{\sqrt{\pi}}\exp[-x_k^2(n)]\right\}\exp[i\vec{\chi}(N)\cdot\vec{\xi}\,]. \tag{5.1}$$

Here, $\vec{\xi}$ is a vector which has a component associated with each lattice point of the b lattice. Hence if we set $\xi_j=1$ for a fixed component j, and all other components of ξ to zero, $S(\vec{\xi},N\epsilon)$ is precisely $S_j(N)$. Thus, a knowledge of $S(\xi,T)$ contains more than the complete information for the S matrix over the b lattice.

Since $|\exp[i\vec{\chi}(N)\cdot\vec{\xi}\,]|=1$, we have from (5.1) that

$$|S(\vec{\xi},N\epsilon)|\le 1,\quad \text{for all } \vec{\xi} \text{ and } N\epsilon. \tag{5.2}$$

Furthermore, if any one of the components ξ_j is very large, the rapid oscillation of the integral in (5.1) makes the integral very small. Thus we have

$$S(\vec{\xi},N\epsilon)\to 0 \quad \text{if any } |\xi_j|\to\infty. \tag{5.3}$$

To derive a partial differential equation for $S(\xi,T)$ in the limit $\epsilon\to 0$, $N\epsilon=T$ fixed, we write (5.1) as

$$S(\xi,(N+1)\epsilon)=\int\prod\left\{\frac{dx_j(N)}{\sqrt{\pi}}\exp[-x_j^2(N)]\right\}\int\left\{\prod_{\substack{j\\ 0<n<N}}\frac{dx_j(n)}{\sqrt{\pi}}\exp[-\chi_j^2(n)]\right\}\exp[i\vec{\chi}(N+1)\cdot\vec{\xi}\,]. \tag{5.4}$$

$\vec{\chi}(N+1)$ obeys a recursion formula which follows from (3.12),

$$\vec{\chi}(N+1)=[1+2\sqrt{\epsilon}\Lambda x(N)]\vec{\chi}(N), \tag{5.5}$$

so that (5.4) can be expanded into a Taylor series of $\sqrt{\epsilon}$, after substituting in Eq. (5.5). We get, after carrying out the integrations over all $x_j(N)$,

$$S(\vec{\xi},(N+1)\epsilon)=S(\vec{\xi},N\epsilon)+\epsilon\sum_{j,k}(\xi_j\Lambda_{j-k})^2\frac{\partial^2}{\partial\xi_k^2}S(\vec{\xi},N\epsilon) + \text{terms of higher order in } \sqrt{\epsilon}. \tag{5.6}$$

In the desired limit $\epsilon\to 0$, this equation becomes

$$\frac{\partial S(\vec{\xi},T)}{\partial T}=\sum_k\left(\sum_j \xi_j\Lambda_{j-k}\right)^2\frac{\partial^2}{\partial\xi_k^2}S(\vec{\xi},T), \tag{5.7}$$

with the initial condition

$$S(\vec{\xi},0)=e^{i\vec{\xi}\cdot\vec{\chi}(0)} \tag{5.8}$$

together with the boundary condition (5.3). The above equation is the partial differential equation we want. We shall solve (5.7) to obtain the asymptotic form of $S(\vec{\xi},T)$ in the limit $T\gg 1$.

Before we attempt to solve this equation, let us first dispose of a related topic. It is of some interest to study the probability function $P(\vec{\chi},N\epsilon)$ defined by

$$P(\vec{\chi},N\epsilon)=\int\prod_{0<n<N}\prod_j\frac{dx_j(n)}{\sqrt{\pi}}e^{-x_j^2(n)}\,\delta(\vec{\chi}-\vec{\chi}(N)). \tag{5.9}$$

The function $P(\vec{\chi},N\epsilon)$ is the distribution function for the eigenvalues of $\vec{\chi}(N)$ as given by (3.12), when the random variables $x_j(n)$ are of Gaussian distributions. This function is, by (5.9) and (5.4), the Fourier transform of $S(\vec{\xi},N\epsilon)$:

$$P(\vec{\chi},N\epsilon)=\int\prod_j\frac{d\xi_j}{2\pi}\exp(-i\vec{\chi}\cdot\vec{\xi})S(\vec{\xi},N\epsilon). \tag{5.10}$$

Thus,

$$S(\vec{\xi},N\epsilon)=\int\prod_j d\chi_j\exp(i\vec{\chi}\cdot\vec{\xi})P(\vec{\chi},N\epsilon). \tag{5.11}$$

In the theory of probability, $S(\vec{\xi},N\epsilon)$ is known as the characteristic function of P.

