SU(3) monopole with magnetic quantum numbers (0,2)

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(Received 4 August 1980)

We prove the existence of a spherically symmetric finite-energy SU(3) solution with magnetic charge 0 and isomagnetic charge 2. Because of the special ansatz this solution cannot satisfy the Bogomolny equations in the Prasad-Sommerfield limit. At the origin the solution is given in the form of a convergent asymptotic series. At infinity we give convergent asymptotic expansions in the Prasad-Sommerfield limit.

I. INTRODUCTION

There are several reasons for the interest in monopoles: Monopoles are in some sense particle-like objects. They resurrect the concept of the Dirac monopole[1] in a new attractive form. And they may even be relevant to color confinement in quantum chromodynamics (QCD).[2] In spite of their importance and the large amount of work which went into this field of research (compare, e.g., Goddard and Olive's review article[3]) only a few regular monopoles are known. To the author's knowledge these consist only of the 't Hooft-Polyakov monopole,[4] its generalizations[5] which lead essentially to the same equations for the regularizing functions, and the solutions of Bais *et al.*[6]

In this paper I add another solution for a nonvanishing Higgs potential which differs essentially from the known ones because it does not satisfy the Bogomolny equations in the Prasad-Sommerfield limit of vanishing Higgs potential. This solution is the regularized version of a pointlike Higgs vacuum found by Corrigan *et al.*[7] Taking their ansatz, in Sec. II of this paper we only exploit the similarity with the 't Hooft-Polyakov solution to prove the existence of a finite-energy solution.

Because one does not know its exact analytic form we study its asymptotic behavior at the origin in Sec. III. In Sec. IV we are able to give convergent asymptotic expansions at infinity in the Prasad-Sommerfield limit where our proof does not apply. Because Kerner's arguments[8] are wrong in the case of the 't Hooft-Polyakov monopole, also in our case no quantization condition holds and we find a continuous spectrum of asymptotic solutions for $r\to\infty$. However, one does not know whether these asymptotic solutions match with finite-energy solutions for $r\to 0$ which in the 't Hooft-Polyakov case is only true for the Prasad-Sommerfield solution.[9]

II. MODEL AND EXISTENCE OF A (0,2) SOLUTION

The theory we are going to study is given by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} G^a_{\mu\nu} G_a^{\mu\nu} + \tfrac{1}{2} D_\mu \phi^a D^\mu \phi_a - V(\phi)\,, \tag{1}$$

where

$$G_{\mu\nu} = G^a_{\mu\nu}\lambda_a/2 = \partial_\mu W_\nu - \partial_\nu W_\mu - i[W_\mu, W_\nu]\,, \tag{2a}$$

$$D_\mu\phi = D_\mu\phi^a\lambda_a/2 = \partial_\mu\phi - i[W_\mu,\phi]\,, \tag{2b}$$

$$\mu,\nu,\ldots = 0,1,2,3,\quad a,b,\ldots = 1,\ldots,8$$

are the field strength and the covariant derivative, respectively, with Gell-Mann matrices λ_a and potentials $W_\mu = W^a_\mu \lambda_a/2$. The Higgs field self-interaction is taken to be

$$V(\phi) = p[1-2\,\mathrm{Tr}(\phi^2)]^2 + q[2\,\mathrm{Tr}(\phi^2) + 8\sqrt{3}\,\mathrm{Tr}(\phi^3) - 8\,\mathrm{Tr}(\phi^2)\,\mathrm{Tr}(\phi^2) + 24\,\mathrm{Tr}(\phi^4)]\,,\quad p,q>0\,, \tag{2c}$$

i.e., we choose the most general renormalizable possibility which breaks the symmetry spontaneously. For convenience, we have set the coupling constant equal to one and assumed that ϕ is normalized in its ground state.

