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Space-time symmetries for theories with extended objects

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We examine the implications of space-time symmetries for quantum field theories with extended objects. It is shown that the existence of the quantum coordinate (collective coordinate) $\vec{q}$ is a direct consequence of the canonical formulation of translational invariance. In 1 + 1 dimensions, Lorentz invariance fixes uniquely the structure of the theory in the no-particle sector. If the tree approximation is used, the structure of the one-particle sector is also uniquely determined. Finally, the techniques developed in this paper allow us to deduce that nonspherically symmetric objects in three dimensions require additional quantum coordinates besides $\vec{q}$.

I. INTRODUCTION

The set of physical systems describable by quantum field theory has been vastly enriched by the realization that the latter can accommodate extended objects in addition to the more familiar field quanta.[1] The theory of such quantum many-body systems in the presence of extended objects has been formulated in two different ways. One formulation uses path-integral techniques[2] while the other relies on conventional quantum field theory and focuses on the relationship between the basic Heisenberg fields ψ and the physical fields ϕ^0 (e.g., in-fields) which describe the Hilbert space of the system.[3-6] This is the method followed in this paper, and we start with a brief outline of the main features relevant to our considerations. A detailed description of the formalism can be found in Refs. 3–6.

One starts by solving the Heisenberg equations of motion under the boundary condition of a homogeneous (space-time-independent) vacuum; the matrix elements of the Heinsenberg field operator ψ in the physical Hilbert space are thus determined and are summarized compactly in the dynamical map:

$$\psi(x)=\psi[x;\phi^0(x)]\,. \tag{1.1}$$

In this relation, $\phi^0(x)$ are the physical fields in the absence of extended objects. The next step consists in performing a boson transformation

$$\phi^0(x)\to\phi^0(x)+f(x)\,, \tag{1.2}$$

where the c-number function $f(x)$ obeys the same (free) field equation as $\phi^0(x)$. Then the Heisenberg field

$$\psi^f(x)=\psi[x;\phi^0(x)+f(x)] \tag{1.3}$$

is also a solution of the original Heisenberg equation of motion.[3,4] Physically, the transformation (1.2) corresponds to a nonhomogeneous condensation of ϕ^0 quanta which creates the extended object. Thus $\psi^f(x)$ describes both the extended object and the field quanta, as well as their mutual interactions. The properties of the extended objects are controlled by the function f in Eq. (1.2); in the following, this function will be assumed to be time independent, thus leading to static extended objects.

We will consider theories whose dynamical equations of motion before the boson transformation (1.2) are invariant under the full Poincaré group. It is clear that after the extended object has been created, the physical system will not manifest translational or Lorentz invariance. On the other hand, since the transformed Heisenberg field ψ^f obeys the original equation of motion, the generators of space-time transformations will be still conserved when expressed in terms of $\psi^f(x)$. The purpose of this paper is to examine the implications of these conservation laws.

It will be shown that the study of space-time-invariant transformations is a very powerful tool in elucidating the structure of systems with extended objects. At least in $1+1$ dimensions and in the tree approximation, it restricts the dependence of ψ^f on x and t, and gives explicit expressions for the Hamiltonian and the generator of Lorentz transformations in the sectors of the Hilbert space with at most one particle quantum present.

The implications of translational invariance are deduced in Sec. II. It is shown there that the existence of a conserved momentum $\vec{P}$ implies the presence of a quantum coordinate $\vec{q}$ conjugate to it. The translation $\vec{x}\to\vec{x}+\vec{a}$ is rearranged into the translation $\vec{q}\to\vec{q}+\vec{a}$ of the quantum coordinate. Thus, any field operator must contain $\vec{x}$ and $\vec{q}$ only in the combination $\vec{x}+\vec{q}$.

Section III examines the consequences of Lorentz invariance for the no-particle sector of the theory. It is found that the time independence of the Lorentz generator in $1+1$ dimensions fixes uniquely the dependence of the momentum and the Hamiltonian on $(q,\dot{q})$. The resulting expressions are those of a relativistic point particle. The explicit form of the Lorentz generator is also obtained.

The implications of Lorentz invariance for the Heisenberg field in $1+1$ dimensions are discussed in Secs. IV and V. In Sec. IV it is shown that the expectation value of ψ^f in the no-particle state (order parameter) depends on x, q, and $\dot{q}$ only through a particular combination, which we call the generalized coordinate X. In Sec. V, we extend our consideration to the one-particle sector. We find that, in the tree approximation, the full Heisenberg field ψ^f depends only on two generalized coordinates X and T. This result is very useful for computations. It implies that the effect of the quantum coordinate in the tree approximation can be neglected in calculating ψ^f: It can be fully restored by the substitution $x \to X$, $t \to T$ in the final result. In the same section, we determine completely the generator of Lorentz transformations and the Hamiltonian in the tree approximation.

Section VI is devoted to a summary of our results and to an indication of possible extensions of the methods developed in this paper. Using these methods, we prove there a powerful theorem: if the only quantum coordinates in $3+1$ dimensions are the three q's conjugate to the momentum operator, the extended object is spherically symmetric in the rest system and, in fact, its energy and momentum are those of a free relativistic point particle. Therefore, objects with more complicated structure must have other quantum coordinates besides $\vec{q}$.

II. TRANSLATION INVARIANCE AND THE QUANTUM COORDINATE

In this section we discuss the implications of translation invariance for our system. If the original Lagrangian is invariant under space translations, there exists a time-independent operator $\vec{P}$ with the property

$$[\psi^f(x), \vec{P}] = i\vec{\nabla}\psi^f(x) . \quad (2.1)$$

If we expand the right-hand side of Eq. (1.3) in a Taylor series around $\phi^0 = 0$, we obtain

$$\psi^f(x) = \sum_n \frac{1}{n!} :(\delta_f)^n \phi^f(x):, \quad (2.2)$$

where

$$\delta_f = \int d^4\sigma\, \phi^0(\sigma) \frac{\delta}{\delta f(\sigma)}, \quad (2.3)$$

$$\phi^f(x) = \langle 0 | \psi^f(x) | 0 \rangle . \quad (2.4)$$

Then

$$\vec{\nabla}\psi^f(x) = \sum_n \frac{1}{n!} :(\delta_f)^n \vec{\nabla}\phi^f(x): . \quad (2.5)$$

If we compare this with Eq. (2.1), we see that *the translation of the Heisenberg field is implemented solely by the translation of* $\phi^f(x)$. In particular, $\vec{P}$ commutes with the creation and annihilation operators for particle quanta. Equations (2.1) and (2.5) lead to

$$[\phi^f(x), \vec{P}] = i\vec{\nabla}\phi^f(x) . \quad (2.6)$$

Thus $\phi^f(x)$ is not a c-number function; it depends on a *quantum coordinate*[7] $\vec{q}$ canonically conjugate to $\vec{P}$:

$$[q_i, P_j] = i\delta_{ij} . \quad (2.7)$$

In fact, Eq. (2.6) implies that ϕ^f depends on $\vec{x}$ and $\vec{q}$ only through the combination $\vec{x}+\vec{q}$:

$$\phi^f = \phi^f(\vec{x}+\vec{q}) . \quad (2.8)$$

Then Eq. (2.2) shows that the same property holds for the Heinsenberg field ψ^f:

$$\psi^f = \psi^f(\vec{x}+\vec{q}) . \quad (2.9)$$

This equation shows how translational symmetry is rearranged in the presence of extended objects: The translation

$$\vec{x} \to \vec{x} + \vec{a} \quad (2.10)$$

is implemented by a shift of the quantum coordinate

$$\vec{q} \to \vec{q} + \vec{a} . \quad (2.11)$$

We can call this phenomenon a "CQ transmutation": The transformation of a classical coordinate is transmuted into a transformation of a quantum-mechanical variable. We will see more examples of this later.