Making a Fourier transform of (5.7) we obtain the partial differential equation for P:

$$\frac{\partial P(\vec{\chi},T)}{\partial T}=\sum_k\left(\sum_j\Lambda_{j-k}\frac{\partial}{\partial\chi_j}\right)^2\chi_k^2P(\vec{\chi},T) \tag{5.12}$$

with the initial condition

$$P(\vec{\chi}, 0) = \delta(\vec{\chi} - \vec{\chi}(0)). \tag{5.13}$$

We also observe from (5.9) that

$$P(\vec{\chi}, T) \geq 0, \tag{5.14}$$

and

$$\int d\vec{\chi}\, P(\vec{\chi}, T) = 1. \tag{5.15}$$

VI. THE COMPLEX ω PLANE

The standard way to deal with the partial differential equations above is to perform with respect to T. Let us define

$$\tilde{S}(\vec{\xi}, \omega) \equiv \int_0^\infty dT\, e^{-\omega T} S(\vec{\xi}, T) \tag{6.1}$$

and

$$\tilde{P}(\vec{\chi}, \omega) \equiv \int_0^\infty dT\, e^{-\omega T} P(\vec{\chi}, T). \tag{6.2}$$

We note that, since $|S(\vec{\xi}, T)|$ is never greater than unity, $\tilde{S}(\vec{\xi}, \omega)$ exists if

$$\mathrm{Re}\,\omega > 0. \tag{6.3}$$

Furthermore, the asymptotic form (5.4) for $S(\vec{\xi}, T)$ leads to

$$\tilde{S}(\vec{\xi}, \omega) \to 0, \quad \text{if any } |\xi_j| \to \infty, \ \mathrm{Re}\,\omega > 0. \tag{6.4}$$

Similarly, from (5.14) and (5.15),

$\tilde{P}(\vec{\chi}, \omega)$ exists for $\mathrm{Re}\,\omega > 0$

and

$$\tilde{P}(\vec{\chi}, \omega) \to 0, \quad \text{if any } |\chi_i| \to \infty, \ \mathrm{Re}\,\omega > 0. \tag{6.5}$$

The functions $S(\vec{\xi}, T)$ and $P(\vec{\chi}, T)$ can be obtained from their Laplace transform by an inverse Laplace transform:

$$S(\vec{\xi}, T) = \int_{L-i\infty}^{L+i\infty} \frac{d\omega}{2\pi i} e^{\omega T} \tilde{S}(\vec{\xi}, \omega) \tag{6.6}$$

and

$$P(\vec{\chi}, T) = \int_{L-i\infty}^{L+i\infty} \frac{d\omega}{2\pi i} e^{\omega T} \tilde{P}(\vec{\chi}, \omega), \tag{6.7}$$

where L is any positive number.

The behaviors of $S(\vec{\xi}, T)$ and $P(\vec{\chi}, T)$ in the limit $T \to \infty$ are very conveniently deduced from (6.6) and (6.7). This is because, in this limit, $e^{\omega T}$ is exponentially small if $\mathrm{Re}\,\omega$ is negative. Indeed, the more negative $\mathrm{Re}\,\omega$ is, the more quickly $e^{\omega T}$ vanishes. Thus, if $T \gg 1$, the integrands in (6.6) and (6.7) become smaller and smaller as we move the contour of integration further and further to the left. The functions of $S(\vec{\xi}, \omega)$ and $P(\vec{\chi}, \omega)$ may have singularities in the left-hand plane $\mathrm{Re}\,\omega \leq 0$, and the contributions of these singularities are picked up as we move the contour to the left. For example, a simple pole in the ω plane gives a term proportional to

$$e^{\omega_0 T} = s^{\omega_0 / 2\pi}, \tag{6.8}$$

where ω_0 is the location of the pole, and similarly for double poles, branch points, etc.