Because of the potential (2c) all monopole solutions are "λ_8-like." If we further restrict our attention to spherically symmetric solutions of this theory we are left with generalizations of the 't Hooft-Polyakov ansatz, corresponding to the "U-spin" embedding of SU(2) in SU(3) and the most general ansatz for the "nuclear physics" embedding[7]

$$\phi = A(r)\varphi_1 + B(r)\varphi_2\,, \tag{3a}$$

$$W_\mu = i[1-D_1(r)][\varphi_1, \partial_\mu\varphi_1] - iD_2(r)[\varphi_2, \partial_\mu\varphi_1]\,, \tag{3b}$$

with

$$(\varphi_1)_{ij} = \hat{r}_i \hat{r}_j - \tfrac{1}{3}\delta_{ij}\,, \quad (\varphi_2)_{ij} = i\,\epsilon_{ijk}\hat{r}_k\,, \qquad i,j,\ldots = 1,2,3. \tag{3c}$$

Within the second class of solutions three Higgs vacuums with magnetic quantum numbers (± 1, 0), (± 1, 1), and (0, 2) (Ref. 7) and in the Prasad-Sommerfield limit one finite-energy solution, the regularized version of the (± 1, 0) Higgs vacuum,[6] are known. This finite-energy solution has regularizing functions

$$A = \frac{\sqrt{3}}{4}(\phi_1 - \phi_2), \quad B = \mp \frac{1}{4\sqrt{3}}(\phi_1 + \phi_2)\,, \tag{4a}$$

$$D_1 = \frac{1}{2\sqrt{2}}(a_1 - a_2), \quad D_2 = \mp \frac{1}{2\sqrt{2}}(a_1 + a_2)\,, \tag{4b}$$

where ϕ_n and a_n are

$$\phi_n = -\frac{Q_n'}{Q_n} + \frac{2\sqrt{3}}{r}, \quad a_n = \frac{r}{\sqrt{3}\,Q_n}(2\,Q_{3-n})^{1/2}\,, \tag{5a}$$

$$Q_{1,2} = \tfrac{2}{3}\left[\pm(r/\sqrt{3} \mp 1)e^{\pm r/\sqrt{3}} + e^{\mp 2r/\sqrt{3}}\right]. \tag{5b}$$

Its asymptotic behavior is

$$(A,B,D_1,D_2) \xrightarrow[r\to\infty]{} \left(\frac{\sqrt{3}}{4}, \pm\frac{\sqrt{3}}{4}, \left(\frac{1}{8}\right)^{1/2}, \mp\left(\frac{1}{8}\right)^{1/2}\right), \tag{6a}$$

whereas for the other two, missing solutions (±1, 1) and (0, 2),

$$(A,B,D_1,D_2) \xrightarrow[r\to\infty]{} \left(\frac{\sqrt{3}}{4}, \pm\frac{\sqrt{3}}{4}, 0, 0\right) \tag{6b}$$

$$\xrightarrow[r\to\infty]{} \left(-\frac{\sqrt{3}}{2}, 0, 0, 0\right) \tag{6c}$$

has to hold, respectively.

If we restrict the ansatz (3) further by putting $B = D_1 = 0$ we are thus restricting our attention to the solution (0, 2). Although it is topologically trivial it is not clear whether such a monopole is trivial from a physical point of view. Under reasonable assumptions the isomagnetic number is quantized[3] and the solution thus cannot be continuously deformed into the vacuum. Besides, the degeneracy is a feature of our restriction to λ_8-like solutions forced upon us by our choice of the Higgs potential with $q \neq 0$. For $q = 0$ the Higgs field can only be SU(3) rotated to a linear combination of λ_3 and λ_8 and the isomagnetic quantum number becomes a topological one. Furthermore, isomagnetic quantum numbers may even be relevant as measurable physical quantities as Goddard *et al.*[10] conjectured.

With our special ansatz which does not satisfy the Bogomolny equations

$$B_i = \tfrac{1}{2}\epsilon_{ijk}G_{jk} = \pm D_i\phi\,, \tag{7}$$

irrespectively of the functions A and D_2, the Euler-Lagrange equations of the Lagrangian (1) reduce to

$$(r^2A')' = 6D_2{}^2A + 4pr^2A(\tfrac{4}{3}A^2 - 1) + \tfrac{16}{3}qr^2A(A + \tfrac{1}{2}\sqrt{3})(A + \tfrac{1}{4}\sqrt{3})\,, \tag{8a}$$

$$r^2D_2'' = D_2(D_2{}^2 + r^2A^2 - 1). \tag{8b}$$

These equations are, on the other hand, the Euler-Lagrange equations of the energy functional

$$E(A,D_2) = 4\pi\int_0^\infty dr\left[4\left(D_2'^2 + \frac{(D_2{}^2-1)^2}{2r^2}\right) + \tfrac{2}{3}r^2A'^2 + 4A^2D_2{}^2 + pr^2(\tfrac{4}{3}A^2-1)^2 + \tfrac{4}{3}qr^2A^2\left(\frac{2}{\sqrt{3}}A + 1\right)^2\right]. \tag{9}$$