The fact that a coordinate translation does not affect the particle quanta becomes less surprising if we recall that the position and spatial distribution of the extended object defines the origin of the coordinate system: A change of origin is effected by moving the extended object. Further comments on this point will be found at the end of Sec. V.

The occurrence of the quantum coordinate and the property expressed in Eq. (2.9) were discussed in Ref. 5 from a different viewpoint. It was shown there that the physical spectrum in the presence of an extended object can be deduced from an eigenvalue equation involving the so-called self-consistent potential. This spectrum contains zero-frequency modes, which correspond to the quantum coordinate $\vec{q}$. There are also scattering and possibly bound-state solutions, whose field operators will be denoted by $\chi^0(x)$ [these should not be confused with $\phi^0(x)$, which denote the physical fields in the absence of extended objects]. One important result of that analysis was that $\vec{q}$, $\vec{P}$ *appear always in a symmetrized form.*[6]

The Hilbert space for our system has the form

$$\mathcal{H}=\mathcal{H}_q\otimes\mathcal{H}_\chi, \tag{2.12}$$

where $\mathcal{H}_\chi$ is the Fock space corresponding to χ^0 and $\mathcal{H}_q$ provides a realization of the canonical pair (q, P). We denote the vacuum of $\mathcal{H}_\chi$ by $|0\rangle$. Therefore, for any operator $\mathcal{H}$,

$$\langle 0|\mathfrak{a}|0\rangle\equiv\overline{\mathfrak{a}} \tag{2.13}$$

is a *quantum-mechanical* operator, and

$$\langle 0|\quad|0\rangle|_{q=p=0}$$

is a classical object. In this paper, the expectation value of operators with respect to $|0\rangle$ will be denoted by a bar over the operator symbol, as in Eq. (2.13).

A consequence of Eq. (2.9) which will be very useful in the sequel is if G is an operator which can be derived from a local density

$$G=\int d^3x\,\rho(\vec{x},t) \tag{2.14}$$

and $\rho(\vec{x},t)$ depends on $\vec{x}$ only through the combination $\vec{x}+\vec{q}$, it is clear that G is independent of $\vec{q}$. In particular, the Hamiltonian is of the form (2.14); therefore,

$$H=H(\vec{P};\chi^0) \tag{2.15}$$

and

$$\overline{H}=\langle 0|H|0\rangle=\overline{H}(\vec{P})\,. \tag{2.16}$$

The quantum coordinate was introduced above by considering the vacuum expectation value of the Heisenberg field:

$$\langle 0|\psi^f|0\rangle=\phi^f(\vec{x}+\vec{q},\dot{\vec{q}})\,. \tag{2.17}$$

Therefore, the time development of q is generated by $\overline{H}(P)$ in Eq. (2.16):

$$\vec{q}=e^{i\overline{H}t}\vec{q}_0e^{-i\overline{H}t}, \tag{2.18}$$

$$\vec{q}_0=\vec{q}(t=0)\,, \tag{2.19}$$

and

$$i\dot{\vec{q}}=[\vec{q},\overline{H}]=i\dot{\vec{q}}(\vec{P})\,. \tag{2.20}$$

It then follows that

$$\ddot{\vec{q}}=0 \tag{2.21}$$

and $\vec{q}(t)$ depends linearly on time:

$$\vec{q}=\vec{q}_0+t\dot{\vec{q}}\,. \tag{2.22}$$

Notice that, although $\vec{q}_0$ is independent of time, its commutator with the Hamiltonian is not zero; in fact, for Eqs. (2.20)–(2.22) we get

$$[\vec{q}_0,\overline{H}]=i\dot{\vec{q}}\,.$$

This is due to the fact that the canonical transformation $\vec{q}\to\vec{q}_0$ is time dependent.

This completes the study of the translational invariance. We found that the invariance of the Heisenberg field equations under spatial translations has the following implications:

(i) Besides the field quanta, there appears a quantum coordinate $\vec{q}$. The dependence of field operators on $\vec{q}$ is obtained by replacing $\vec{x}\to\vec{x}+\vec{q}$ everywhere.

(ii) The operator conjugate to $\vec{q}$ is the *total* momentum operator $\vec{P}$ (i.e., the generator of space translations for the system). As a result, the coordinate translation $\vec{x}\to\vec{x}+\vec{a}$ is implemented by the shift $\vec{q}\to\vec{q}+\vec{a}$.

(iii) The quantum coordinate $\vec{q}$ is a linear function of time.

Let us also note that since $\vec{q}$ and the quanta of χ^0 correspond to different degrees of freedom, $\vec{q}$ and $\dot{\vec{q}}$ commute with the field operators χ^0.

III. LORENTZ TRANSFORMATION PROPERTIES OF THE QUANTUM COORDINATE

For the remainder of this paper, we restrict ourselves to relativistic models in one space dimension. We also deal exclusively with the static extended objects; therefore,

$$\phi^f(\vec{x}+\vec{q},\dot{\vec{q}})\equiv\langle 0|\psi^f|0\rangle \tag{3.1}$$

has no explicit time dependence; it depends on t only through q. The generator of Lorentz transformations is given by

$$M_{x0}=\int dx(xT_{00}-tT_{x0})\,, \tag{3.2}$$

where $T_{\mu\nu}$ is the canonical energy-stress tensor (we deal with a scalar field theory, so that $T_{\mu\nu}$ is symmetric). The Hamiltonian and the generator of space translations are

$$H=\int dx\,T_{00}\,, \tag{3.3}$$

$$P=-\int dx\,T_{x0}\,. \tag{3.4}$$

As we saw in Sec. II, P coincides with the canonical conjugate of the quantum coordinate q.

Using (3.3) and (3.4) we can rewrite M_{x0} as

$$M_{x0}=F(\dot{q},\chi^0)+tP-qH\,, \tag{3.5}$$

where

$$F\equiv\int dx(x+q)T_{00}\,. \tag{3.6}$$

Since F is of the form (2.14), it is independent of q, as indicated explicitly in Eq. (2.5).

We are interested in the Lorentz transformation properties of the quantum coordinate; we can therefore restrict ourselves to the vacuum sector of $\mathcal{H}_\chi$. Using the definition (2.13), Eq. (3.5) becomes in this sector

$$\overline{M}_{x0}=\overline{F}(\dot{q})+tP-q\overline{H}\,. \tag{3.7}$$

The Lorentz invariance of the Heisenberg equation of motion implies that $\overline{M}_{x0}$ is a constant of motion:

$$\frac{d}{dt}\overline{M}_{x0}=0\,. \tag{3.8}$$

Then since $\dot{q}$, P, and $\overline{H}$ are independent of time, Eq. (3.7) gives the relation

$$P=\dot{q}\overline{H}\,, \tag{3.9}$$

and since (q, P) are canonical conjugates,

$$i=[q,p]=[q,\dot{q}]\overline{H}+i\dot{q}^2\,. \tag{3.10}$$

On the other hand,

$$i\dot{q}=[q,\overline{H}]=[q,\dot{q}]\frac{\partial\overline{H}}{\partial\dot{q}}\,. \tag{3.11}$$

Comparing Eqs. (3.10) and (3.11) we see that $\overline{H}$ obeys the equation

$$\frac{\partial\overline{H}}{\partial\dot{q}}=\frac{\dot{q}}{1-\dot{q}^2}\overline{H} \tag{3.12}$$

with solution

$$\overline{H}=\frac{m}{(1-\dot{q}^2)^{1/2}}\,. \tag{3.13}$$

m is a c-number constant. Then from (3.9)

$$P=\frac{m\dot{q}}{(1-\dot{q}^2)^{1/2}}\,, \tag{3.14}$$

$$\overline{H}=(p^2+m^2)^{1/2}\,. \tag{3.15}$$

This relation shows that m is the mass of the *classical* extended object.

$$m=\overline{H}_{P=q=0}\,. \tag{3.16}$$

It was shown in Ref. 6 that Eq. (3.15), although true for any (1+1)-dimensional relativistic model, does not generally hold in 3+1 dimensions: This point will be discussed in Sec. VI.