The behavior of (6.8) is reminiscent of the Regge behavior. The singularities in the complex ω plane determine the asymptotic behaviors of $S(\vec{\xi}, T)$ and $P(\vec{\chi}, T)$ in the same way that Regge singularities control the asymptotic form of the scattering amplitude. It is therefore of interest to determine the location of singularities in the ω plane. The differential equations satisfied by $\tilde{S}(\vec{\xi}, \omega)$ and $\tilde{P}(\vec{\chi}, \omega)$ are easily obtained by making Laplace transforms of (5.7) and (5.12) with respect to T. We get

$$(H - \omega)\tilde{S}(\vec{\xi}, \omega) = -e^{i\vec{\xi}\cdot\vec{\chi}(0)} \tag{6.9}$$

and

$$(H' - \omega)\tilde{P}(\vec{\chi}, \omega) = -\delta(\vec{\chi} - \vec{\chi}(0)). \tag{6.10}$$

Thus,

$$H \equiv \sum_k \Big(\sum_j \xi_j \Lambda_{j-k} \Big)^2 \frac{\partial^2}{\partial \xi_k^{\,2}} \tag{6.11}$$

and

$$H' \equiv \sum_k \Big(\sum_j \Lambda_{j-k} \frac{\partial}{\partial \chi_j} \Big)^2 \chi_k^{\,2}. \tag{6.12}$$

The solutions of (6.9) and (6.10) are, respectively,

$$\tilde{S}(\vec{\xi}, \omega) = -(H - \omega)^{-1} e^{i\vec{\xi}\cdot\vec{\chi}(0)} \tag{6.13}$$

and

$$\tilde{P}(\vec{\chi}, \omega) = -(H' - \omega)^{-1} \delta(\vec{\chi} - \vec{\chi}(0)). \tag{6.14}$$

VII. SOME SIMPLE EXAMPLES

In order to gain some familiarity with the problem, we shall give in this section results of two simplified examples. Derivations of some of these results are given in the Appendix.

A. Case of one lattice point

Consider first the case in which the lattice in the $\vec{b}$ space has only one lattice point.[7] In this case, $\vec{\xi}$ and $\vec{\chi}$ have only one component and we shall denote them as ξ and χ, respectively. The solutions of (5.7) and (5.12) have been obtained in closed forms[8]:

$$S(\xi, T) = 1 + e^{-\Lambda^2 T/4} \int_{-\infty}^{\infty} \frac{dp}{2\pi} (\chi_0\, \xi e^{-i\pi/2})^{1/2} \\ -ip\, e^{-p^2 \Lambda^2 T}\, \Gamma(-\tfrac{1}{2} + ip) \tag{7.1}$$

and

$$P(\chi, T)=\frac{1}{(4\pi\Lambda^2 T)^{1/2}}\frac{1}{\chi}\exp\left\{-\frac{[\ln(\chi/\chi_0)+\Lambda^2 T]^2}{4\Lambda^2 T}\right\}. \tag{7.2}$$

In (7.1) and (7.2), Λ is the (only) matrix element of the Λ matrix and χ_0 is the (only) component of $\vec{\chi}(0)$.

From (7.1), we see that as $T\to\infty$, $S(\xi, T)$ approaches unity. More precisely the function $[S(1,T)-1]$, measuring the amount of scattering, vanishes roughly like a negative power of s as $s\to\infty$.

As for $P(\chi, T)$, we find from (7.2) that the mean values of $\ln(\chi/\chi_0)$ is equal to $-\Lambda^2 T$. Thus as $T\to\infty$, $P(\chi, T)$ is nonzero only if χ is roughly equal to $\chi_0 e^{-\Lambda^2 T}$, a very small number. This means that the large number of random terms in the eikonal cancel one another as $T\to\infty$.