To prove the existence of a solution one therefore has only to show that the energy functional (9) attains its minimal value. This is an easy task because we can cast the energy functional in the form

$$E(\sigma,\tau) = 4\pi\int_0^\infty dr\left(\sigma'^2 + \frac{r^2\tau'^2}{2} + \mu^2 + 2\nu^2 + \rho^2 + \kappa^2\right), \tag{10}$$

with

$$\sigma = 2D_2, \quad \tau = \frac{2}{\sqrt{3}}A - 1,$$

$$\mu = \frac{\sqrt{3}}{2}\sigma(\tau+1), \quad \nu = \frac{1}{r}\left(\frac{\sigma^2}{4} - 1\right),$$

$$\rho = r\sqrt{p}\,\tau(\tau+2), \quad \kappa = r\sqrt{q}\,(\tau+1)(\tau+2)\,,$$

which is the starting point of Tyupkin *et al.*[11] for their proof in the 't Hooft-Polyakov case.

Except for some unimportant factors the proof can be transcribed literally and yields the existence of a finite-energy solution with $\lim_{r\to\infty}A(r) = -\frac{1}{2}\sqrt{3}$ and $\lim_{r\to\infty}D_2(r) = 0$ in the weak sense. That is to say, we have proven that there exist functions $h(r) = rA(r)$ and $k(r) = D_2(r)$ with weak derivatives such that for any pair of test functions $\varphi_{1,2} \in C_0^\infty(R_+)$ with compact support in the open half-line $R_+ = (0,\infty)$ the following equations hold:

$$\int_0^\infty \left[h'(r)\varphi_1'(r) + \frac{6k^2(r)}{r^2} h(r)\varphi_1(r) + \frac{4ph(r)}{r^2}[\tfrac{4}{3}h^2(r) - r^2]\varphi_1(r) + \frac{16}{3}\frac{qh(r)}{r^2}\left(h(r) + \frac{\sqrt{3}}{2}r\right)\left(h(r) + \frac{\sqrt{3}}{4}r\right)\varphi_1(r)\right] dr = 0\,, \tag{11a}$$

$$\int_0^\infty \left[k'(r)\varphi_2'(r) + \frac{k^2(r) + h^2(r) - 1}{r^2} k(r)\varphi_2(r)\right] dr = 0\,. \tag{11b}$$

III. TAYLOR EXPANSION AT THE ORIGIN

As well as in the 't Hooft-Polyakov case we do not know the existing solution in terms of familiar functions. To obtain some knowledge of its characteristics and a starting point for numerical calculations we therefore study its asymptotic behavior at the origin proceeding as follows: First, it is proven that h, k, and all their derivatives exist at the origin. Then, the coefficients of the Taylor expansions are given by recurrence relations and finally, the convergence of the series is proven.

For the first step we again take advantage of the close analogy to the Prasad-Sommerfield case.[9] In both cases general theorems guarantee that the differential equations (8) hold in the strong sense on the open half-line $(0,\infty)$ and $h(r)$ and $k(r)$ are even C^∞ functions there. To integrate Eqs. (8a) and (8b) one uses the Green's functions

$$G_h(r,s) = -\frac{1}{5}\left(\frac{s^3}{r^2}\Theta(r-s) + \frac{r^3}{s^2}\Theta(s-r)\right)\,, \tag{12a}$$

$$G_k(r,s) = -\frac{1}{3}\left(\frac{s^2}{r}\Theta(r-s) + \frac{r^2}{s}\Theta(s-r)\right) \tag{12b}$$

for the differential operators $L_h = d^2/dr^2 - 6/r^2$ and $L_k = d^2/dr^2 - 2/r^2$ which describe the leading behavior of Eqs. (8a) and (8b), respectively, and the solutions r^3, $1/r^2$ and r^2, $1/r$ of the corresponding homogeneous equations.