Equations (3.11) and (3.13) also give

$$[q,\dot{q}]=i(1-\dot{q}^2)\overline{H}^{-1}=\frac{i}{m}(1-\dot{q}^2)^{3/2}\,. \tag{3.17}$$

We now have all the ingredients to discuss the Lorentz transformation of q and $\dot{q}$. We define the Lorentz-transformed $\dot{q}$ by

$$\dot{q}(\theta)=e^{i\theta\overline{M}_{x0}}\dot{q}e^{-i\theta\overline{M}_{x0}}\,, \tag{3.18}$$

where θ is related to the boosting velocity v by the formula

$$\tanh\theta=v\,. \tag{3.19}$$

Then

$$\frac{\partial}{\partial\theta}\dot{q}(\theta)=e^{i\theta\overline{M}_{x0}}i[\overline{M}_{x0},\dot{q}]e^{-i\theta\overline{M}_{x0}} \tag{3.20}$$

and, using Eqs. (3.7) and (3.17),

$$i[\overline{M}_{x0},\dot{q}]=1-\dot{q}^2\,.$$

Therefore,

$$\frac{\partial}{\partial\theta}\dot{q}(\theta)=1-\dot{q}^2(\theta) \tag{3.21}$$

and

$$\dot{q}(\theta)=\tanh(A+\theta)\,. \tag{3.22}$$

The operator A is defined by

$$A=\tanh^{-1}\dot{q}\,. \tag{3.23}$$

Next, define

$$q(\theta)=e^{i\theta\overline{M}_{x0}}qe^{-i\theta\overline{M}_{x0}}=q_0(\theta)+t\dot{q}(\theta)\,, \tag{3.24}$$

where

$$q_0(\theta)=e^{i\theta\overline{M}_{x0}}q_0e^{-i\theta\overline{M}_{x0}} \tag{3.25}$$

and q_0 was defined by Eq. (2.22):

$$q_0=q(t=0)\,.$$

From (3.7) and (3.17) we get

$$i[\overline{M}_{x0},q]=\frac{1}{m}(1-\dot{q}^2)^{3/2}\frac{\partial\overline{F}}{\partial\dot{q}}+t-q\dot{q}\,,$$

and therefore

$$i[\overline{M}_{x0},q_0]=\frac{1}{m}(1-\dot{q}^2)^{3/2}\frac{\partial\overline{F}}{\partial\dot{q}}-\dot{q}q_0\,. \tag{3.26}$$

Then Eq. (3.25) leads to

$$\frac{\partial}{\partial\theta}q_0(\theta)=\frac{1}{m}[1-\dot{q}^2(\theta)]^{3/2}\left(\frac{\partial\overline{F}}{\partial\dot{q}}\right)_\theta-\dot{q}(\theta)q_0(\theta)\,, \tag{3.27}$$

where

$$\left(\frac{\partial\overline{F}}{\partial\dot{q}}\right)_\theta\equiv\left(\frac{\partial\overline{F}}{\partial\dot{q}}\right)_{\dot{q}=\dot{q}(\theta)}\,.$$

We also have

$$\frac{\partial\overline{F}(\dot{q}(\theta))}{\partial\theta}=\left(\frac{\partial\overline{F}}{\partial\dot{q}}\right)_\theta\frac{\partial\dot{q}(\theta)}{\partial\theta}=[1-\dot{q}^2(\theta)]\left(\frac{\partial\overline{F}}{\partial\dot{q}}\right)_\theta$$

and we can rewrite (3.27) as

$$\frac{\partial}{\partial\theta}q_0(\theta)=\frac{1}{m}[1-\dot{q}^2(\theta)]^{1/2}\frac{\partial\overline{F}(\dot{q}(\theta))}{\partial\theta}-\dot{q}(\theta)q_0(\theta)\,. \tag{3.28}$$

Using the formulas [see Eq. (3.22)]

$$[1-\dot{q}^2(\theta)]^{1/2}=\frac{1}{\cosh(A+\theta)},$$

$$\frac{\partial}{\partial\theta}\left[\frac{\overline{F}(\dot{q}(\theta))}{\cosh(A+\theta)}\right]=\frac{1}{\cosh(A+\theta)}\frac{\partial\overline{F}(\dot{q}(\theta))}{\partial\theta}-\frac{\overline{F}(\dot{q}(\theta))}{\cosh(A+\theta)}\tanh(A+\theta)$$
$$=[1-\dot{q}^2(\theta)]^{1/2}\frac{\partial\overline{F}(\dot{q}(\theta))}{\partial\theta}-\frac{\overline{F}(\dot{q}(\theta))}{\cosh(A+\theta)}\ddot{q}(\theta),$$

we can integrate (3.28)

$$q_0(\theta)=\frac{1}{\cosh(A+\theta)}\left[B+\frac{1}{m}\overline{F}(\dot{q}(\theta))\right], \tag{3.29}$$

where the operator B is independent of t and θ and is given by

$$B=\cosh A q_0-\frac{1}{m}\overline{F}(\dot{q})=\frac{1}{(1-\dot{q}^2)^{1/2}}q_0-\frac{1}{m}\overline{F}(\dot{q}). \tag{3.30}$$

The operator B has the following curious property: From Eqs. (3.7), (3.9), and (3.13) we obtain

$$\overline{M}_{x0}=\overline{F}(\dot{q})+t(P-\dot{q}\overline{H})-q_0\overline{H}$$
$$=\overline{F}(\dot{q})-\frac{mq_0}{(1-\dot{q}^2)^{1/2}},$$

and comparision with (3.30) gives

$$\overline{M}_{x0}=-mB. \tag{3.31}$$

Equations (3.22), (3.24), and (3.29) determine the Lorentz-transformation properties of q, $\dot{q}$ completely.

Compared to the transformation law for $\dot{q}(\theta)$, the transformation of $q_0(\theta)$ is quite complicated. This suggests that there may exist a time-independent canonical transformation leading to a new quantum coordinate with simpler Lorentz-transformation properties. This is indeed the case; define the *covariant quantum coordinate* $\hat{q}$ by

$$\hat{q}=q-\frac{1}{m}(1-\dot{q}^2)^{1/2}\overline{F}(\dot{q})=q-\overline{H}^{-1}\overline{F}(\dot{q}). \tag{3.32}$$

Since

$$[\hat{q},P]=[q,P], \tag{3.33}$$

the transformation $q\to\hat{q}$ is canonical. We also have

$$[\hat{q},H]=[q,H], \tag{3.34}$$

$$\dot{\hat{q}}=\dot{q}. \tag{3.35}$$

The Lorentz-transformation law for $\hat{q}$ is quite simple:

$$\hat{q}(\theta)=\hat{q}_0(\theta)+i\dot{q}(\theta)t, \tag{3.36}$$

$$\hat{q}_0(\theta)=\frac{B}{\cosh(A+\theta)}, \tag{3.37}$$

$$\dot{q}(\theta)=\tanh(A+\theta). \tag{3.38}$$

The constant operators A and B are given by Eqs. (3.23) and (3.30):

$$A=\tanh^{-1}\dot{q}, \tag{3.39}$$

$$B=\frac{1}{(1-\dot{q}^2)^{1/2}}\hat{q}_0. \tag{3.40}$$

Another remarkable result is the following: By definition,

$$\left\langle 0\left|\int dx(x+q)T_{00}\right|0\right\rangle=\overline{F}(\dot{q})=\overline{F}(\dot{q})\overline{H}^{-1}\overline{H}$$
$$=(q-\hat{q})\overline{H}. \tag{3.41}$$

