PHYSICAL REVIEW D VOLUME 23, NUMBER 6 15 MARCH 1981

Coincident anharmonic oscillators

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(Received 17 November 1980)

We discuss and extend an observation of Zinn-Justin that the double-well potential in one dimension and the anharmonic oscillator in two dimensions have coinciding perturbation expansions.

I. INTRODUCTION

Zinn-Justin noticed a remarkable coincidence for anharmonic oscillators[1]: The double-well (DW) potential in one dimension and the spherically symmetric $\vec{x}^4$ anharmonic oscillator (AO) in two dimensions have, up to an over-all constant and an alternating sign, identical perturbation coefficients for the ground state. This observation is based on computer calculations. It was found to hold to an accuracy of at least 12 places and to all orders computed. The reason for this coincidence is not known.

Our results are as follows: First, we sharpen and extend the observation of Zinn-Justin. Second, we develop a method for handling certain coincident perturbation expansions. However, we only handle finite orders of perturbation theory. More precisely:

(a) For the DW in one dimension and the spherically symmetric AO in two dimensions, the coincidence of perturbation coefficients for eigenvalues holds analytically (in contradistinction to numerically) at least up to order g^{11} (there is strong evidence that this result is in fact true at least up to order g^{15}). Moreover, it holds for *all* levels of the DW and their corresponding levels in the L=0 sector of the AO (an in particular for the ground state).

(b) We map the DW in one dimension and the $\vec{x}^4$ AO in two dimensions on a common space.

(c) We analyze general AO's in one and two dimensions and determine the coupling constants so that the two AO's have coincident expansions to a given order. This is done explicitly up to g^5. By extending to higher orders one would either prove that the example found by Seznec and Zinn-Justin is unique or manufacture other such examples. We have not succeeded in carrying the computations to sufficiently high order to achieve either goal.

II. COINCIDENT EXPANSION

The perturbation coefficients $\{E_N\}$ for the lth level of the Hamiltonian

$$H(g)=\sum_{j=0}^{N} H_j g^j \tag{2.1}$$

are defined by the recursion relations

$$\begin{aligned} E_N &= (\psi_0, H_N\psi_0)+\sum_{j=1}^{N-1} (\psi_0, (H_j - E_j)\psi_{N-j}),\\ \psi_N &= \left(\frac{1}{E_0-H_0}\right)_{\text{red}} \sum_{j=1}^{N} (H_j - E_j)\psi_{N-j}, \end{aligned} \tag{2.2}$$

where $[(E_0-H_0)^{-1}]_{\text{red}}$ is the reduced resolvent. The subscript 0 denotes unperturbed quantities. For H_0 the harmonic oscillator (HO) and $H_j=\alpha_j|\vec{x}|^{j+2}$, $j\geq 1$ it is known that $|E_N|\leq C^N(N/2)!$. By symmetrizing Eq. (2.2) one shows that $\psi(g)$ to order N determines $E(g)$ to order $2N+1$. Consider the two Hamiltonians $H(g)$ and $h(g)$ of the form (2.1) and suppose there is

$$S(g)=1+\sum_{j=1}^{\infty} S_j g^j \tag{2.3}$$

such that

$$\sum_{j=0}^{k} (H_{k-j}S_j - S_j h_{k-j})=0, \quad k=1,\ldots,M \tag{2.4}$$

then we say that $H(g)$ *and* $h(g)$ *coincide to order* M. Equation (2.4) implies that if $\psi(g)$ of $H(g)$ is given to order M then $\phi(g)=S(g)\psi(g)$ is a perturbative eigenfunction of $h(g)$ to order M. In particular, the perturbation coefficients for the eigenvalues of the two Hamiltonians, $\{E_j\}=\{e_j\}$, coincide for all $j\leq 2M+1$, for all levels. With $H_0=h_0=2n$, where n is the one-dimensional HO, Eq. (2.4) can be written as a recursion relation for the S_k,

$$[2n,S_k]=-\sum_{j=0}^{k-1} (H_{k-j}S_j - S_j h_{k-j}). \tag{2.5}$$

To solve (2.5) we make use of the *Friedrichs* Γ *operator*, defined on the operator B with $\langle m|B|m\rangle=0$, where $|m\rangle$ is the one-dimensional HO eigenfunction, by

$$[2n,\Gamma(B)]=B. \tag{2.6}$$

In particular,

$$\Gamma(b^k)=\frac{b^k}{2k},\quad b^k=\begin{cases}a^{\dagger k}, & k>0\\ a^{|k|}, & k<0.\end{cases} \tag{2.7}$$

