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Massless particles, conformal group, and de Sitter universe

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We first review a recent result on the uniqueness of the extension to the conformal group of massless representations of the Poincaré group. By restricting these representations to SO(3,2) we obtain a unique definition of massless particles in de Sitter space. This definition is compared with the concept of masslessness that arises from considerations of gauge invariance. Next, we recall the startling fact that the direct product of two Dirac singleton representations of SO(3,2) decomposes into a direct sum of the massless representations of SO(3,2). A theory of interacting singleton fields is developed and a simple expression is given for the intertwining operator between massless fields and two-singleton fields. Finally, we discuss the behavior of these massless representations with respect to the contraction of the de Sitter group to the Poincaré group.

INTRODUCTION

In 1909, Cunningham[1] discovered the role of the conformal group as a covariance group in relativistic physics. Since that time interest in this group has been periodically renewed. Indeed, the conformal group has both a geometrical meaning, arising from its definition, and a dynamical significance, associated with its representations, with conformally invariant field equations, scaling behavior, etc. In both directions much progress has been made.

In particular, the concept of conformal compactification $(S^1\times S^3)/Z_2$ of Minkowski space, and its universal covering $R^1\times S^3$ have recently been clarified; field equations and their properties have been extensively studied (this includes, at least in the analogous Euclidean case, the Yang-Mills field equations, etc.[2-4]). The compactifications make it possible to deal with a global action of the conformal group $\mathcal{C}$ or of its universal covering. These actions are known[5] to be locally causal, i.e., a neighborhood of the Poincaré group $\mathcal{P}$ in $\mathcal{C}$ acts causally on a neighborhood of the origin in Minkowski space.

The (projective) representations of $\mathcal{C}$ have also been studied for some time with various degrees of completeness. In particular, it was known more than 10 years ago that the so-called most degenerate (ladder) representations of $\mathcal{C}$ remain irreducible when restricted to $\mathcal{P}$, giving rise to massless, discrete-helicity representations.[6] Furthermore, those unitary representations of $\mathcal{P}$ that correspond to massless particles (zero mass, discrete helicity) are the only ones that can be extended to the conformal group. (It is obvious that the mass has to be zero, and examination of the so-called continuous spin representations of $\mathcal{P}$ shows that they do not qualify.) A complete classification of all irreducible projective representations of $\mathcal{C}$ and of SO(3, 2), with characterization of the unitary ones, has recently been obtained by one of us.[7]

Given a massless (=zero mass, discrete-helicity) representation V of $\mathcal{P}$, we thus know that it is the restriction to $\mathcal{P}$ of a unitary irreducible representation (UIR) of $\mathcal{C}$; but the uniqueness of the extension was not known. Worse, even the unitarity of every extension to $\mathcal{C}$ was an open question, and in some applications unitarity had to be proved in each particular case. This situation arose, in particular, in the study of conformally invariant field equations. Furthermore, the uniqueness of the extension within an equivalence class of representations was also open.

In Sec. I we shall recall a recent result[8] that settles all this uncertainty: When a unitary, irreducible representation V of $\mathcal{P}$ has an extension to $\mathcal{C}$, then there is a unique equivalence class of UIR's of $\mathcal{C}$ having V as its restriction to $\mathcal{P}$. With V given, there is in fact a unique extension; the operational form of this extension may be given explicitly. This is due to the fact that V is irreducible; the intertwining operator between two equivalent representations of $\mathcal{C}$ having the same restriction must commute with V and is therefore the identity.

In Sec. II we describe the projective representation $\mathcal{T}_n^+$ of $\mathcal{C}$ whose restriction to $\mathcal{P}$ has mass zero, helicity n, and positive energy. Section III is preparation for Sec. IV and contains a description of the relevant representations of SO(3, 2).

In Sec. IV we consider the restriction of $\mathcal{T}_n^+$ to the subgroup SO(3, 2). This restriction is irreducible for $n \neq 0$. Associating these irreducible representations of SO(3, 2) with elementary particles, we obtain an unambiguous definition of masslessness in de Sitter space. In Sec. V we show that this definition coincides, for $|n| \geq 1$, with the concept of masslessness that one obtains by considerations of gauge invariance.[9,10]

Massless particles in de Sitter space differ from massless particles in Minkowski space, in that they can be interpreted as two-particle states. In Sec. VI we recall the most exciting property of the Dirac singleton representations of SO(3, 2): the direct product of two positive-energy singletons reduces to a sum of massless representations of SO(3, 2).[11] The field theories associated with singletons are discussed in Sec. VII in somewhat greater detail than in previous publications.[9,11] This discussion is continued in Sec. VIII where we obtain a simple and direct relation between massless fields and two-singleton fields.

Finally, in Sec. IX we investigate the contraction[12] of the massless representations of SO(3, 2) back to the starting point of massless representations of $\mathcal{P}$. An interesting aspect of this contraction is that helicities n and $-n$ arise by contraction of one and the same representation of SO(3, 2).