Equation (3.32) was used in the last step. Therefore,

$$\left\langle 0\left|\int dx(x+\hat{q})T_{00}\right|0\right\rangle=\left\langle 0\left|\int dx(x+q)T_{00}\right|0\right\rangle+(\hat{q}-q)\overline{H}$$
$$=0. \tag{3.42}$$

This implies that, if we write the Lorentz generator as

$$\overline{M}_{x0}=\int dx(x+\hat{q})T_{00}+tP-\hat{q}\overline{H}, \tag{3.43}$$

we have

$$\overline{M}_{x0}=tP-\hat{q}\overline{H}. \tag{3.44}$$

We can also rewrite Eqs. (3.36)–(3.38) in a suggestive form by introducing

$$\gamma=\frac{1}{(1-v^2)^{1/2}}=\cosh\theta. \tag{3.45}$$

Then

$$\hat{q}(\theta)=\frac{1}{\gamma[1+v\dot{q}]}[\hat{q}_0+\gamma(v+\dot{q})t]. \tag{3.46}$$

This has the familiar form of the Lorentz transformation of the coordinate of a particle in uniform motion with velocity $\dot{q}$ (i.e., $\hat{q}=\hat{q}_0+\dot{q}t$). It shows that $\dot{q}$ is not related to the velocity of the coordinate system; the latter is given by v.

It is clear from the above that the covariant quantum coordinate is the natural quantum coordinate for relativistic systems. Starting from any original choice of q, $\hat{q}$ can be calculated by means of the formulas

$$\hat{q}=q-\frac{1}{m}(1-\dot{q}^2)^{1/2}\overline{F}(\dot{q}),$$
$$\overline{F}(\dot{q})=\left\langle 0\left|\int dx(x+q)T_{00}\right|0\right\rangle.$$

IV. THE GENERALIZED COORDINATE

We saw in Sec. III that the existence of a conserved Lorentz generator has far-reaching consequences: It determines completely the dependence of P and $\overline{H}$ on $\dot{q}$. In this section, we show that it also fixes uniquely the way in which q and $\dot{q}$ appear in

$$\langle 0|\psi^f|0\rangle = \phi^f(x+q,\dot{q})\,. \tag{4.1}$$

Under a Lorentz transformation with parameter θ,

$$x \to x' = x\cosh\theta + t\sinh\theta, \tag{4.2a}$$

$$t \to t' = x\sinh\theta + t\cosh\theta\,, \tag{4.2b}$$

and

$$\phi^f(x'+q(t'),\dot{q}) = \langle 0|e^{i\theta M_{x0}}\psi^f e^{-iM_{x0}}|0\rangle\,. \tag{4.3}$$

Since a Lorentz transformation should not change the total number of particles,

$$\langle 0|e^{i\theta M_{x0}}\psi^f e^{-i\theta M_{x0}}|0\rangle = e^{i\theta\overline{M}_{x0}}\phi^f e^{-i\theta\overline{M}_{x0}} = \phi^f(x+q(\theta,t),q(\theta)), \tag{4.4}$$

where $q(\theta,t)$ and $\dot{q}(\theta)$ are the transformed $q(t)$ and $\dot{q}$, given explicitly in Eqs. (3.22), (3.24), and (3.29). If we combine (4.3) and (4.4), we obtain the condition

$$\phi^f(x'+q(t'),\dot{q}) = \phi^f(x+q(\theta,t),\dot{q}(\theta))\,. \tag{4.5}$$

This puts severe restrictions on the dependence of ϕ^f on x, q, and $\dot{q}$. It implies that

$$\phi^f = \phi^f(X)\,,$$

where X is a function of $x+q$, $\dot{q}$ with the property

$$X(x'+q(t'),q) = X(x+q(\theta,t),\dot{q}(\theta))\,. \tag{4.6}$$

Let us assume that X has the general form

$$X = L(q)(x+q) + M(\dot{q})t + N(\dot{q}) = L(\dot{q})(x+q_0) + [M(\dot{q}) + \dot{q}L(\dot{q})]t + N(\dot{q})\,, \tag{4.7}$$

where L, M, and N are arbitrary functions of $\dot{q}$ (they cannot depend on q, since q must appear always in the combination $x+q$). Strictly speaking, the t dependence of ϕ^f can only come from $q(t)$, and there is no need for the $M(\dot{q})$ term in (4.7); by including it, we generalize (4.6) to include the possibility that X depends explicitly on t.

Under an infinitesimal transformation,

$$\begin{aligned} &X(x',q(t'),q) = X(x,q(t),q) + \theta\delta_1 X\,,\\ &\delta_1 X = L(\dot{q})t + [M(\dot{q}) + \dot{q}L(\dot{q})]x\,. \end{aligned} \tag{4.8}$$

Similarly, using (3.22), (3.24), and (3.29), we find that

$$\begin{aligned} &X(x+q(\theta,t),\dot{q}(\theta)) = X(x+q(t),\dot{q}) + \theta\delta_2 X\,,\\ &\delta_2 X = (1-\dot{q}^2)\Big[L'(x+q_0) + (L+L'\dot{q}+M')t + N' + \frac{1}{m}(1-\dot{q}^2)^{1/2}\overline{F}'L - \frac{\dot{q}}{1-\dot{q}^2}q_0 L\Big]\,, \end{aligned} \tag{4.9}$$

where, e.g.,

$$L' \equiv \frac{\partial L(\dot{q})}{\partial\dot{q}}\,.$$

According to (4.6),

$$\delta_1 X = \delta_2 X\,.$$

Since this should be true identically for x, t, and q_0, it leads to the differential equations

$$L'(\dot{q}) = \frac{\dot{q}}{1-\dot{q}^2}L(\dot{q})\,, \tag{4.10a}$$

$$(1-\dot{q}^2)L'(\dot{q}) = M(\dot{q}) + \dot{q}L(\dot{q})\,, \tag{4.10b}$$

$$(1-\dot{q}^2)[\dot{q}L'(\dot{q}) + L(\dot{q}) + M'(\dot{q})] = L(\dot{q})\,, \tag{4.10c}$$

$$N'(\dot{q}) + \frac{1}{m}(1-\dot{q}^2)^{1/2}\overline{F}'(\dot{q})L(\dot{q}) = 0 \tag{4.10d}$$

The initial conditions are provided by the requirement

$$X|_{q=\dot{q}=0} = x\,. \tag{4.11}$$

They are

$$L(0) = 1\,, \tag{4.12a}$$

$$M(0) = N(0) = 0\,. \tag{4.12b}$$

The system of equations (4.10) with these initial conditions has the unique solution

$$L(\dot{q}) = \frac{1}{(1-\dot{q}^2)^{1/2}}\,,$$

$$M(\dot{q}) = 0\,,$$

$$N(\dot{q}) = -\frac{1}{m}\overline{F}(\dot{q})\,.$$

Therefore, X is determined to be

$$X = \frac{1}{(1-\dot{q}^2)^{1/2}}(x+q) - \frac{1}{m}\overline{F}(\dot{q})\,. \tag{4.13}$$

The condition (4.6) uniquely determines X; even if the initial conditions (4.11) had not been imposed, X would have been fixed up to an irrelevant overall multiplicative constant. Equation (4.6) provides another example of the concept of CQ transmutation introduced in Sec. II.