With this input one obtains for $0 < r_0 \le r \le r_1 < \infty$

$$h(r) = \frac{r^3}{5r_1^3}[r_1 h'(r_1) + 2h(r_1)] + \frac{r_0^2}{5r^2}[3h(r_0) - r_0 h'(r_0)] - \frac{1}{5r^2}\int_{r_0}^{r} sV_1(s)\,ds - \frac{r^3}{5}\int_r^{r_1}\frac{V_1(s)}{s^4}\,ds\,, \tag{13a}$$

$$\tilde{k}(r) = \frac{r^2}{3r_1^2}[r_1\tilde{k}'(r_1) + \tilde{k}(r_1)] + \frac{r_0}{3r}[2\tilde{k}(r_0) - r_0\tilde{k}'(r_0)] - \frac{1}{3r}\int_{r_0}^{r} V_2(s)\,ds - \frac{r^2}{3}\int_r^{r_1}\frac{V_2(s)}{s^3}\,ds \tag{13b}$$

with

$$\tilde{k}(r) = k(r) - 1\,, \tag{14a}$$

$$V_1(r) = 6\tilde{k}(\tilde{k}+2)h + 4ph(\tfrac{4}{3}h^2 - r^2) + \tfrac{16}{3}qh(h + \tfrac{1}{2}\sqrt{3}\,r)(h + \tfrac{1}{4}\sqrt{3}\,r)\,, \tag{14b}$$

$$V_2(r) = (1+\tilde{k})h^2 + 3\tilde{k}^2 + \tilde{k}^3\,. \tag{14c}$$

Following the same line of reasoning as in the Prasad-Sommerfield case we can first set $r_0 = 0$, find next $\lim_{r\to 0} h/r = \lim_{r\to 0}\tilde{k}/r = 0$ and prove finally the existence of $\lim_{r\to 0} h/r^2$ and $\lim_{r\to 0}\tilde{k}/r^2$ and by general theorems the existence of all derivatives at $r = 0$.

Equations (13a) and (13b) now also permit us to write recurrence relations for the coefficients of Taylor expansions at the origin. To find these relations we set $r_0 = 0$ and $r_1 = r$. The first thing to notice is that

$$h(r) = \rho r^3 + O(r^4) \tag{15a}$$

and

$$\tilde{k}(r) = \sigma r^2 + O(r^3) \tag{15b}$$

hold with arbitrary coefficients ρ and σ. Then we conclude by induction that all even terms of the Taylor expansion of h and all odd terms of the Taylor expansion of $\tilde{k}$ vanish. In fact, if we assume

$$h = \sum_{m=1}^{N} h_m r^{2m+1} + O(r^{2N+2}) \tag{16a}$$

and

$$\tilde{k} = \sum_{m=1}^{N} k_m r^{2m} + K r^{2N+1} + O(r^{2N+2})\,, \tag{16b}$$

we obtain from Eq. (13b) $(2N-1)K = 0$ and therefore $K = 0$. If we assume on the other hand

$$h = \sum_{m=1}^{N} h_m r^{2m+1} + H r^{2N+2} + O(r^{2N+3}) \tag{17a}$$

and

$$\tilde{k} = \sum_{m=1}^{N+1} k_m r^{2m} + O(r^{2N+3})\,, \tag{17b}$$

we obtain from Eq. (13a) $(2N-1)H = 0$ and $H = 0$ as well.

We thus have

$$h(r) = \sum_{m=1}^{\infty} h_m r^{2m+1} \tag{18a}$$

and

$$k(r) = 1 + \sum_{m=1}^{\infty} k_m r^{2m} . \tag{18b}$$

Plugging this once more into Eqs. (13a) and (13b) yields the recurrence relations

$$h_m = \frac{1}{2m^2 + m - 3} \Bigg[6 \sum_{m_1, m_2 \geq 1} k_{m_1} h_{m_2} \delta_{m, m_1 + m_2} + 3 \sum_{m_1, m_2, m_3 \geq 1} k_{m_1} k_{m_2} h_{m_3} \delta_{m, m_1 + m_2 + m_3} + \tfrac{8}{3}(p+q) \sum_{m_1, m_2, m_3 \geq 1} h_{m_1} h_{m_2} h_{m_3} \delta_{m, m_1 + m_2 + m_3 + 1} - (2p - q) h_{m-1} + 2\sqrt{3}\, q \sum_{m_1, m_2 \geq 1} h_{m_1} h_{m_2} \delta_{m, m_1 + m_2 + 1} \Bigg], \tag{19a}$$

$$k_m = \frac{1}{2(2m^2 - m - 1)} \Bigg[\sum_{m_1, m_2 \geq 1} (3 k_{m_1} k_{m_2} \delta_{m, m_1 + m_2} + h_{m_1} h_{m_2} \delta_{m, m_1 + m_2 + 1}) + \sum_{m_1, m_2, m_3 \geq 1} (k_{m_1} k_{m_2} k_{m_3} \delta_{m, m_1 + m_2 + m_3} + k_{m_1} h_{m_2} h_{m_3} \delta_{m, m_1 + m_2 + m_3 + 1}) \Bigg] . \tag{19b}$$