The emphasis that this paper places on massless particles is justified by the observation that all physical theories that pretend to be fundamental make use of masslessness in one form or another. Thus massless photons and gravitons are basic to electrodynamics and to gravitation. Massless neutrinos probably play the same role in weak interactions, although this expectation has not yet been realized. Currently popular models of both weak and strong interactions employ massless fields in an essential way. In our opinion, all these theories suffer from a basic limitation: they are all conceived in a far too unimaginative imitation of electrodynamics. Thus, the masslessness of neutrinos has not yet been fully exploited. Perhaps this will happen only when neutrinos are integrated with photons and gravitons in the picture that is based on singletons.

This picture, which represents massless particles as two-singleton states,[11] has much in common with the quark model of hadrons. Let us stress, however, an important feature of singleton dynamics that is quite unlike quark dynamics. If by confinement of quarks one means that quarks cannot be observed directly as individual particles, then singletons may also be said to be confined. But singletons are unobservable, in practice if not in principle, for purely kinematical reasons. The relation energy $= \rho^{1/2} \times$ angular momentum (where ρ is the curvature of space-time and hence very small) means that an apparatus of cosmic dimensions is required to detect singleton absorption by energy balance. An absolute (super) selection rule forbids transitions between elementary particles by one-singleton emission. Thus singleton "confinement" is automatic and natural, and independent of the dynamics.

I. MASSLESS REPRESENTATIONS OF THE POINCARÉ GROUP HAVE UNIQUE CONFORMAL EXTENSIONS

Massless particles in Minkowski space are associated with unitary, irreducible representations of the Poincaré group, with zero mass and with discrete helicity; we shall refer to such representations as "massless representations."

It is well known that the only unitary, irreducible representations of the Poincaré group that have extensions to the conformal group are the massless ones. More surprisingly, it turns out[8] that any system that is invariant under a massless representation of the Poincaré group is invariant under a uniquely determined, unitary, irreducible representation of [the fourfold covering SU(2, 2) of] the conformal group. The precise statement is the following:

Theorem 1.[8] Let $\mathcal{T}$ be a projective representation of the conformal group $\mathcal{C}$, such that the restriction $\mathcal{T}|\mathcal{P}$ of $\mathcal{T}$ to the Poincaré group $\mathcal{P}$ is unitary and irreducible. Then $\mathcal{T}$ is a unitary representation of SU(2, 2), the operational form of which is uniquely determined by that of $\mathcal{T}|\mathcal{P}$.

Outline of proof. Let $\{L_{AB}\}$, $A, B = 0, 1, 2, 3, 5, 6$ with $L_{AB} = -L_{BA}$, be a basis for the Lie algebra associated with $\mathcal{T}$. This Lie algebra is isomorphic to so(4, 2) and the basis will be chosen so that the structure takes the familiar form

$$[L_{AB}, L_{CD}] = i(\eta_{BC} L_{AD} + \eta_{AD} L_{BC} - \eta_{AC} L_{BD} - \eta_{BD} L_{AC}). \tag{1.1}$$

The metric tensor is diagonal with $\eta_{00} = \eta_{55} = +1$, $\eta_{11} = \eta_{22} = \eta_{33} = \eta_{66} = -1$. These commutation relations, and all relations written below, are valid on a common invariant domain of differentiable vectors for $\mathcal{T}$, dense in the Hilbert space $\mathcal{H}$ of the representation.

The Lie algebra $\mathcal{L}(\mathcal{P}) = \mathrm{so}(3, 1) \times t_4$ is the subalgebra of so(4, 2) spanned by $\{L_{\mu\nu}\}$, $\mu, \nu = 0, 1, 2, 3$ (Lorentz transformations) and by $\{P_\mu \equiv L_{\mu 5} + L_{\mu 6}\}$, $\mu = 0, 1, 2, 3$ (translations). The operator $D \equiv L_{56}$ is the dilation operator. The restriction $\mathcal{T}|\mathcal{P}$ is massless; this implies that

$$\epsilon_{\mu\nu\lambda\rho} P_\nu L_{\lambda\rho} - 2nP_\mu = 0, \quad P_\mu P_\mu = 0, \tag{1.2}$$

where n is the helicity ($2n$ fixed integer). (We use Feynman's summation convention, e.g., $K_{AA} = K_{AB}\eta^{AB}$. Also $\epsilon_{0123} = \epsilon_{012356} = +1$.) These expressions generate an ideal of the enveloping algebra of so(4, 2), which implies that

$$\epsilon_{ABCDEF} L_{CD} L_{EF} + 8nL_{AB} = 0, \tag{1.3}$$

$$L_{AB} L_{AC} + L_{AC} L_{AB} = \tfrac{2}{3}\eta_{BC} C, \tag{1.4}$$

where $C \equiv \frac{1}{2} L_{AB} L_{AB}$ is the second-order Casimir operator for so(4, 2). Equation (1.3) gives, in particular,