Thus invariance arguments restrict the functional form of ϕ^f to a remarkable extent. *A priori*, ϕ^f could be a function of three variables, x, q, $\dot{q}$. Translational invariance implies that it is a function of two variables only, $x+q$ and $\dot{q}$. Finally, Lorentz invariance specifies that it is a function of the single variable X. We will call this variable the *generalized coordinate*.

As a result of this analysis, we have the following prescription for including the quantum coordinate in ϕ^f: (i) Compute $\phi^f(x)$ by ignoring the quantum coordinate, (ii) replace $x \to X$ everywhere, and (iii) expand in q, $\dot{q}$ and symmetrize fully with respect to q, $\dot{q}$. This symmetrization rule, which can be proved on general grounds, allows us to ignore questions of ordering between q and $\dot{q}$ in this section; such questions never arise.

What can we say about the dependence of the Heisenberg field on q and $\dot{q}$? This question is partially answered by the results of this section: The inclusion of the quantum coordinate necessitates the replacement $x \to \tilde{X}$ everywhere, where

$$\tilde{X} = X + (\text{terms depending on } \chi^0)\,. \tag{4.14}$$

Since ψ^f carries the prescription of normal ordering with respect to χ^0, the second term in (4.14) is consistent with the result of this section. A more complete specification of the dependence of ψ^f on $(q, \dot{q})$ will be given in Sec. V.

Finally, let us note that, if the covariant quantum coordinate $\hat{q}$ is used instead of q, Eqs. (3.32) and (4.13) give

$$X = \frac{1}{(1-\dot{q}^2)^{1/2}}(x+\hat{q})\,. \tag{4.15}$$

The generalized coordinate in this form was found in Ref. 6 starting from different considerations, although the fact that $\hat{q}$ must be the covariant quantum coordinate was not realized there.

The simple expression (4.15) shows again that $\hat{q}$ is the natural choice for quantum coordinate. We shall use it exclusively in the sequel, and for notational convenience we will drop the caret: *q from now on stands for the covariant quantum coordinate.*

V. THE GENERALIZED COORDINATE T

The main result of Sec. IV was that the inclusion of the quantum coordinate has the effect of replacing x in ψ^f by the generalized coordinate X (up to χ^0-dependent terms). The question naturally arises: Is there also a generalized coordinate T that replaces t? This section is devoted to the investigation of this problem.

It is clear that an answer to the above equation cannot be found if one stays within the vacuum sector of $\mathcal{H}_\chi$, since for static extended objects ϕ^f has no explicit t dependence. We are led therefore to examine the one-particle sector of $\mathcal{H}_\chi$. The tree approximation will be used throughout this and the following sections.

Let us first recall how the one-particle states are constructed.[5] If the Heisenberg field before the boson transformation obeys the equation of motion,

$$(-\partial^2 - \mu^2)\psi(x) = F[\psi(x)]\,, \tag{5.1}$$

$$\partial^2 = \partial_t^{\,2} - \partial_x^{\,2}\,, \tag{5.2}$$

then according to the boson transformation theorem,

$$(-\partial^2 - \mu^2)\psi^f(x) = F[\psi^f(x)]\,. \tag{5.3}$$

In the tree approximation, where contractions among physical fields are ignored, this leads to the following equation for $\phi^f(x)$:

$$(-\partial^2 - \mu^2)\phi^f(x) = F[\phi^f(x)]\,. \tag{5.4}$$

Now recall Eq. (2.2):

$$\psi^f(x) = \sum_n \frac{1}{n!} : \psi_f^{(n)}(x):\,,$$

$$\psi_f^{(n)}(x) = (\delta_f)^n \phi^f(x)\,,$$

$$\delta_f \equiv \int d\sigma\, \phi^0(\sigma) \frac{\delta}{\delta f(\sigma)}\,.$$

Since (5.4) is valid for all $f(x)$, and both $f(x)$ and $\phi^0(x)$ satisfy

$$(-\partial^2 - \mu^2)f(x) = (-\partial^2 - \mu^2)\phi^0(x) = 0\,,$$

we can deduce from (5.4) that

$$\begin{aligned}(-\partial^2 - \mu^2)\psi_f^{(1)}(x) &= \delta_f F[\phi^f(x)] \\ &= V(X)\psi_f^{(1)}(x)\,,\end{aligned} \tag{5.5}$$

where

$$V[X] \equiv \frac{\partial}{\partial \phi^f(x)} F[\phi^f(x)] \tag{5.6}$$

is the self-consistent potential induced by the extended object. In Eq. (5.6) we used the fact that ϕ^f is a function of the generalized coordinate only.

For an extended object of finite size, the potential $V(X)$ will have a finite range R (in the one-dimensional models under consideration, R is of the order of μ^{-1}). Thus Eq. (5.5) will have scattering solutions, which we denote by χ^0:

$$[-\partial^2 - \mu^2 - V(X)]\chi^0(x+q, t, \dot{q}) = 0\,. \tag{5.7}$$

They are characterized by the boundary condition that, for $|x| \gg R$,

$$\chi^0(x+q, t, q) = \frac{1}{\sqrt{2\pi}} \int \frac{dK}{\sqrt{2W_K}} (e^{iK(x+q) - iW_K t}\alpha_K + \text{H.c.})\,. \tag{5.8}$$

In this expression, α_K are the annihilation operators for particle quanta in mode K, and W_K, K are functions of $\dot{q}$ satisfying the condition

$$(W_K + \dot{q}K)^2 - K^2 = \mu^2\,. \tag{5.9}$$

This condition is obtained by substituting (5.8) in (5.7) with $V=0$.

We introduce the generalized time coordinate T by the requirement that all of the $(q,\dot q)$ dependence in (5.8) should come from X and T:

$$K(x+q)-W_K t=kX-w_k T \tag{5.10}$$

and

$$T|_{q=\dot q=q}=t . \tag{5.11}$$

Since we also have

$$X|_{q=\dot q=0}=x , \tag{5.12}$$

it follows from Eq. (5.10) that

$$K(\dot q=0)=k , \tag{5.13a}$$

$$W_K(\dot q=0)=w_k , \tag{5.13b}$$

and from Eq. (5.9)

$$w_k^2-k^2=\mu^2 . \tag{5.14}$$

If we assume that T is a linear function of $(x+q)$, and t,

$$T=A(\dot q)(x+q)+B(\dot q)t+C(\dot q) ,$$

the requirements (5.9)–(5.11) and (5.14) fix the coefficient functions $A(\dot q)$, $B(\dot q)$, and $C(\dot q)$ uniquely and we obtain

$$T=(1-\dot q^2)^{1/2}t+\frac{\dot q}{(1-\dot q^2)^{1/2}}(x+q) . \tag{5.15}$$

We also find that

$$K=\frac{k-\dot q w_k}{(1-\dot q^2)^{1/2}} , \tag{5.16}$$

$$W_K=(1-\dot q^2)^{1/2}w_k , \tag{5.17}$$

from which we deduce

$$\frac{\partial(K,W_K)}{\partial(k,w_k)}=1 . \tag{5.18}$$

The pair of generalized coordinates (X,T) have the following important properties:

$$(\partial_t^2-\partial_x^2)f(x+q,t,\dot q)=(\partial_T^2-\partial_X^2)f(x+q,t,\dot q) , \tag{5.19}$$

$$\frac{\partial(X,T)}{\partial(x,t)}=1 . \tag{5.20}$$

In Eq. (5.19), f is an arbitrary function of the indicated arguments. Equation (5.19) allows us to rewrite (5.7) as

$$[\partial_T^2-\partial_X^2+\mu^2+V(X)]\chi^0=0 . \tag{5.21}$$

Therefore the assumption that χ^0 depends on $(q,\dot q)$ only through (X,T) yields a solution of this equation. Similarly, (5.3) can be written as

$$(-\partial_T^2+\partial_X^2-\mu^2)\psi^f=F[\psi^f] ,$$

and this has a solution of the form

$$\psi^f(x+q,t,\dot q)=\psi^f(X,T) . \tag{5.22}$$

We will assume in the following that (5.22) is the physically correct solution, at least in the tree approximation. Several consistency checks of (5.22) will be made in the remainder of this section.