Equation (2.5) now reads

$$S_k=-\Gamma(T_k),\qquad T_k=\sum_{j=0}^{k-1}(H_{k-j}S_j-S_jh_{k-j}). \tag{2.8}$$

Thus if T_k is in the domain of Γ for $1\le k\le M-1$, then $H(g)$ and $h(g)$ coincide to order M.

III. ANHARMONIC OSCILLATORS IN DIFFERENT DIMENSIONS

Perhaps the most surprising fact about the coincidence Seznec and Zinn-Justin found is the way in which different dimensions enter. We start with coincident spectra for harmonic oscillators in one and d dimensions.

Let $\vec{a}=(\vec{x}+i\vec{p})/\sqrt{2}$,

$$H_0^d=\vec{a}^\dagger\cdot\vec{a},\quad \vec{x}\in R^d. \tag{3.1}$$

The scalar $A=\frac{1}{2}\vec{a}\cdot\vec{a}$ is a ladder operator for H_0^d:

$$[H_0^d,A^\dagger]=A^\dagger,\quad d\ge 2. \tag{3.2}$$

Henceforth H_0^d will be restricted to $\oplus_n(A^\dagger)^n|0)_d$, which is the $\vec{L}=0$ sector of the space of states of the d-dimensional harmonic oscillator. The spectrum for $d\ge 2$ is the non-negative integers. The spectrum of $2H_0^1$ in one dimension is also the non-negative intergers. The coincident spectra and degeneracies give a unitary map between one and d dimensions.

Let

$$|n\rangle=\frac{(a^\dagger)^n}{(n!)^{1/2}}|0\rangle, \tag{3.3}$$

$$|n)_d=\frac{(A^\dagger)^n|0)_d}{||(A^\dagger)^n|0)_d||}=\frac{(A^\dagger)^n|0\rangle|0\rangle\cdots|0\rangle}{||(A^\dagger)^n|0)_d||}. \tag{3.4}$$

Then,

$$U\equiv\sum_{n=0}^{\infty}|n\rangle(n| \tag{3.5}$$

is unitary and

$$U\vec{a}^\dagger\cdot\vec{a}U^{-1}=2a^\dagger a. \tag{3.6}$$

Lemma 3.1

$$||(A^\dagger)^m|0)||^2=m!\left(m+\frac{d-2}{2}\right)!\Big/\left(\frac{d-2}{2}\right)!.$$

Proof:

Using the tensor product property and Pythagoras's law,

$$\begin{aligned}||e^{\alpha A^\dagger}|0)_d||^2&=||\prod_{i=1}^{d}e^{(\alpha/2)a_i^{\dagger 2}}|0\rangle_i||^2\\&=||e^{(\alpha/2)a_0^{\dagger 2}}|0\rangle||^{2d}\\&=\left[\sum_{m=0}^{\infty}\left(\frac{\alpha}{2}\right)^{2m}\frac{(2m)!}{(m!)^2}\right]^d\\&=(1-\alpha^2)^{-d/2}\\&=\sum_{m=0}^{\infty}\frac{\alpha^{2m}}{m!}\left(m+\frac{d-2}{2}\right)!\Big/\left(\frac{d-2}{2}\right)!.\end{aligned} \tag{3.7}$$

On the other hand,

$$||e^{\alpha A^\dagger}|0)_d||^2=\sum_m\frac{\alpha^{2m}}{(m!)}||A^{\dagger m}|0)_d||^2. \tag{3.8}$$