$$2nD = \epsilon_{0ikl} L_{0i} L_{kl} \tag{1.5}$$

for $n \neq 0$. (The case $n=0$ can be treated in similar fashion,[8] using the relation $P_0 D = P_k L_{0k} - iP_0$, but will be ignored here.) It also follows easily from (1.4) that $W = J^2$, where $J^2 \equiv \frac{1}{2} L_{kl} L_{kl}$ is the Casimir operator for the rotation subalgebra (k, l summed over 1, 2, 3) and $W \equiv L_{05}{}^2 - L_{06}{}^2 - D^2$ is the Casimir operator for the so(2, 1) subalgebra generated by L_{05}, L_{06}, and $D = L_{56}$. [The corresponding subgroups of the universal covering group of $\mathcal{C}$ will be referred to as SU(2) and $\widetilde{\mathrm{SO}}(2,1)$ in what follows, while R_2 will denote the solvable subgroup of $\widetilde{\mathrm{SO}}(2,1)$ generated by P_0 and D.]

We may realize the representation space as $\mathcal{H} = L^2(\mathrm{R}^3, d^3p/|\vec{p}|)$ and decompose it according to the eigenvalues of J^2 as $\mathcal{H} = \overline{\oplus}_j \mathcal{H}_j$, j taking the values $s = |n|$, $s+1, \ldots$, and $J^2 = W$ having the value $j(j+1)$ in $\mathcal{H}_j$. Now direct inspection of the operator D in $\mathcal{H}_j$ shows that the subgroup $R_2 \times \mathrm{SU}(2)$ acts irreducibly and unitarily in $\mathcal{H}_j$. Since $J^2 = W$, $\mathcal{H}_j$ is stable under $\widetilde{\mathrm{SO}}(2,1)$; therefore, $\widetilde{\mathrm{SO}}(2,1) \times \mathrm{SU}(2)$ acts irreducibly in $\mathcal{H}_j$. The problem is thus reduced to one of lower dimension: to find all representations of $\widetilde{\mathrm{SO}}(2,1)$ that have the value $j(j+1)$ for the Casimir operator and are unitary irreducible when restricted to the subgroup R_2.

The solution of this simpler problem is trivial. In any irreducible Harish Chandra module of so(2, 1) the compact operator L_{05} has a discrete spectrum. The restriction to R_2 is irreducible if and only if the spectrum of L_{05} is either positive definite or negative definite. The restriction of $\mathcal{T}$ to $\widetilde{\mathrm{SO}}(2,1)$ is therefore equivalent to one of the unitary representations of the discrete series in which the spectrum of L_{05} is either $j+1, j+2, \ldots$ or $-j-1, -j-2, \ldots$. Since $\mathcal{T}|R_2$ is given explicitly in terms of $\mathcal{T}|\mathcal{P}$, and is unitary, the intertwining operator is in fact the identity and $\mathcal{T}|\widetilde{\mathrm{SO}}(2,1)$ is thus uniquely determined by $\mathcal{T}|\mathcal{P}$. It follows that $\mathcal{T}$ is uniquely determined.

Let V_n^+ and V_n^- denote the unitary, irreducible representations of $\mathcal{P}$ with zero mass, helicity n, and positive and negative energy, respectively. The corresponding representations of $\mathcal{C}$ will be denoted $\mathcal{T}_n^+$ and $\mathcal{T}_n^-$. We shall now describe these representations of $\mathcal{C}$.

II. DESCRIPTION OF $\mathcal{T}_n^\pm$

The operational form of the representation $\mathcal{T}_n^\pm$ in $\mathcal{H} = L^2(\mathrm{R}^3, d^3p/|\vec{p}|)$ is given by the usual expressions for the Poincaré generators $L_{\mu\nu}$ and P_μ in V_n;

$$L_{\mu 5} + L_{\mu 6} \equiv P_\mu = p_\mu, \tag{2.1}$$

with $p_0 = \pm(p_1{}^2 + p_2{}^2 + p_3{}^2)^{1/2}$,

$$L_{hk} = -i(p_h\partial_k - p_k\partial_h) + nS_{hk}, \tag{2.2}$$

$$L_{0k} = -ip_0\partial_k + nT_k, \tag{2.3}$$

where $h, k = 1, 2, 3$, $\partial_k = \partial/\partial p_k$, and we have chosen, e.g.,

$$\begin{aligned} &S_{12} = 1, \quad T_3 = 0, \\ &S_{23} = p_1(p_0 + p_3)^{-1} = T_2, \\ &S_{13} = -p_2(p_0 + p_3)^{-1} = T_1. \end{aligned} \tag{2.4}$$

The five other generators are then given by

$$L_{56} \equiv D = -i(p_k\partial_k + 1), \tag{2.5}$$

$$L_{05} - L_{06} \equiv Q_0 = p_0{}^{-1}[D(D+i) + J^2], \tag{2.6}$$

$$L_{k5} - L_{k6} \equiv Q_k = i[L_{0k}, Q_0]. \tag{2.7}$$

In this representation, relation (1.3) holds in the enveloping algebra of so(4, 2) which shows (taking $B=6$) that when $n \neq 0$ the latter coincides with the enveloping algebra of so(3, 2) in the representation $\mathcal{T}_n^\pm$. From this it follows that the restriction of $\mathcal{T}_n^\pm$ to the (3+2) de Sitter group generated by the L_{AB} ($A, B \neq 6$) is irreducible when $n \neq 0$; this restriction will be denoted by $D(s+1, s)$, $s = |n|$, in Sec. III.