To prove the convergence of the Taylor series we now show by induction that

$$|h_m| \leq \frac{M^m}{(m+1)^2} , \tag{20a}$$

$$|k_m| \leq \frac{M^m}{(m+1)^2} \tag{20b}$$

hold for sufficiently large m and $M \geq 1$. For this purpose, we estimate the sums as in the following example:

$$\begin{aligned} \Big| \sum_{m_1, m_2 \geq 1} k_{m_1} k_{m_2} \delta_{m, m_1 + m_2} \Big| &\leq M^m \sum_{m_1 = 1}^{m-1} \frac{1}{(m_1 + 1)^2} \frac{1}{(m - m_1 + 1)^2} \\ &\leq M^m \int_{1/2}^{m - 1/2} dx \frac{1}{(1+x)^2 (m - x + 1)^2} \\ &= M^m \frac{4}{(m+2)^2} \left(\frac{1}{3} - \frac{1}{2m+1} + \frac{1}{m+2} \ln \frac{2m+1}{3} \right) \leq \frac{M^m}{(m+2)^2} O(1) . \end{aligned} \tag{21}$$

Treating the other sums analogously we arrive at

$$|h_m| \leq \frac{M^m}{(m+1)^2 (2m^2 + m - 3)} O(1), \tag{22a}$$

$$|k_m| \leq \frac{M^m}{(m+1)^2\, 2(2m^2 - m - 1)} O(1). \tag{22b}$$

This proves the inequalities (20) and at last the covergence of the series (18) for $r \leq 1/\sqrt{M}$.

IV. ASYMPTOTIC EXPANSIONS AT INFINITY IN THE PRASAD-SOMMERFIELD LIMIT

We now look for asymptotic expansions at infinity. Because in the preceding section we already found a starting point for numerical calculations in the case where our existence proof holds and are mainly interested in a comparison with Kerner's work[8] we restrict our attention to the Prasad-Sommerfield limit of vanishing Higgs potential. The solutions we are looking for hence have to satisfy the equations

$$r^2 h'' = 6 k^2 h, \tag{23a}$$

$$r^2 k'' = (k^2 + h^2 - 1) k \tag{23b}$$

and have to fulfill $h \sim_{r \to \infty} -\frac{1}{2}\sqrt{3}\, r$ and $k \sim_{r \to \infty} 0$. [Because of the form of Eqs. (23a) and (23b) a change of scale and sign is always permitted in the Prasad-Sommerfield limit.] We shall furthermore attempt to find expansions in exponential functions with polynomially bounded coefficient functions.

If we assume that k falls off as an inverse power in the asymptotic region, Eq. (23b) together with the linearity of h leads to a contradiction. Therefore, in an exponential expansion k falls off exponentially and $6k^2 h$ does not contribute to the zero-order term of h which thus reads

$$h_0(r) = -(\tfrac{1}{2}\sqrt{3}\, r + \alpha) . \tag{24}$$

Using Eq. (23b) this on the other hand yields for the zero-order term of k ($z = \sqrt{3}\, r$, $\gamma^2 = \alpha^2$

$-\frac{3}{4}$)

$$\frac{d^2K_0}{dz^2}=\left(\frac{1}{4}+\frac{\alpha}{z}+\frac{\gamma^2-\frac{1}{4}}{z^2}\right)K_0. \tag{25}$$