Equating powers of α gives the result. Since

$$\begin{aligned}A^\dagger|m)_d&=\frac{||(A^\dagger)^{m-1}|0)_d||}{||(A^\dagger)^m|0)_d||}|m+1)_d\\&=U^\dagger\left(n+\frac{d-2}{2}\right)^{1/2}a^\dagger|m\rangle\\&=U^\dagger\left(n+\frac{d-2}{2}\right)^{1/2}a^\dagger U|m)_d,\end{aligned} \tag{3.9}$$

we find

$$UA^\dagger U^\dagger=\left(n+\frac{d-2}{2}\right)^{1/2}a^\dagger,\qquad UAU^\dagger=a\left(n+\frac{d-2}{2}\right)^{1/2}. \tag{3.10}$$

Proposition 3.2:

$$U\vec{x}^2U^\dagger=a\left(n+\frac{d-2}{2}\right)^{1/2}+\left(n+\frac{d-2}{2}\right)^{1/2}a^\dagger+2n+d/2, \tag{3.11}$$

$$U\vec{p}^2U^\dagger=\frac{d-2}{2}-a\left(n+\frac{d-2}{2}\right)^{1/2}-\left(n+\frac{d-2}{2}\right)^{1/2}a^\dagger. \tag{3.12}$$

Proof:

Equation (3.11) follows from the identity

$$\vec{x}^2=A+A^\dagger+H_0^d+d/2 \tag{3.13}$$

and Eqs. (3.10) and (3.6).

Equation (3.12) then follows from Eq. (3.6) and

$$H_0^d=\tfrac{1}{2}(\vec{p}^2+\vec{x}^2-d).$$

Proposition 3.2 and the functional calculus give an explicit expression for $U[H_0^d+V(\vec{x}^2)]U^\dagger$ in the Fock space $\oplus_{n=0}^{\infty}|n\rangle$ of the one-dimensional oscillator. In the next section we shall use this to construct recursion relations.

IV. RECURSION AND CONSISTENCY RELATIONS

Let

$$H(\vec{c},g)\equiv(p^2+x^2-1)+\frac{1}{g^2}\sum_{j=3}^{2N}c_j^{j-2}(gx)^j,\quad x\in R^1,$$

$$h^d(\vec{e},g)\equiv\tfrac{1}{2}(\vec{p}^2+\vec{x}^2-d)+\frac{1}{g^2}\sum_{j=3}^{2N}e_j^{j-2}(g\vec{x})^j,\qquad(4.1)$$

$$e_{2j+1}\equiv 0,\quad \vec{x}\in R^d.$$

h^d is restricted to $\vec{L}=0$. The two Hamiltonians coincide in the sense of perturbation theory if T_k is in the domain of Γ and

$$T_k\equiv\sum_{j=0}^{k-1}\left(S_j c_{k+2-j}^{k-j}x^{2+k-j}-e_{k+2-j}^{k-j}\vec{x}^{k+2-j}S_j\right),$$
$$S_k=-\Gamma(T_k).\qquad(4.2)$$

We used the abbreviation $\vec{x}^j$ for $U\vec{x}^jU^\dagger$. We call the equations

$$\langle m|T_k|m\rangle\equiv 0,\quad 1\leqslant k\leqslant M-1\qquad(4.3)$$

consistency relations for coincidence to order M. Equation (4.3) can be viewed as a set of equations for the unknowns $\{\vec{c}\}$ and $\{\vec{e}\}$.

One proves by induction that if S_k and T_k exist, their parity under reflection $x\to-x$ is $(-)^k$. Since the HO eigenfunctions $|m\rangle$ have definite parity we get the following:

Proposition 4.1:

Suppose T_k, $k=1,\ldots,2M$ satisfies the consistency condition (4.3), then T_{2M+1} also satisfies (4.3). It follows that (4.3) is nontrivial for k even only.

We shall now consider the first few consistency relations for even orders.