Moreover the two (Casimir) generators of the center of the enveloping algebra of the Lorentz Lie algebra so(3, 1) generated by the $L_{\mu\nu}$ [common to Poincaré and so(3, 2)] can be expressed in terms of n (or s) and D only, in the representation $\mathcal{T}_n^\pm$ by the relation

$$L_{0k} L_{0k} - J^2 = D^2 - s^2 + 1, \tag{2.8}$$

which follows from (1.4) and by a special case of (1.3)

$$\epsilon_{0hkl} L_{0h} L_{kl} = 2nD. \tag{2.9}$$

We give a brief account of the (already well-known) weight diagram of $\mathcal{T}_n^+$.

Consider the subalgebra so(4) = so(3)×so(3) of so(4, 2) that is spanned by $\{L_{ab}\}$, $a, b = 1, 2, 3, 6$. The two summands are spanned by $L_i^\pm \equiv \frac{1}{2}(L_{ij} \pm L_{k6})$, where ijk is a cyclic permutation of 123. Using Eqs. (1.3) and (1.4) one easily shows that

$$\sum (L_i^+ L_i^+ - L_i^- L_i^-) = n L_{05} \quad [\text{from } (1.3)] ,$$

$$\sum (L_i^+ L_i^+ + L_i^- L_i^-) = \tfrac{1}{2} L_{05}{}^2 + \tfrac{1}{6} C \quad [\text{from } (1.4)] .$$

The value of C in $\mathcal{T}_n^\pm$ is $6n^2 - 3$, so

$$C^\pm \equiv \sum L_i^\pm L_i^\pm = \tfrac{1}{4}(L_{05} \pm n + 1)(L_{05} \pm n - 1) .$$

The discussion for the preceding section shows that the spectrum of L_{05} is $j+1, j+2, \ldots$ for each $j = s = |n|, s+1, \ldots$. The restriction of $\mathcal{T}$ to the compact subalgebra so(4) is thus given by

$$\mathcal{H} = \overline{\bigoplus_{E = s+1, s+2, \ldots}} \mathcal{H}_E ,$$

where $\mathcal{H}_E$ is the eigenspace of L_{05} with eigenvalue E, and so(4) acts in $\mathcal{H}_E$ by the irreducible representation $D(k^+, k^-)$, $k^\pm = (E \pm n - 1)/2$, in which the Casimir operators $C^\pm$ take the values $k^\pm(k^\pm + 1)$.

III. REPRESENTATIONS OF SO(3,2)

Minkowski space, according to the general theory of relativity, is the point of departure of a sequence of approximations to a Riemannian space, the metric of which satisfies Einstein's equations with vanishing cosmological constant. In quantum field theory (in its present imperfect form) the Minkowski metric is the vacuum expectation value of the Riemannian metric. It seems unsafe to restrict the attention arbitrarily to the special case of vanishing cosmological constant, for this case is unstable to deformations; a nonzero cosmological constant may, for example, appear spontaneously through renormalization. In that case the zeroth approximation (or vacuum expectation value) of the metric cannot be Minkowski, but must be de Sitter. Our previous analysis must therefore be modified by the substitution of the de Sitter group for the Poincaré group. There are two de Sitter groups, SO(4, 1) and SO(3, 2); here we consider only the latter. More precisely, what we call the de Sitter group is the universal covering of the connected component $SO_0(3, 2)$; for convenience we abuse the notation and refer to this group simply as SO(3, 2). We must now describe some of its representations.

For our purposes it is convenient to define the Lie algebra so(3, 2) as the subalgebra of so(4, 2) that is spanned by $\{L_{\alpha\beta}\}$, $\alpha, \beta = 0, 1, 2, 3, 5$. These 10 generators satisfy commutation relations similar to (1.1) obtained by replacing indices $A, B, \ldots$ with range 0, 1, 2, 3, 5, 6 by indices $\alpha, \beta, \ldots$ with range 0, 1, 2, 3, 5. The subalgebra so(3, 1) spanned by $\{L_{\mu\nu}\}$, $\mu, \nu = 0, 1, 2, 3$, plays the same role as the Lorentz subalgebra of $\mathcal{P}$ and contains the usual rotation algebra so(3) spanned by $\{L_{kl}\}$, $k, l = 1,2,3$. The role of translations is taken over by the generators $\{L_{\mu 5}\}$, $\mu = 0, 1, 2, 3$, and L_{05} corresponds to time translations.