Notice also that Eq. (5.8) can be written as

$$\chi^0(X,T)=\frac{1}{\sqrt{2\pi}}\int\frac{dk}{\sqrt{2w_k}}(e^{i(kX-w_kT)}\alpha_k+\text{H.c.}) \tag{5.23}$$

for $|x|\gg R$. Here α_K is denoted by α_k, since there is a one-to-one correspondence between K and k [cf. (5.16) and (5.17)]; Eq. (5.18) was also used in deriving (5.23).

It should be noted that (5.22) does not imply that q and $\dot q$ occur through X and T in all field operators; operators other than ψ^f can have additional $q,\dot q$ dependence. For example,

$$\frac{\partial}{\partial x}\psi^f=\frac{\partial X}{\partial x}\partial_X\psi^f+\frac{\partial T}{\partial x}\partial_T\psi^f$$
$$=\frac{1}{(1-\dot q^2)^{1/2}}(\partial_X+\dot q\partial_T)\psi^f(X,T) .$$

We now turn to the consequences of the ansatz (5.22). Consider first the matrix element (for $|x|\gg R$)

$$\langle 0|\psi^f(X,T)|k\rangle=\frac{Z^{1/2}}{(2\pi 2w_k)^{1/2}}e^{i(kX-w_xT)} , \tag{5.24}$$

where Z is a wave-function renormalization constant and $|k\rangle$ is the state of a single particle in mode k. The generator of time translations is the total Hamiltonian, which can be written as

$$H=\bar H+H'(\dot q,\chi^0) . \tag{5.25}$$

H' represents the contribution of χ^0 particles to the energy. Since $\bar H$ depends only on $\dot q$, which commutes with χ^0,

$$[\bar H,H']=0 . \tag{5.26}$$

H' annihilates the $\mathcal{H}_\chi$ vacuum.

Let us also define the time-translated X and T:

$$X_a=\frac{1}{(1-\dot q^2)^{1/2}}[x+q(t+a)]=X+\frac{a\dot q}{(1-\dot q^2)^{1/2}} , \tag{5.27}$$

$$T_a=\frac{\dot q}{(1-\dot q^2)^{1/2}}[x+q(t+a)]+(1-\dot q^2)^{1/2}(t+a)$$
$$=T+\frac{a}{(1-\dot q^2)^{1/2}} . \tag{5.28}$$

Then we obtain from (5.24)

$$\langle 0|e^{iHa}\psi^f(X,T)e^{-iHa}|k\rangle=\frac{Z^{1/2}}{(2\pi 2w_k)^{1/2}}e^{i(kX_a-w_kT_a)} .$$

which leads to

$$e^{i\overline{H}a}e^{i(kX-w_kT)}e^{-i\overline{H}a}e^{-iE'_k a}=e^{i(kX_a-w_kT_a)}\,. \tag{5.29}$$

Here $H'|k\rangle=E'_k|k\rangle$. Since

$$i[\overline{H},X]=\frac{\dot q}{(1-\dot q^2)^{1/2}}\,,$$

$$i[\overline{H},T]=\frac{\dot q^2}{(1-\dot q^2)^{1/2}}\,,$$

we have

$$\begin{aligned}e^{i\overline{H}a}(kX-w_kT)e^{-i\overline{H}a}&=k\left[X+\frac{a\dot q}{(1-\dot q^2)^{1/2}}\right]\\&\quad-w_k\left[T+\frac{a\dot q^2}{(1-\dot q^2)^{1/2}}\right]\\&=kX_a-w_k[T_a-a(1-\dot q^2)^{1/2}]\end{aligned}$$

and

$$e^{i\overline{H}a}e^{i(kX-w_kT)}e^{-i\overline{H}a}=e^{ikX_a-iw_k[T_a-a(1-\dot q^2)^{1/2}]}\,. \tag{5.30}$$

Comparison of (5.30) and (5.29) yields the result that, in the tree approximation,

$$H'=(1-\dot q^2)^{1/2}H_\chi\,, \tag{5.31}$$

where

$$H_\chi=\int dk\,w_k\alpha_k^\dagger\alpha_k+\sum_i w_i\alpha_i^\dagger\alpha_i^\dagger\,. \tag{5.32}$$

The sum on the right-hand side of this relation runs over the bound-state solutions of Eq. (5.5).

Equation (5.31) is in agreement with Eq. (5.17). It shows how the energies of χ^0 quanta are influenced by the quantum fluctuations of the extended object: The energy of mode k is given by $(1-\dot q^2)^{1/2}w_k$.

We can also derive Eq. (5.31) by purely algebraic means as follows: from

$$\begin{aligned}[\psi^f,H]&=i\frac{\partial}{\partial t}\psi^f=i\left(\frac{\partial T}{\partial t}\partial_T+\frac{\partial X}{\partial t}\partial_X\right)\psi^f\\&=\frac{i}{(1-\dot q^2)^{1/2}}(\partial_T+\dot q\partial_X)\psi^f\end{aligned} \tag{5.33}$$

and

$$\begin{aligned}[\psi^f,\overline{H}]&=[q,\overline{H}]\frac{\partial}{\partial q}\psi^f=i\dot q\frac{\partial}{\partial x}\psi^f\\&=i\dot q\left(\frac{\partial X}{\partial x}\partial_X+\frac{\partial T}{\partial x}\partial_T\right)\psi^f\\&=i\frac{\dot q}{(1-\dot q^2)^{1/2}}(\partial_X+\dot q\partial_T)\psi^f\,,\end{aligned} \tag{5.34}$$

we obtain

$$[\psi^f,H']=i(1-\dot q^2)^{1/2}\partial_T\psi^f \tag{5.35}$$

with H' defined in Eq. (5.25).

When the quantum coordinate is ignored,

$$\psi^f(x,t)=\sum_n\frac{1}{n!}\int d\sigma_1\ldots d\sigma_n c_f(x,t;\sigma_1\ldots\sigma_n):\chi^0(\sigma_1)\ldots\chi^0(\sigma_n): \tag{5.36}$$

and for a static extended object, c_f depends on t and $t_{\sigma_1},\ldots,t_{\sigma_n}$ only through the differences $t-t_{\sigma_1},\ldots,t-t_{\sigma_n}$. Thus

$$\begin{aligned}i\frac{\partial}{\partial t}\psi^f(x,t)&=-i\sum_n\frac{1}{n!}\int d\sigma_1\ldots d\sigma_n\sum_i\left[\frac{\partial}{\partial t_{\sigma_i}}c_f(x,t;\sigma_1\ldots\sigma_n)\right]:\chi^0(\sigma_1)\ldots\chi^0(\sigma_n):\\&=i\sum_n\frac{1}{n!}\int d\sigma_1\ldots d\sigma_n c_f(x,t;\sigma_1\ldots\sigma_n)\sum_i:\chi^0(\sigma_1)\ldots\dot\chi^0(\sigma_i)\ldots\chi^0(\sigma_n):\\&=[\psi^f,H_\chi]\,.\end{aligned} \tag{5.37}$$

Since H_χ does not depend on t, the inclusion of the quantum coordinate (i.e., the replacement $x\to X$, $t\to T$) gives

$$i\partial_T\psi^f=[\psi^f,H_\chi] \tag{5.38}$$

and comparison with (5.35) leads again to (5.31). Finally, we consider the generator of Lorentz transformations

$$M_{x0}=\overline{M}_{x0}+M'_{x0}\,, \tag{5.39}$$

where M'_{x0} contains the contribution of χ^0 quanta and annihilates the $\mathcal{H}_\chi$ vacuum. $\overline{M}_{x0}$ is given by Eq. (3.44).