As in the 't Hooft-Polyakov case[8] we arrive at Whittaker's equation of the second kind. [This comes as no surprise because h_0 differs only by a scale factor and Eq. (23b) is the same compared to the 't Hooft-Polyakov case.] However, we do not arrive at Kerner's quantization condition which is wrong but find exponentially decreasing solutions for every α, namely,

$$\begin{aligned}K_0(r)&=\beta W_{-\alpha,\gamma}(z)\\&=\beta\frac{\Gamma(2\gamma)}{\Gamma(\alpha+\gamma+\frac{1}{2})}M_{-\alpha,-\gamma}(z)\\&\quad+\beta\frac{\Gamma(-2\gamma)}{\Gamma(\alpha-\gamma+\frac{1}{2})}M_{-\alpha,\gamma}(z)\\&=k_0(r)e^{-\sqrt{3}r/2}\equiv\beta e^{-\sqrt{3}r/2}r^{-\alpha}M_\alpha(r),\end{aligned} \tag{26a}$$

with

$$\begin{aligned}M_{-\alpha,\gamma}(z)&=e^{-z/2}z^{\gamma+1/2}\left(1+\frac{\alpha+\gamma+\frac{1}{2}}{2\gamma+1}z\right.\\&\quad\left.+\frac{(\alpha+\gamma+\frac{1}{2})(\alpha+\gamma+\frac{3}{2})}{(2\gamma+1)(2\gamma+2)}\frac{z^2}{2!}+\cdots\right)\end{aligned} \tag{26b}$$

and the asymptotic behavior

$$W_{-\alpha,\gamma}(z)\underset{r\to\infty}{\sim}e^{-z/2}z^{-\alpha}[1+O(z^{-1})]. \tag{27}$$

For quantized $\alpha=-(n^2+n+1)/(2n+1)$, $\alpha+\gamma+\frac{1}{2}=-n$, only the solution (26) becomes especially simple because the series (26b) is finite and $M_{-\alpha,\gamma}$ is a solution with the appropriate asymptotic behavior in this case.

Using Eq. (23a) one can now conclude that the function H_1 in

$$h=h_0+H_1e^{-\sqrt{3}r/2}+O(e^{-\sqrt{3}r})$$

satisfies

$$H_1''-\sqrt{3}H_1'+\tfrac{3}{4}H_1=0, \tag{28}$$

therefore takes the form

$$H_1=H_1^1re^{\sqrt{3}r/2}+H_1^2e^{\sqrt{3}r/2} \tag{29}$$

and has to be identically zero because (29) is not polynomially bounded. In the next step one derives for the function K_1 of order $e^{-\sqrt{3}r}$ in k Whittaker's differential equation (25). However, we already know that the solution of this equation is of the order $e^{-(\sqrt{3}/2)r}$. We thus find that K_1 has to vanish and by induction finally

$$h(r)=\sum_{m=0}^{\infty}h_m(r)e^{-\sqrt{3}mr} \tag{30a}$$

and

$$k(r)=\sum_{m=0}^{\infty}k_m(r)e^{-(\sqrt{3}/2)(2m+1)r}, \tag{30b}$$

where h_0 and k_0 are given by (24) and (26), respectively.

To find the functions h_m and k_m for $m=2,3,\ldots$ we write the recursive differential equations

$$\begin{aligned}h_m''-2\sqrt{3}mh_m'+3m^2h_m&=\frac{6}{r^2}\sum_{m_1,m_2,m_3\geq0}k_{m_1}k_{m_2}h_{m_3}\delta_{m,m_1+m_2+m_3+1}\\&\equiv\eta_m,\end{aligned} \tag{31a}$$

$$\begin{aligned}&k_m''-\sqrt{3}(2m+1)k_m'\\&\quad+\left[\tfrac{3}{4}(2m+1)^2-\frac{3}{4}-\frac{\sqrt{3}\alpha}{r}-\frac{\alpha^2-1}{r^2}\right]k_m\\&\quad=\frac{1}{r^2}\sum_{m_1,m_2,m_3\geq0}k_{m_1}k_{m_2}k_{m_3}\delta_{m,m_1+m_2+m_3+1}\\&\quad+\frac{1}{r^2}\sum_{\substack{m_1,m_2,m_3\geq0\\m_1+m_2>0}}h_{m_1}h_{m_2}k_{m_3}\delta_{m,m_1+m_2+m_3}\equiv\kappa_m.\end{aligned} \tag{31b}$$