The representations of the Poincaré group that have direct relevance to elementary particles are characterized by the fact that the energy spectrum is positive definite. It is expected, therefore, that the representations of so(3, 2) of the most immediate interest to us are those in which the spectrum of L_{05} is positive definite. [Such representations have no analog in so(4, 1) which is the main reason why we reject the other de Sitter group.] In an irreducible representation of this type, in a Hilbert space $\mathcal{H}'$, let E_0 denote the lowest eigenvalue of L_{05}; then

$$\mathcal{H}' = \overline{\bigoplus_{E = E_0, E_0+1, \ldots}} \mathcal{H}'_E ,$$

where $\mathcal{H}'_E$ is an eigenspace of L_{05} with eigenvalue E. Now it is easy to verify that so(3) acts irreducibly in $\mathcal{H}'_{E_0}$. Let $D(s)$ be the representation of so(3) that appears here, $2s$ being a fixed non-negative integer. In $\mathcal{H}'_{E_0}$ we may diagonalize L_{12}; the eigenspace with highest eigenvalue s corresponds to an extremal weight $(L_{05}, L_{12}) \to (E_0, s)$ of the representation. This extremal weight uniquely defines the whole representation of so(3, 2) up to equivalence and we may thus denote the equivalence classes of interest by $D(E_0, s)$. This corresponds to the labeling of representations of $\mathcal{P}$ by mass and spin, as will be seen later.

Concerning the problem of the unitarity of $D(E_0, s)$, the following was known. (i) The inequality $E_0 \geq s+2$ is a sufficient condition for $D(E_0, s)$ to be equivalent to a unitary representation[13]; (ii) the inequality $E_0 > s$ is a necessary condition[14]; (iii) when $s = 0$ the necessary and sufficient condition is $E_0 \geq \frac{1}{2}$,[15] and when $s = \frac{1}{2}$ the necessary and sufficient condition is $E_0 \geq 1$[16]; (iv) $D(s+1, s)$ is unitary for all s. This left out most of the interval $s < E_0 < s+2$ for $s \geq 1$. The question was resolved recently and the result is given by the following.

Proposition 1.[7] $D(E_0, s)$ is unitary if and only if one of the following conditions holds: (i) $s = 0$, $E_0 \geq \frac{1}{2}$. (ii) $s = \frac{1}{2}$, $E_0 \geq 1$. (iii) $s \geq 1$, $E_0 \geq s+1$.

The proof is given in Ref. 7.

The spectrum of L_{05} is E_0, $E_0 + 1, \ldots$ in all cases, and E denotes any one of these eigenvalues. We give the irreducible representations $D(j)$ of SO(3) that occur in each eigenspace $\mathcal{H}'_E$, including the multiplicities. In all cases $s+j$ is an integer, $|j-s|\leq E-E_0$ and $j\geq 0$. The following special cases are distinguished by the fact that all the multiplicities are equal to one:

$D(E_0,0)$, $E_0>\frac{1}{2}$, $j=E-E_0, E-E_0-2,\ldots,$ 1 or 0 (Ref. 15)

$D(E_0,\frac{1}{2})$, $E_0>1$, $j-\frac{1}{2}=E-E_0, E-E_0-1,\ldots,$ 0 (Ref. 16)

$D(s+1,s)$, $s\geq 1$, $j=s, s+1,\ldots, E-1$ (Ref. 9)

$D(\frac{1}{2},0)=\mathrm{Rac}$ and $D(1,\frac{1}{2})=\mathrm{Di}$, the two Dirac singleton representations, with the drastically reduced weight diagram given by $j=E-\frac{1}{2}$ (Refs. 15 and 16). Notice the lower cutoff on j in $D(s+1,s)$; this is strongly reminiscent of a property of massless wave propagation in Minkowski space. As will be seen, these are the massless representations. For $D(E_0,s)$, $E_0>s+1\geq 2$, the multiplicities are all equal to those of the reducible representation $D(s+1,s)\oplus D(E_0+1,s-1)$. (This is an easy consequence of the facts discussed in Sec. V.) The highest multiplicity is thus $s+1$.

IV. RESTRICTION OF $\mathcal{T}_n^+$ TO SO(3,2)

Recall that the eigenspace $\mathcal{H}_E$ of L_{05} in $\mathcal{T}_n^+$ carries an irreducible representation $D(k^+,k^-)$ of SO(4). In identifying SO(3, 2) with a subgroup of SO(4, 2), we relate the rotation subgroup SO(3) of SO(3, 2) to the subgroup of SO(4) generated by $\{L_{kl}\}$, $k,l=1,2,3$. Restriction of $D(k^+,k^-)$ to SO(3) gives $\oplus_j D(j)$, with the summation restricted to $|k^+-k^-|\leq j\leq|k^++k^-|$ or $s=|n|\leq j\leq E-1$. This is, for $s\neq 0$, precisely the spectrum of j in the eigenspace $\mathcal{H}'_E$ of the representation $D(s+1,s)$ of SO(3, 2). In the case of $s=0$, it is the spectrum of j in the eigenspace $\mathcal{H}'_E$ of the representation $D(1,0)\oplus D(2,0)$. Hence we conclude the following.