Let us define

$$X'=(X)_{x\to x',\,t\to t'}\,,$$

$$T'=(T)_{x\to x',\,t\to t'}\,,$$

where x',t' are the Lorentz-transformed coordinates [see Eq. (4.2)]. Then, by steps similar to those that led to Eq. (5.29), we obtain the condition

$$\begin{aligned}\langle 0|e^{i(\overline{M}_{x0}+M'_{x0})\theta}e^{i(kX-w_kT)}\alpha_k e^{-i(\overline{M}_{x0}+M'_{x0})\theta}|k'\rangle\\=e^{i(kX'-w_kT')}\langle k|k'\rangle\,.\end{aligned} \tag{5.40}$$

α_k is the annihilation operator for state $|k\rangle$. We restrict ourselves to infinitesimal θ; then

$$X' = X + \theta \frac{t + \dot{q}x}{(1-\dot{q}^2)^{1/2}},$$

$$T' = T + \theta \frac{x + \dot{q}t}{(1-\dot{q}^2)^{1/2}},$$

and (5.40) gives

$$[\overline{M}_{x0}, e^{i(kX - w_kT)}]\langle k|k'\rangle - e^{i(kX-w_kT)}\langle k|M'_{x0}|k'\rangle = \frac{1}{(1-\dot{q}^2)^{1/2}}[k(t+\dot{q}x) - w_k(x+\dot{q}t)] \times e^{i(kX-w_kT)}\langle k|k'\rangle . \tag{5.41}$$

Notice that, since $e^{i(kX-w_kT)}$ came from the Fourier expansion of the field χ^0, it is understood to be fully symmetrized with respect to $q, \dot{q}$. Using Eqs. (3.44), (4.15), (5.15), and the commutation relation (3.17), we find that

$$[\overline{M}_{x0}, X] = -i\frac{t+\dot{q}x}{(1-\dot{q}^2)^{1/2}}, \tag{5.42}$$

$$[\overline{M}_{x0}, T] = -i\frac{x+\dot{q}t}{(1-\dot{q}^2)^{1/2}} - i(q-\dot{q}t)(1-\dot{q}^2)^{1/2}. \tag{5.43}$$

Then Eq. (5.41) reduces to

$$(q-\dot{q}t)(1-\dot{q}^2)^{1/2}w_k\langle k|k'\rangle + \langle k|M'_{x0}|k'\rangle = 0 .$$

Therefore,

$$M'_{x0} = -(q-\dot{q}t)(1-\dot{q}^2)^{1/2}H_\chi . \tag{5.44}$$

We have thus succeeded in determining explicitly all the generators of space-time transformations in the tree approximation. They are given by

$$H = \frac{m}{(1-\dot{q}^2)^{1/2}} + (1-\dot{q}^2)^{1/2}H_\chi , \tag{5.45}$$

$$M_{x0} = tP - \frac{mq}{(1-\dot{q}^2)^{1/2}} - (q-\dot{q}t)(1-\dot{q}^2)^{1/2}H_\chi . \tag{5.46}$$

H_χ is defined in Eq. (5.32).

If we compare (5.46) with (3.5),

$$M_{x0} = \int dx(x+q)T_{00} + tP - qH ,$$

we obtain the result

$$\int dx(x+q)T_{00} = \dot{q}t(1-\dot{q}^2)^{1/2}H_\chi . \tag{5.47}$$

This may be an artifact of the tree approximation. As a consistency check, we verify the commutation relations

$$[M_{x0}, P] = -iH , \tag{5.48}$$

$$[M_{x0}, H] = -iP . \tag{5.49}$$

Equation (5.48) follows immediately from (5.46). The second relation merits a little discussion. Reverting to the notation

$$\overline{H} = \frac{m}{(1-\dot{q}^2)^{1/2}},$$

we have

$$[M_{x0}, H] = [tP - q\overline{H}, \overline{H}] + [tP - q\overline{H}, 1-\dot{q}^2]H_\chi - [(q-\dot{q}t)(1-\dot{q}^2)^{1/2}, \overline{H}]H_\chi - [(q-\dot{q}t)(1-\dot{q}^2)^{1/2}, (1-\dot{q}^2)^{1/2}]H_\chi^2 . \tag{5.50}$$

We can easily compute that

$$[tP - q\overline{H}, \overline{H}] = -i\dot{q}\overline{H} = -iP , \tag{5.51a}$$

$$[tP - qH, (1-\dot{q}^2)^{1/2}] = i\dot{q}(1-\dot{q}^2)^{1/2}, \tag{5.51b}$$

$$[(q-\dot{q}t)(1-\dot{q}^2)^{1/2}, \overline{H}] = i\dot{q}(1-\dot{q})^{1/2} . \tag{5.51c}$$

The last term in Eq. (5.50) must be dropped in our approximation. The reason is that H_χ contains w_k, and therefore is of order h (h is the Planck constant). Since we calculated M_{x0} and H to this order only, it is not consistent to keep a term proportional to H_χ^2, which is of order h^2. Then combining (5.50) with Eqs. (5.51) we verify that Eq. (5.49) is satisfied. The discussion connected with Eq. (5.49) shows that the expressions for H and M_{x0} given by Eqs. (5.45) and (5.46) are not expected to hold beyond the tree approximation.

Next, we examine the change of the one-particle states under a Lorentz transformation. Using (5.44), we find that

$$e^{i\theta M'_{x0}}|k\rangle = e^{i\theta(q-\dot{q}t)(1-\dot{q}^2)^{1/2}w_k}|k\rangle . \tag{5.52}$$

Thus the transformed state differs from the original one only by a state-dependent phase factor. This is reminiscent of our previous result, that the generator of space translations does not act on particle states. Both are manifestations of the rearrangement of space-time symmetries caused by the presence of the extended object. Equation (5.46) implies that H_χ is unchanged by Lorentz transformations; at least in the tree approximation

$$[M_{x0}, H_\chi] = 0 . \tag{5.53}$$

This gives us a better physical understanding of the meaning of the energy and momentum of quanta in the presence of the extended object: The coordinate system is fixed by the extended object, and coordinate transformations are implemented by transformations of variables characterizing the extended object (i.e., q and $\dot{q}$). The energy and momentum of particle quanta are to be understood relative to the extended object itself. For this reason the particle quanta do not contribute to the total momentum of the system. An example of this is a crystal, where it is well known that the phonons give no contribution to the total momentum.

VI. CONCLUSIONS AND EXTENSIONS

In the previous sections, we examined the consequences of space-time translation and Lorentz invariance for systems of interacting extended objects and particle quanta. We found that such invariance considerations are very powerful; let us list the main results here, in order of decreasing generality.