These equations are solved recursively by

$$h_m(r)=\int_r^\infty dr'\,e^{\sqrt{3}m(r-r')}(r'-r)\eta_m(r'), \tag{32a}$$

$$\begin{aligned}k_m(r)&=\int_r^\infty dr'r^{-\alpha}r'^{-\alpha}e^{\sqrt{3}m(r-r')}\\&\quad\times e^{-\sqrt{3}r'}\frac{M_\alpha(r)}{M_\alpha(r')}k(r,r')\kappa_m(r'),\end{aligned} \tag{32b}$$

where

$$k(r,r')=\int_r^{r'}dx\,\frac{e^{\sqrt{3}x}}{x^{-2\alpha}}\frac{M_\alpha^2(r)}{M_\alpha^2(x)}, \tag{32c}$$

with the functions M_α of Eq. (26). The solutions (32) are unique because the solutions of the homogeneous equations corresponding to (31), which can always be added to the special solutions (32), are not polynomially bounded. The solutions (32) on the other hand are polynomially bounded which can be shown by induction and the integrals exist for sufficiently large r. [Notice that $M_\alpha(r)>0$ holds for sufficiently large r because of (27).]

To prove the convergence of the expansions (30) we proceed again by induction. We assume

$$\sup_{r>r_0}|h_me^{-(\sqrt{3}/2)mr}|<\frac{C_0M^{\sqrt{3}m}}{(m+1)^2}, \tag{33a}$$

$$\sup_{r>r_0}|k_me^{-(\sqrt{3}/4)(2m+1)r}|<\frac{C_0M^{(\sqrt{3}/2)(2m+1)}}{(m+1)^2} \tag{33b}$$

for $m<N$. We then conclude

$$\sup_{r>r_0} |h_N e^{-(\sqrt{3}/2)Nr}| < \sup_{r>r_0} \int_r^\infty dr' e^{(\sqrt{3}/2)N(r-r')} (r'-r) \sup_{r'>r_0} |\eta_N(r') e^{-(\sqrt{3}/2)Nr'}|$$

$$= \frac{4}{3N^2} \sup_{r'>r_0} |\eta_N(r') e^{-(\sqrt{3}/2)Nr'}| . \quad (34)$$

To estimate $\sup|\eta_N e^{-(\sqrt{3}/2)Nr}|$ one uses the same technique as in the preceding section. For sufficiently large N we can thus prove (33a).

For the functions k_N we estimate $\sup|\kappa_N(r')\exp[-\sqrt{3}(2N+1)r'/4]|$ analogously and notice that by partial integration and because of $M_\alpha(r)/M_\alpha(r')<2$ for sufficiently large r and $r'>r$

$$\sup_{r>r_0} \int_r^\infty dr' r^{-\alpha} r'^{-\alpha} e^{(\sqrt{3}/4)(2N-1)(r-r')} e^{-\sqrt{3}r'} \frac{M_\alpha(r)}{M_\alpha(r')} k(r,r') < \frac{64(2N+2)}{3(2N-1)(2N+3)} \left[1+O\left(\frac{1}{Nr_0}\right)\right] \quad (35)$$

holds. Hence, we arrive at the following result. For sufficiently large $r>r_0$ and $r>2\ln M$ solutions of the Eqs. (23) are given by the expansion (30) with functions h_m, k_m given by (24), (26), and (32).

V. CONCLUSION

We have proven the existence of a monopole with magnetic quantum numbers (0, 2) and given a two-parameter set of Taylor expansions at the origin which contains this solution. Which parameters belong to the solution could be answered by a computer calculation. The proof does not apply to the Prasad-Sommerfield limit, but because for small coupling constants p, q the Higgs potential only contributes for large r and the solutions are kept fixed for all $p, q>0$ at infinity we conjecture that a nontrivial finite-energy solution also exists in the Prasad-Sommerfield limit.

In this limit we found at infinity a two parameter set of solutions. However, we do not know whether a solution or, if our conjecture is right, which solution matches with a finite-energy solution for $r\to 0$. It is also not obvious, at least not to the author, how to apply similar techniques as in the 't Hooft-Polykov case[9] and reduce the problem to the solution of two first-order equations. The best one can do at the moment is thus a computer check.

ACKNOWLEDGMENTS

I would like to thank Dr. Y. M. Cho and Dr. D. Maison for many clarifying discussions, Dr. D. Maison in particular for pointing out to me that Kerner's quantization condition is wrong. Furthermore, I would like to thank Professor W. Rühl for useful advice and a careful reading of the manuscript.

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