Proposition 2. The restriction of the massless representation $\mathcal{T}_n^+$ of the conformal group to SO(3, 2) is the massless representation $D(s+1,s)$ of SO(3, 2), $s=|n|$, except that the restriction of $\mathcal{T}_0^+$ is the sum of $D(1,0)$ and $D(2,0)$.

This, of course, is one of our reasons for calling $D(s+1,s)$ a "massless" representation, but it is not the only reason. In fact, those representations are the only ones among the particlelike representations $D(E_0,s)$ that are associated with the usual type of gauge invariance.[9,10,17] Another type of gauge invariance arises in connection with singletons.[9,11]

In de Sitter space one finds no precise analog of the concept of helicity. The quantum number s is non-negative and is more appropriately called spin, it is related to the absolute value of the helicity, as we have seen and as we shall confirm later, but the sign of the helicity is not an attribute of a representation of SO(3, 2). (Chirality can nevertheless be defined in de Sitter space, at least for neutrino fields.[17]) We wish to explain how this comes about.

As earlier, we denote by V_n^+ (V_n^-) the unitary, irreducible representation of the Poincaré group $\mathcal{P}$ with mass zero, helicity n, and positive (negative) energy. Recall that V_n^+ (V_n^-) has $\mathcal{T}_n^+$ ($\mathcal{T}_n^-$) as a unique extension to the conformal group—more precisely SU(2, 2)—and that $\mathcal{T}_n^\pm$ are given explicitly in $\mathcal{H}=L^2(\mathbb{R}^3, d^3p/|\vec{p}|)$ by the formulas (2.1)–(2.3). Now the conformal group contains two classes of Poincaré subgroups, one that contains $\mathcal{P}$ (generated by $L_{\mu\nu}$ and P_μ) and another that contains the subgroup $\mathcal{P}'$ generated by $L_{\mu\nu}$ and Q_μ. The Cartan involution $L_{AB}\to L^{AB}$ of so(4, 2) exchanges P_μ with Q^μ but is given by an outer automorphism of $\mathcal{C}$. However, the restriction of $\mathcal{T}_n^+(\mathcal{T}_n^-)$ to $\mathcal{P}'$ is equivalent to $V_n^+(V_n^-)$. This follows from the relations

$$W_\mu\equiv\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}L_{\nu\rho}P_\sigma=nP_\mu\,,$$

$$\tilde{W}_\mu\equiv\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}L_{\nu\rho}Q_\sigma=nQ_\mu\,,$$

where we adopt the same definition $\epsilon_{\mu\nu\rho\sigma}=\epsilon_{\mu\nu\rho\sigma 56}$ for both $\mathcal{P}$ and $\mathcal{P}'$. The involution defined by the exchange of the coordinates 5 and 6 changes P_μ into P_μ, Q_μ into $-Q_\mu$, and $\epsilon_{\mu\nu\rho\sigma}$ into $-\epsilon_{\mu\nu\rho\sigma}$, thereby changing n into $-n$. In the representation space, this involution transforms $V_n^\pm$ into $V_{-n}^\pm$ and changes the signs of part of the conformal generators. The subgroups $\mathcal{P}$ and $\mathcal{P}'$ are conjugate only in the full, nonconnected group SO(4, 2) and in the representations extended from $\mathcal{T}_n^\pm$ are transformed one into the other by the involution $\theta=P\exp(i\pi L_{05})$, where P is the spatial reflection operator P: $f(\vec{p})\to f(-\vec{p})$, $\vec{p}\in\mathbb{R}^3$. On the other hand, the conformal group contains only one conjugacy class of SO(3, 2) subgroups, and θ commutes with the action of SO(3, 2) in $\mathcal{H}$.

From the above, and from the knowledge of the Poincaré conjugacy classes within the conformal group, it follows that no UIR of the latter will have both V_n^+ and V_{-n}^+ as its Poincaré restriction. The distinction between V_n^+ and V_{-n}^+, characteristic to both $\mathcal{P}$ and $\mathcal{C}$, has no analog for SO(3, 2): in de Sitter space, massless particles are completely characterized by the absolute value of the helicity (the "spin" $s=|n|$).

Proposition 2 shows us that the representations D_s^+, where we define

$$D_s^+=D(s+1,s)\quad\text{if } s>0\,,$$

$$D_0^+=D(1,0)\oplus D(2,0)\,,$$

are restrictions to SO(3, 2) of a (unitary) representation of $\mathcal{C}$, namely $\mathcal{T}_n^+$ where $n=\pm s$. We may now ask the same question as in theorem 1, namely, the (operatorial) uniqueness of the extension from SO(3, 2) to $\mathcal{C}$. The answer is what can be expected.

Theorem 1'. The only extensions of D_s^+ from SO(3, 2) to a projective group representation of $\mathcal{C}$ are $\mathcal{T}_{\pm s}^+$. When $s\neq 0$, there are two possible inequivalent extensions, each operatorially uniquely determined by its restriction to SO(3, 2). For $s=0$ there are two possible equivalent realizations of the extension.