(i) The existence of a conserved generator of space translations $\vec{P}$ implies that the set of physical observables includes a quantum coordinate $\vec{q}$, which is canonically conjugate to $\vec{P}$. The translation $\vec{x} \to \vec{x} + \vec{a}$ is implemented by the operation $\vec{q} \to \vec{q} + \vec{a}$, and therefore $\vec{x}$ and $\vec{q}$ appear always in the combination $\vec{x} + \vec{q}$.

(ii) In $1+1$ dimensions, the existence of a conserved generator of Lorentz transformations completely determines the structure of the theory in the no-particle sector. The momentum and Hamiltonian are functions of $\dot{q}$ only, and their dependence on $\dot{q}$ is that of relativistic point particle. The Lorentz generator is also completely determined as a function of q and $\dot{q}$. Finally, the order parameter (i.e., the expectation value of the Heisenberg field operator in the particle vacuum) depends on x, q, and $\dot{q}$ only through the generalized space coordinate X.

(iii) In $1+1$ dimensions and in the tree approximation, we found a consistent solution of the theory in the one-particle sector. The Heisenberg field operator depends on x, t, q, and $\dot{q}$ only through the generalized space and time coordinates X and T. The momentum, Hamiltonian, and Lorentz generator were constructed explicitly.

The considerations of this paper can be extended in various directions.

(i) One can consider models in $3+1$ dimensions. Although the arguments become more complicated, the techniques developed here allow us to establish an important result, which we present here.

According to Sec. II, there appears a quantum coordinate $\vec{q}$ canonically conjugate to $\vec{P}$:

$$[q_i, P_j] = i\delta_{ij} \,. \tag{6.1}$$

If the q_i are the only quantum coordinates,

$$\overline{H} \equiv \langle 0|H|0\rangle = \overline{H}(\vec{P})$$

and the $\dot{q}_i$ are time independent and commute among themselves. The commutator $[q_i, \dot{q}_j]$ is obtained from the generator of Lorentz transformations by the method of Sec. III: Denoting by a bar the vacuum expectation value,

$$\begin{aligned} &\overline{M}_{i0} = \overline{F}_i(\dot{\vec{q}}) \quad tP_i - q_i\overline{H} \,, \\ &\overline{F}_i(\dot{q}) = \left\langle 0 \left| \int d^3x (x+q)_i T_{00} \right| 0 \right\rangle . \end{aligned} \tag{6.2}$$

Then the time independence of $\overline{M}_{i0}$ implies that

$$P_i = \dot{q}_i \overline{H} \tag{6.3}$$

and substitution of this result in (6.1) gives

$$[q_i, \dot{q}_j] = i(\delta_{ij} - \dot{q}_i\dot{q}_j)\overline{H}^{-1} \,. \tag{6.4}$$

The dependence of $\overline{H}$ on $\dot{\vec{q}}$ can be obtained from

$$i\dot{q}_i = [q_i, \overline{H}] = \sum_j [q_i, \dot{q}_j] \frac{\partial \overline{H}}{\partial \dot{q}_j} \,. \tag{6.5}$$

Using (6.4), this gives

$$\dot{q}_i = \sum_j (\delta_{ij} - \dot{q}_i\dot{q}_j) \frac{\partial \ln \overline{H}}{\partial \dot{q}_j}$$

or

$$\sum_j \dot{q}_j \frac{\partial \ln \overline{H}}{\partial \dot{q}_j} = \frac{\dot{\vec{q}}^{\,2}}{1 - \dot{\vec{q}}^{\,2}} \,. \tag{6.6}$$

The only analytic solutions of this equation are spherically symmetric

$$\overline{H} = \overline{H}(\dot{\vec{q}}^{\,2}) \,. \tag{6.7}$$

In fact, we have the explicit result

$$\overline{H} = \frac{m}{(1 - \dot{\vec{q}}^{\,2})^{1/2}} \,. \tag{6.8}$$

Therefore, we get the following theorem: *If $\vec{q}$ is the only quantum coordinate, the extended object is spherically symmetric and behaves like a point particle.* More complicated extended objects will require other quantum coordinates besides $\vec{q}$.

(ii) One can explain nonrelativistic models, and replace Lorentz invariance by Galilean invariance. In $1+1$ dimensions, we expect that the techniques developed in this paper will be as powerful as in the relativistic case. This extension of our work could be physically very interesting, since there are indications that one-dimensional systems with solitons may occur in organic conductors.

(iii) One can extend the work presented here beyond the tree approximation. Although precise results in this direction are still missing, one general feature seems to emerge from the calculations presented in this paper. In Sec. II, we defined the quantum coordinate $q(t)$ by Eq. (2.18):

$$q(t) = e^{i\overline{H}t} q(0) e^{-i\overline{H}t} \,.$$

However, for the fully interacting system of quanta + extended object, the time developed should be governed by the full Hamiltonian H:

$$Q(t)=e^{iHt}q(0)e^{-iHt}\,. \tag{6.9}$$

Although the complete H is not known, we have a partial result in the tree approximation [Eq. (5.45)]:

$$H=\overline{H}+(1-\dot{q}^2)^{1/2}H_x\,. \tag{6.10}$$

This indicates that, when the quantum corrections are properly included, the Hamiltonian contains χ^0-dependent terms with coefficients depending on P. Then the full quantum coordinate, defined through Eq. (6.9), will contain terms depending on the creation and annihilation operators of χ^0. In particular,

$$\dot{Q}=i[H,Q]=\dot{Q}(P,\chi^0) \tag{6.11}$$

and the fluctuations in position of the extended object will be influenced by the χ^0 quanta present in the state. Physically, this is quite reasonable: It corresponds to a kind of Brownian motion of the extended object. It also restores a symmetry in the interaction between quanta and extended object; we saw earlier that the energy of the quanta is influenced by the presence of the extended object. It is not known at present whether (6.11) leads to a linear time dependence for Q, i.e., whether $\ddot{Q}=0$.

Work along the three lines of research suggested above is now in progress.

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[1]See, for example, review articles: S. Coleman, in *New Phenomena in Subnuclear Physics*, proceedings of the 14th Course of the International School of Subnuclear Physics, 1975, edited by A. Zichichi (Plenum, New York, 1977); R. Jackiw, Rev. Mod. Phys. 49, 681 (1977).

[2]J. L. Gervais and B. Sakita, Phys. Rev. D 11, 2943 (1975); J. L. Gervais, A. Jevicki, and B. Sakita, *ibid.* 12, 1038 (1975); C. G. Callan, Jr. and D. J. Gross, Nucl. Phys. B93, 29 (1975); J. L. Gervais and A. Jevicki, *ibid.* B110, 93 (1975).

[3]L. Leplae, F. Mancini, and H. Umezawa, Phys. Rev. B 2, 3594 (1970); H. Matsumoto, N. J. Papastamatiou, and H. Umezawa, Nucl. Phys. B82, 45 (1974); B97, 90 (1975).

[4]H. Matusmoto, P. Sodano, and H. Umezawa, Phys. Rev. D 19, 511 (1979).

[5]H. Matsumoto, G. Oberlechner, M. Umezawa, and H. Umezawa, J. Math. Phys. 20, 2088 (1979).

[6]H. Matusmoto, G. Semenoff, H. Umezawa, and M. Umezawa, J. Math. Phys. 21, 1761 (1980).

[7]This is also known as the collective coordinate. See, for example, articles in Refs. 2 and G. Wentzel, Helv. Phys. Acta 13, 269 (1940); D. Bohm and D. Pines, Phys. Rev. 92, 609 (1953); N. H. Christ and T. D. Lee, Phys. Rev. D 12, 1606 (1975).