Proposition 4.2:

If the Hamiltonians $H^{(1)}(\vec{c},g)$ and $h^{(d)}(\vec{e},g)|_{\vec{L}=0}$ have coincident perturbation expansion to second order (and then also to third order) then

(a) $d=2$ implies $(c_3^2,c_4^2,e_4^2)=\lambda^2(4,1,-1)$, λ arbitrary;

(b) $d\neq 2$ implies $(c_3^2,c_4^2,e_4^2)=\vec{0}$.

Remark: No condition is imposed on c_j, e_j with $j\geqslant 5$.

Proof:

$$S_1=-\Gamma(T_1)=-c_3\Gamma(x^3),$$
$$T_2=c_4^2x^4+c_3x^3S_1-e_4^2\vec{x}^4.\qquad(4.4)$$

Now,

$$\langle m|x^4|m\rangle=\|\,|x^2|m\rangle\|^2=\tfrac{3}{4}(2m^2+2m+1),$$
$$\langle m|x^3\Gamma(x^3)|m\rangle=\tfrac{1}{16}(30m^2+30m+11),\qquad(4.5)$$
$$\langle m|\vec{x}^4|m\rangle=\|\,|\vec{x}^2|m\rangle\|^2=6m^2+3dm+\frac{d}{2}\left(1+\frac{d}{2}\right).$$

The equation for consistency, $\langle m|T_2|m\rangle=0$, translates to the set of linear equations for (c_3^2,c_4^2,e_4^2):

$$\begin{bmatrix}-\frac{30}{16}&\frac{3}{2}&-6\\-\frac{30}{16}&\frac{3}{2}&-3d\\-\frac{1}{16}&\frac{3}{4}&-\frac{1}{2}d(1+d/2)\end{bmatrix}\begin{bmatrix}c_3^2\\c_4^2\\e_4^2\end{bmatrix}=0.\qquad(4.6)$$

The determinant of the matrix is $\frac{63}{8}(2-d)$. Thus a nontrivial solution exists only for $d=2$ and then

$$(c_3^2,c_4^2,e_4^2)=\lambda^2(4,1,-1).\qquad(4.7)$$

Remarks: 1. λ is a trivial scale parameter. 2. Setting $c_j=e_j=0$, $j\geqslant5$, and c_3,c_4,e_4 as above, gives the coincident Hamiltonians found by Zinn-Justin.

The recursion relations for the operators S_k in Eq. (4.2) can be written as recursion relations for functions $\{s_j^{(k)}\}$ introducing

$$S_k=\sum_j s_j^k(n)b^j,\qquad(4.8)$$

b^j as in (2.7). This is convenient for computer calculation via formal languages, such as FORMAC. The following result is a consequence of such a calculation, going up to $k=5$.

Proposition 4.3:

For the (c_3,c_4,e_4) as in Proposition (4.2) and $e_i=c_i=0$, $i\geqslant5$, giving the DW and the $\vec{x}^4$ AO, coincidence holds at least to order g^5. Moreover,

$$\ln S(g)=-ig(\tfrac{2}{3}p^3+xpx)$$
$$+\tfrac{1}{2}g^2\Big(\frac{i}{4}\{x^2,D\}-\frac{i}{4}\{p^2,D\}-iD+i\{n,D\}+\tfrac{1}{2}[a^2(n^2-n)^{1/2}-\text{H.c.}]+4(an^{3/2}-\text{H.c.})$$
$$+\tfrac{1}{72}[164n^3+246n^2+256n+89]\Big)+O(g^3).$$

Here $D=\frac{1}{2}(xp+px)$ generates dilations, and $\{,\}$ is the anticommutator.

Remarks: 1. The output listed S_k and T_k and the consistency condition was checked by hand. We also checked the consistency numerically (i.e., letting the machine compute $\langle m|T_k|m\rangle$ for specific m). This check was performed up to $k=7$. 2. The tedium in such calculations may be appreciated by the 300 seconds it took IBM 370/168 to iterate the recursion to $k=7$ and the 450 full length lines of print for the functions $\{s_j^k\}$.