B. Case of two lattice points

We consider the next simplest case in which the lattice in the $\vec{b}$ space has two lattice points. In this example, the Λ matrix is a 2×2 matrix in the form of

$$\Lambda=\begin{bmatrix}\Lambda_0 & \Lambda_1\\ \Lambda_1 & \Lambda_0\end{bmatrix} \tag{7.3}$$

and $\vec{\xi}$ has two components ξ_1 and ξ_2.

In the limit $T\to\infty$, we get

$$S(\vec{\xi}, T)\to 1,\quad \text{if } \Lambda_0{}^2>\Lambda_1{}^2 \tag{7.4}$$

$$\to 0,\quad \text{if } \Lambda_0{}^2<\Lambda_1{}^2 \tag{7.5}$$

$$\to \exp\left[\frac{i(a\mp b)(\xi_1\pm\xi_2)-|(a\mp b)(\xi_1\mp\xi_2)|}{2}\right],\quad \text{if } \Lambda_0=\pm\Lambda_1 \tag{7.6}$$

where a and b are constants which appear in the initial condition

$$S(\vec{\xi}, 0)=\exp(ia\xi_1+ib\xi_2). \tag{7.7}$$

From (7.4) we find that complete cancellation happens when the condition $\Lambda_0{}^2>\Lambda_1{}^2$. Physically, a target particle becomes transparent (no scattering) in the high-energy limit if $\Lambda_0{}^2>\Lambda_1{}^2$. A special case of $\Lambda_0{}^2>\Lambda_1{}^2$ is the case of $\Lambda_1=0$. In this special case, Λ is proportional to the identity matrix. Thus the two lattice points are decoupled and the problem is reduced to that of one lattice point treated in subsection A. The result there can therefore be considered as a special case of (7.4).

From (7.5), we find that the cancellation does not happen if $\Lambda_0{}^2<\Lambda_1{}^2$. Physically, it means that, if $\Lambda_0{}^2<\Lambda_1{}^2$, the target particle becomes black (completely absorptive) in the high-energy limit as a result of particle creation and annihilation.

From (7.6), we find that the cancellation is partial if $\Lambda_1{}^2=\Lambda_0{}^2$. Physically, it means that a target particle becomes gray (partially absorptive) in the high-energy limit.

VIII. CASE OF THREE SPATIAL DIMENSIONS

In this section, we shall determine the behavior of the S matrix for the case we set out to solve: Eq. (5.7) in the limit the lattice constant d goes to zero. This is the case in which we have a continuous, two-dimensional $\vec{b}$ space.

The key observation here is that, since Λ_{jk} is proportional to d [see (3.8)], we may neglect, in the limit $d\to 0$, the diagonal matrix element Λ_{jj}. This is because we can drop one infinitesimally small term from a sum which is an approximation of an integral.[9]

If we neglect Λ_{jj} (or equivalently, set $\Lambda_{jj}=0$), then the operator

$$H=\sum_k\left(\sum_j \xi_j\Lambda_{j-k}\right)^2\frac{\partial^2}{\partial\xi_k{}^2}$$

is self-adjoint, as $\sum\xi_j\Lambda_{j-k}$ commutes with $\partial/\partial\xi_k$. Furthermore, the eigenvalues of H are nonpositive, as we have, after an integration by parts,

$$\int d\vec{\xi}\;\phi^*(\vec{\xi})H\phi(\vec{\xi})=-\int d\vec{\xi}\sum_k\left(\sum_j\xi_j\Lambda_{j-k}\right)^2\left|\frac{\partial\phi(\vec{\xi})}{\partial\xi_k}\right|^2. \tag{8.1}$$

The right side of (8.1) is equal to zero if $\phi(\xi)$ is equal to a constant.