The proof is basically the same as for theorem 1, and we adopt the same notations. We can still decompose the representation space $\mathcal{H}=\overline{\oplus}_j\mathcal{H}_j$ ($j=s, s+1, \ldots$) according to the eigenvalues $j(j+1)$ of J^2, and look for the subgroup $\tilde{S}O(2,1)$ of $\mathcal{C}$ generated by the compact generator L_{05} of so(3, 2), D, and L_{06}. In each $\mathcal{H}_j$ the spectrum of L_{05} is semibounded and of the form $\{(j+n+1);\ n\in \mathbf{N}\}$. This determines uniquely the representation of $\tilde{S}O(2,1)$ in question. If we denote by L_{05}, L_{56}', and L_{06}' a set of generators of this so(2, 1) Lie algebra that satisfy the same commutation relations as L_{05}, L_{56}, and L_{06}, the only ambiguity in the extension is the identification $D\equiv L_{56}=\epsilon L_{56}'$ and $L_{06}=\epsilon L_{06}'$ with the same $\epsilon=\pm 1$. Whichever choice is made, relation (2.8) will hold in the enveloping algebra of so(4, 2), which gives us the (unique) operatorial expression of D^2 from the restriction of D_s^+ to the Lorentz subgroup generated by the $L_{\mu\nu}$.

If $s\neq 0$, we may then, on the basis of (2.9), choose

$$2sL_{56}'=\epsilon_{0hkl}L_{0h}L_{kl}$$

and define $n=\epsilon s$. For each choice of ϵ we then obtain the extension $\mathcal{T}_n^+$. [The operatorial uniqueness of each extension follows also from the irreducibility of $D(s+1,s)$.] The involution θ, which commutes with the generators of so(3, 2) and anticommutes in particular with D, transforms one of these extensions into the other one. On the other hand, when $s=0$, the enveloping algebra of so(3, 1) determines only D^2 and we have a decomposition $\mathcal{H}_j=\mathcal{H}_j^1\oplus\mathcal{H}_j^2$, hence $\mathcal{H}=\mathcal{H}^1\oplus\mathcal{H}^2$, according to the eigenvalues of the involution θ. Each $\mathcal{H}^\tau$ ($\tau=1,2$) is invariant under SO(3, 2) represented by $D(\tau,0)$. L_{05} and D^2 leave $\mathcal{H}_j^\tau$ invariant ($\frac{1}{2}L_{05}$ having for spectrum $[\frac{1}{2}(j+\tau)+n,\ n\in\mathbf{N}]$ on $\mathcal{H}_j^\tau$, $j\in\mathbf{N}$) while D transforms $\mathcal{H}_j^\tau$ into $\mathcal{H}_j^{3-\tau}$. Therefore, we obtain in this case, operatorially, two possible extensions, each equivalent to $\mathcal{T}_0^+$ and transformed one into another by θ. A similar result is true in the negative-energy case (representations $\mathcal{T}_n^-$).

V. GAUGE INVARIANCE

Gauge invariance arises from the association between representations and tensor fields. The following is a brief sketch of ordinary gauge invariance in de Sitter space. Singletons will be discussed in Sec. VI.

Integer spins.[9] As in flat space, $D(E_0,s)$ is naturally associated with symmetric tensor fields of rank s on de Sitter space:

$$h=\{h_{\mu_1\cdots\mu_s}(x)\},\quad \mu_1,\ldots,\mu_s=0,1,2,3,$$

satisfying the wave equation

$$\left[\tfrac{1}{2}\sum_{\alpha,\beta}L_{\alpha\beta}{}^2-E_0(E_0-3)-s(s+1)\right]h=0 \qquad (5.1)$$

and the subsidiary conditions

$$g^{\mu\nu}h_{\mu\nu\ldots}=0,\quad g^{\mu\nu}\nabla_\mu h_{\nu\ldots}=0. \qquad (5.2)$$

Here $L_{\alpha\beta}$, by abuse of notation, is the Lie derivative associated with an infinitesimal SO(3,2) transformation of de Sitter space, g is the de Sitter metric, and ∇ the covariant derivative determined by the metric connection.

If $E_0\neq s+1$ (when $s=0$, if $E_0\neq\frac{1}{2}$), the solutions of these equations with suitable boundary conditions carry the irreducible representation $D(E_0,s)$. But if $E_0=s+1$, $s>0$, then there exists a subspace of solutions of the form of gauge fields:

$$h_{\mu_1\cdots}=\sum\nolimits_1\nabla_{\mu_1}\xi_{\mu_2\cdots},\quad \xi'=0. \qquad (5.3)$$

[Notation: ξ is symmetric and the trace ξ' is $g^{\mu\nu}\xi_{\mu\nu\cdots}$. The sum $\sum_1$ is over the $s=s!/(s-1)!$ essential reorderings of the s indices.]