We shall now return to a systematic study of the consistency conditions for higher order. The purpose of this exercise is to see whether the Zinn-Justin example is a unique solution to this set or whether there are also other solutions for higher-order polynomial self-interaction. We have not succeeded in going far enough to establish either result.

Consistency for T_4 is

$$\langle m | c_6^4 x^6 + c_5^3 x^5 S + c_4^2 x^4 S_2 + c_3 x^3 S_3 - e_6^4 \vec{x}^6 - e_4^2 S_2 \vec{x}^4 | m\rangle \equiv 0\,. \quad (4.9)$$

S_1 and S_2 are independent of c_i, e_i, $i \geq 5$, while

$$S_3 = -c_5^3 \Gamma(x^5) - \Gamma(c_4^2 x^4 S_1 + c_2 x^3 S_2 - e_4^2 S_1 \vec{x}^4)\,. \quad (4.10)$$

By Proposition (4.3), Eq. (4.9) holds for $c_i, e_i = 0$, $i \geq 5$. It remains therefore to consider

$$\langle m | c_6^4 x^6 - c_3 c_5^3 [x^5 \Gamma(x^3) + x^3 \Gamma(x^5)] - e_6^4 \vec{x}^6 | m\rangle \equiv 0\,. \quad (4.11)$$

To evaluate (4.11) the following is useful.

Lemma 4.4:

Let $l_{1,2}$ be odd, then

$$\langle m | x^{l_2}\Gamma(x^{l_1}) - x^{l_1}\Gamma(x^{l_2}) | m\rangle \equiv 0\,.$$

Proof:

Write

$$x^m = \sum_{k=1}^{m} [\chi_k^{(m)}(n) a^{\dagger k} + \text{H.c.}]\,. \quad (4.12)$$

Then

$$\langle m | x^{l_2}\Gamma(x^{l_1}) | m\rangle = \left\langle m \left| \sum_{k=1}^{l_2} \left(-\chi_k^{(l_2)}(n) \frac{a^{\dagger k} a^k}{k} \chi_k^{(l_1)}(n) + k^{-1} a^k \chi_k^{(l_2)}(n) \chi_k^{(l_1)}(n) a^{\dagger k} \right) \right| m \right\rangle \quad (4.13)$$

which is symmetric in l_1 and l_2.

Using

$$\langle m | x^6 | m\rangle = \| \, |x^3 | m\rangle \|^2 = m^3 + \tfrac{3}{2} m^2 + \tfrac{11}{4} m + \tfrac{9}{8}\,,$$

$$\langle m | x^5 \Gamma(x^3) | m\rangle = \tfrac{1}{32}(146 m^3 + 219 m^2 + 203 m + 65)\,,$$

$$\langle m | \vec{x}^6 | m\rangle = 20 m^3 + 15 d m^2 + (3d^2 + 3d + 4) m + \frac{d}{8}(d^2 + 6d + 8) \quad (4.14)$$

gives

$$\begin{pmatrix} 1 & -\frac{146}{16} & -20 \\ \frac{3}{2} & -\frac{219}{16} & -15d \\ \frac{11}{4} & -\frac{203}{16} & -(3d^2+3d+4) \\ \frac{9}{8} & -\frac{65}{16} & -\frac{1}{8}d(d^2+6d+8) \end{pmatrix} \begin{pmatrix} c_6^4 \\ c_3 c_5^3 \\ e_6^4 \end{pmatrix} = 0\,. \quad (4.15)$$

The rank of the 3×4 matrix is 2 if $d=2$ and 3 otherwise. For $d=2$ a one-parameter family of solutions is

$$(c_6^4, c_3, c_5^3, e_6^4) = \tilde{\mu}(106, 66, -\tfrac{397}{16})\,. \quad (4.16)$$

A summary follows:

Proposition 4.5:

$H^{(1)}(\vec{c}, g)$ and $h^{(d)}(\vec{e}, g)|_{\vec{L}=0}$ coincide to fourth order, and then also to fifth order, if

(a) $d=2$: $(c_3^2, c_4^2, e_4^2) = \lambda^2(4, 1, -1)$,

$(2\lambda c_5^3, c_6^4, e_6^4) = (\lambda\mu)^4(66, 106, -\tfrac{397}{16})$,

λ and μ are arbitrary;

(b) $d \neq 2$: $c_i, e_i = 0$, $i \leq 6$.