We shall show that this eigenfunction of zero eigenvalue does not contribute to the asymptotic form of $S(\vec{\xi}, T)$; hence, $S(\vec{\xi}, T)$ vanishes in the limit $T\to\infty$ with $\vec{\xi}$ fixed. We observe that the operator H is equidimensional. Hence it is useful to introduce the spherical coordinates

$$r \equiv \left(\sum_j \xi_j^2\right)^{1/2},$$
$$\hat{\xi}_j \equiv \frac{\xi_j}{r} . \tag{8.2}$$

The operator H operating on $r^{-\eta}$ times a function of $\hat{\xi}_j$ is always equal to $r^{-\eta}$ times another function of $\hat{\xi}_j$. We may therefore define the operator $\mathcal{K}(\eta)$,

$$Hr^{-\eta}F(\hat{\xi}_j) \equiv r^{-\eta}\mathcal{K}(\eta)F(\hat{\xi}_j) , \tag{8.3}$$

where $\mathcal{K}(\eta)$ involves only the angular variables $\hat{\xi}_j$.

The expression (6.13) for the Laplace transform $\tilde{S}(\vec{\xi}, \omega)$ is therefore best handled if we express $e^{i\vec{\xi}\cdot\vec{\chi}(0)}$ by its Mellin transform integral:

$$e^{i\vec{\xi}\cdot\vec{\chi}(0)} = \int_{L-i\infty}^{L+i\infty} \frac{d\eta}{2\pi i} [r\hat{\xi}\cdot\vec{\chi}(0)]^{-\eta} e^{i\pi\eta/2}\Gamma(\eta) , \tag{8.4}$$

where L is any positive constant.[8] Then (6.13) becomes

$$\tilde{S}(\vec{\xi}, \omega) = \int_{L-i\infty}^{L+i\infty} \frac{d\eta}{2\pi i} r^{-\eta} e^{i\pi\eta/2}\Gamma(\eta) \frac{1}{\mathcal{K}(\eta)-\omega} [\hat{\xi}\cdot\vec{\chi}(0)]^{-\eta} . \tag{8.5}$$

As a consequence of (8.5) the fact that H is self-adjoint, $\tilde{S}(\vec{\xi}, \omega)$ is an entire function of ω. To see this, let us for the moment imagine that the lattice in the b space has M lattice points. (Eventually, we will take the limit $M \to \infty$.) Then, since H is self-adjoint, we have

$$\int \psi^*(\vec{\xi})H\phi(\vec{\xi})d^M\xi = \left[\int \phi^*(\vec{\xi})H\psi(\vec{\xi})d^M\xi\right]^* \tag{8.6}$$

for any ψ and ϕ satisfying proper boundary conditions. Let us choose

$$\phi(\xi) = r^{-\eta}F_1(\hat{\xi}) \tag{8.7}$$

and

$$\psi(\xi) = r^{-M+\eta'^*}F_2(\hat{\xi}) .$$

Substituting (8.7) into (8.6), and carrying out the integration over r, we get, after factoring out a common factor $\delta(\eta-\eta')$,

$$\int F_2^*(\hat{\xi})\mathcal{K}(\eta)F_1(\hat{\xi})d\Omega = \left[\int F_1^*(\hat{\xi})\mathcal{K}(M-\eta)F_2(\hat{\xi})d\Omega\right]^*, \tag{8.8}$$

where $d\Omega$ denotes the integration over all angular variables. From (8.8), we conclude that

$$\mathcal{K}(\eta) = \mathcal{K}^\dagger(M-\eta) . \tag{8.9}$$

In particular, $\mathcal{K}(M/2+ip)$ is Hermitian if p is real.

If we now move the contour of integration in (8.5) into the line

$$\eta = \frac{M}{2} + ip ,$$

then we have

$$\tilde{S}(\vec{\xi}, \omega) = -\int_{-\infty}^{\infty} \frac{dp}{2\pi} (re^{i\pi/2})^{-M/2-ip}\Gamma(M/2+ip) \times \frac{1}{\mathcal{K}(M/2+ip)-\omega} [\hat{\xi}\cdot\vec{\chi}(0)]^{-\eta} . \tag{8.10}$$

From (8.10), we find that $\tilde{S}(\vec{\xi}, \omega)$ has singularities at the points of ω equal to the (real-valued) eigenvalues of $\mathcal{K}(M/2+ip)$.