Remarks: 1. Choosing $\nu = \lambda\mu$ can be done without loss since $c_3 = 0$ implies $c_i = e_i = 0$, $i \leq 6$. The form we choose has the right scaling properties. 2. Zinn-Justin's case corresponds to choosing $\mu = 0$.

Coincidence to sixth and seventh order determines μ. We have not succeeded in completing the computations to this order, but we can still say something on the general character of the equations.

Coincidence to sixth order requires

$$\langle m | T_6 | m\rangle \equiv 0$$

with

$$T_6 = c_8^6 x^8 + c_7^5 x^7 S_1 + c_6^4 x^6 S_2 + c_5^3 x^5 S_3 + c_4^2 x^4 S_4 + c_3 x^3 S_5 - e_4^2 S_4 \vec{x}^4 - e_6^4 S_2 \vec{x}^6 - e_8^6 \vec{x}^8\,. \quad (4.17)$$

We shall not write explicitly the s_j appearing in (4.16) as this is partly unnecessary and partly very space consuming. The consistency equation turns out to be a quadratic equation in (μ^4),

$$\lambda^6 A(m)\mu^8 + \lambda^6 B(m)\mu^4 + c(m) = 0\,. \quad (4.18)$$

Here

$$A(m) = -(33)^2 \langle m | x^5 \Gamma(x^5) | m\rangle \quad (4.19)$$

is a polynomial in m of degree 4. Besides μ, Eq. (4.17) has an additional three undetermined parameters (c_7^5, c_8^6, e_8^6). (λ is a scale and thus not a free parameter.) By Zinn-Justin, $\mu = c_7 = c_8 = e_8 = 0$ has to solve (4.17). This gives

$$c(m) = \langle m | c_8^6 x^8 - 2\lambda c_7^5 [x^7 \Gamma(x^3) + x^3 \Gamma(x^7)] - e_8^6 \vec{x}^8 | m\rangle\,. \quad (4.20)$$

Since

$$\langle m|x^8|m\rangle=\tfrac{1}{16}(70m^4+140m^3+350m^2+253m+105)\,,$$
$$\langle m|\tilde{x}^8|m\rangle=70m^4+140m^3+170m^2+100m+24\,, \quad (4.21)$$
$$\langle m|x^7|\Gamma(x^3)|m\rangle=\tfrac{7}{32}(120m^4+270m^3+420m^2+270m+75)\,,$$

$c(m)$ is also a polynomial of degree 4.

Now either $B(m)$ is a polynomial of, at most, 4 or not. In the second case (4.18) holds only if $\mu=0$, and by (4.21), $c_7=c_8=e_8=0$ and Zinn-Justin's example is, in a sense, unique. On the other hand, if $B(m)$ is a polynomial of at most, degree 4, Eq. (4.18) reduces to five equations with four unknowns. Generically, such a system is overdetermined, but since this system appears to be rather particular it should not be too surprising if one finds solutions to (4.18) other than the trivial $\mu=0$.

This type of analysis can, in principle, be continued to arbitrary order. Unfortunately, the computations become so tedious that we suspect that further progress along these lines can only be made with the help of a computer.

ACKNOWLEDGMENT

We are indebted to E. Brezin for first telling us about the observation of Zinn-Justin. We are greatly indebted to Ran Gilády who wrote for us a computer program in FORMAC leading to Proposition 4.3. This work was started while J.A. was a member and R.S. a guest of the Physics Department of the Technion in Haifa. The support of this institute is greatly appreciated.

[1]R. Seznec and J. Zinn-Justin, J. Math. Phys. 20, 1398 (1979).