As $M \to \infty$, all of these eigenvalues become negatively infinite. This is because they are also the eigenvalues of H corresponding to the eigenfunctions of the form $r^{-M/2-ip}$ times a function of $\hat{\xi}_j$. The radial derivatives of these eigenfunctions are equal to infinity as $M \to \infty$. Thus from (8.1), the corresponding eigenvalues go to $-\infty$ as $M \to \infty$.

Since $\tilde{S}(\vec{\xi}, \omega)$ is entire, we have

$$S(\vec{\xi}, T) = \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} e^{\omega T} S(\vec{\xi}, \omega) \to 0 \text{ as } T \to \infty . \tag{8.11}$$

We therefore conclude that a target particle becomes completely absorptive as $T \to \infty$.

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APPENDIX

In this Appendix, we derive the closed forms of $S(\xi, T)$ and $P(\chi, T)$ for the case in which the lattice in the $\vec{b}$ space has only one lattice point.

In the case of only one lattice point, the eikonal as given by (3.13) is equal to a product of random numbers, not random matrices. Thus the logarithm of the eikonal is equal to a sum of random numbers. This problem can therefore be easily treated by standard techniques in probability theory, e.g., by applying the central limit theorem. Let us, however, utilize the formalism we have set up. We have

$$\Lambda^2\xi^2 \frac{\partial^2}{\partial\xi^2} \tilde{S}(\xi, \omega) - \omega\tilde{S}(\xi, \omega) = -e^{i\xi C} \tag{A1}$$

and

$$\Lambda^2 \frac{\partial^2}{\partial\chi^2} \chi^2 \tilde{P}(\chi, \omega) - \omega\tilde{P}(\chi, \omega) = -\delta(\chi - C) , \tag{A2}$$

where Λ and C are constants. The boundary conditions are

$$\tilde{S}(\xi,\omega)\to 0,\quad |\xi|\to\infty,\quad \mathrm{Re}\,\omega>0\ , \tag{A3}$$

$$\tilde{P}(\chi,\omega)\to 0,\quad |\chi|\to\infty,\quad \mathrm{Re}\,\omega>0. \tag{A4}$$

We first solve for $\tilde{S}(\xi,\omega)$. Since the operator

$$H\equiv \Lambda^2\xi^2\frac{\partial^2}{\partial\xi^2} \tag{A5}$$

is equidimensional, Hx^η is equal to a constant times x^η. Thus (A1) is easily solved by expressing $e^{i\xi C}$ by its Mellin transform integral

$$e^{i\xi C}=\int_{L-i\infty}^{L+\infty}\frac{d\eta}{2\pi i}\,(C\xi e^{-i\pi/2})^{-\eta}\Gamma(\eta)\ , \tag{A6}$$

where L is any positive constant. Then (A1) and (A6) give

$$\tilde{S}(\xi,\omega)=-\int_{L-i\infty}^{L+i\infty}\frac{d\eta}{2\pi i}\,\frac{(C\xi e^{-i\pi/2})^{-\eta}\Gamma(\eta)}{\eta(\eta+1)\Lambda^2-\omega}\ . \tag{A7}$$

By moving the contour of integration in (A7) to the line $\mathrm{Re}\,\eta=-\frac{1}{2}$, and picking up the contribution of the pole at $\eta=0$, we get

$$\tilde{S}(\xi,\omega)=\frac{1}{\omega}+\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\frac{(C\xi e^{-i\pi/2})^{1/2-ip}\Gamma(-\frac{1}{2}+ip)}{\Lambda^2(\frac{1}{4}+p^2)+\omega}\ . \tag{A8}$$

$\tilde{S}(\xi,\omega)$ has a simple pole at $\omega=0$. By making an inverse Laplace transform of (A8), we get $S(\xi,T)$ as given by (7.1).

It is also interesting to study the Green's function $G(\xi,\xi',\omega)$ which satisfies

$$\left(\Lambda^2\xi^2\frac{\partial^2}{\partial\xi^2}-\omega\right)G(\xi,\xi',\omega)=\delta(\xi-\xi')\ . \tag{A9}$$

The solution is, for $\xi'>0$,

$$G(\xi,\xi',\omega)=-\frac{1}{\Lambda^2(1+4\omega/\Lambda^2)}\ \frac{1}{\xi'}\left(\frac{\xi}{\xi'}\right)^{1-(1+4\omega/\Lambda^2)^{1/2}}\ ,\quad \xi>\xi' \tag{A10}$$

$$=-\frac{1}{\Lambda^2(1+4\omega/\Lambda^2)^{1/2}}\ \frac{1}{\xi'}\left(\frac{\xi}{\xi'}\right)^{1+(1+4\omega/\Lambda^2)^{1/2}}\ ,\quad 0<\xi<\xi' \tag{A11}$$

$$=0,\quad \xi<0\ . \tag{A12}$$

A similar expression holds for $\xi'<0$. It is seen that the only singularity for $G(\xi,\xi',\omega)$ is a branch point at $\omega=-\Lambda^2/4$. The function $\tilde{S}(\xi,\omega)$, equal to the integral

$$\int_{-\infty}^{\infty}d\xi'\,G(\xi,\xi',\omega)e^{i\xi' C}\ , \tag{A13}$$

has additional singularities due to the divergence of integration at $\xi'=0$. From (A10) and (A13), we find that these singularities occur at

$$\frac{1-(1+4\omega/\Lambda^2)^{1/2}}{2}=0,1,2,\ldots\ . \tag{A14}$$

The first root of (A14) (with the right side of the equation set to zero) is

$$\omega=0\ ,$$

which is the singularity responsible for the vanishing of scattering. All other roots of (A14) are located in the second Riemann sheet of ω.

Finally, we turn our attention to $\tilde{P}(\chi,\omega)$. By comparing (A2) and (A9), we find that

$$\tilde{P}(\chi,\omega)=-\frac{C^2}{\chi^2}\,G(\chi,C,\omega)\ . \tag{A15}$$

Thus, if $C>0$, we have from (A10)–(A12) and (A15) that

$$\tilde{P}(\chi,\omega)=\frac{1}{\Lambda^2(1+4\omega/\Lambda^2)^{1/2}}\,\frac{1}{\chi}\left(\frac{C}{\chi}\right)^{1+(1+4\omega/\Lambda^2)^{1/2}},\quad \chi>C$$

$$=\frac{1}{\Lambda^2(1+4\omega/\Lambda^2)^{1/2}}\left(\frac{C}{\chi}\right)^{1-(1+4\omega/\Lambda^2)^{1/2}},\quad 0<\chi<C$$

$$=0,\quad \chi<0\,. \tag{A16}$$

Again, the only singularity for $\tilde{P}(\chi,\omega)$ is a branch point at $\omega=-\Lambda^2/4$. By making an inverse Laplace transform of (A16) we obtain $P(\chi,T)$ as given by (7.2).

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[8]To make the integral in (7.1) convergent, we shall give ξ a positive phase angle, i.e., we put $\xi=|\xi|e^{i\theta}$. The case of positive ξ is obtained by taking the limit $\theta\to 0$ and the case of negative ξ is obtained by taking the limit $\theta\to\pi$.

[9]Another way to justify this is to observe that all high-energy amplitudes from diagrammatic calculations are in the form of a power of $\ln s$ times *convergent* integrals over transverse distances. Thus these amplitudes are unchanged if we put in a cutoff for small transverse distances and then let the cutoff go to zero. Now if we expand $\exp(i\vec{\chi}\cdot\vec{\xi})$ in (5.11) into a power series of $\vec{\chi}\cdot\vec{\xi}$, the $(\vec{\chi}\cdot\vec{\xi})^N$ term is proportional to the V^N terms from diagrammatic calculations (Refs. 5 and 6). It is also the nth moment of P. Thus the moments of P remain the same if we put in an infinitesimal cutoff. The S matrix is the characteristic function of P, and is hence unchanged with an infinitesimal cutoff.