The representations $D(s+1,s)$, $s=1,2,\ldots$, are thus further distinguished among the $D(E_0,s)$ by the fact that relativistic wave fields exhibit gauge phenomena. Gauge-invariant wave equations have been found[9] for all spins; they describe electrodynamics when $s=1$,[17] linearized gravitation when $s=2$.[17]

Half-integer spins.[10] Again as in flat space, $D(E_0,s)$ is associated with Rarita-Schwinger spinor-tensor fields of tensorial rank n on de Sitter space:

$$h=\{h_{\mu_1\cdots\mu_n}(x)\},\quad \mu_1,\ldots,\mu_n=0,1,2,3,\quad n=s-\tfrac{1}{2},$$

satisfying the wave equation

$$[i\gamma^\mu D_\mu-\rho^{1/2}(E_0-\tfrac{3}{2})]h=0 \qquad (5.4)$$

and the subsidiary conditions

$$\gamma^\mu h_{\mu\ldots}=0,\quad g^{\mu\nu}\nabla_\mu h_{\nu\ldots}=0. \qquad (5.5)$$

Here D_μ is the spinor-covariant derivative and $\Delta_\mu=D_\mu-\gamma_\mu\rho^{1/2}/2i$. Again one finds that the case $E_0=s+1$, $s\geq\frac{3}{2}$, is distinguished by the existence of a space of gauge field solutions of the form

$$h_{\mu_1\cdots}=\sum_1\nabla_{\mu_1}\xi_{\mu_2\cdots},\quad \xi'=0. \qquad (5.6)$$

Here ξ' is the spinor trace $\gamma^\mu\xi_{\mu\cdots}$. Gauge-invariant wave equations have been found[10] for all half-integer spins as well; for $s=\frac{3}{2}$ they describe the spin $=\frac{3}{2}$ sector of linearized supergravity (with cosmological constant).

The representations $D(s+1,s)$ for $s\geq 1$ have two attributes of masslessness: gauge invariance and extensions to $\mathcal{T}^{+}_{\pm s}$. No gauge invariance seems to be associated with $D(\frac{3}{2},\frac{1}{2})$, but this representation is unique in that the field is chiral.[17] As a matter of fact, gauge invariance appears with the lowest value of E_0 that is compatible with unitarity (cf. Proposition 1) in the family $D(E_0, s)$. This value, for $s\geq 1$, is $E_0=s+1$ while for $s=0,\frac{1}{2}$ it is $E_0=s+\frac{1}{2}$. The gauge invariance of the last two (singleton) representations is explicated in Sec. VI.

VI. SINGLETONS

The singletons Rac $=D(\frac{1}{2},0)$ and Di $=D(1,\frac{1}{2})$ have the following wonderful properties.[11]

Theorem 3.

$$\mathrm{Rac}\otimes\mathrm{Rac}=\bigoplus_{s=0,1,\ldots} D(s+1,s)\,,$$

$$\mathrm{Rac}\otimes\mathrm{Di}=\bigoplus_{2s=1,3,\ldots} D(s+1,s)\,,$$

$$\mathrm{Di}\otimes\mathrm{Di}=\bigoplus_{s=1,2,\ldots} D(s+1,s)\oplus D(2,0)\,.$$

Massless particles can therefore be thought of as composite objects. Note that this is possible in de Sitter space as long as the curvature is nonzero, but not in flat space. We believe, nevertheless, that it may be very useful, even within the strict context of physics in Minkowski space, to recognize that massless particles are composite in some limiting sense. What makes this especially attractive is that singletons are practically unobservable. First of all, absorption of a single Di or Rac particle by an apparatus of earthly dimensions involves an energy so small as to make detection by energy balance quite impossible. Detecting singleton emission or absorption by zero-energy spin transitions between massless particles (e.g., photon $\to$ neutrino + singleton) is forbidden because of a special property of singletons: for massless particles $E-j$ is always an integer, but for singletons $E-j=\frac{1}{2}$. (It was this feature that drew Dirac's attention to singletons in the first place.[18])

In view of this relationship between singletons and massless fields, it is not surprising to find that singleton fields are encumbered by gauge invariance. To describe this phenomenon we embed de Sitter space in R^5 by means of a differentiable and locally invertible map given by $(x^0,x^1,x^2,x^3)\to(y^0,y^1,y^2,y^3,y^5)$ with $y^2\equiv\eta_{\alpha\beta}y^\alpha y^\beta=1/\rho$. (The metric η is the same as in Sec. I and ρ is a positive constant related to the cosmological constant.)

The Rac.[9,11] Consider scalar fields ϕ on the hyperboloid $y^2=1/\rho$ satisfying the wave equation

$$(\tfrac{1}{2}L_{\alpha\beta}L^{\alpha\beta}+\tfrac{5}{4})\phi=0\,, \tag{6.1}$$

$$L_{\alpha\beta}\phi\equiv i(y_\alpha\partial_\beta-y_\beta\partial_\alpha)\phi\,. \tag{6.2}$$

Here $-\frac{5}{4}$ is the value $E_0(E_0-3)+s(s+1)$ of the Casimir operator in the case $E_0=\frac{1}{2}$, $s=0$. Note that $L_{\alpha\beta}$ is a vector field on the hyperboloid. Now let us extend the definition of ϕ to R^5 by fixing the degree of homogeneity to be $-\frac{1}{2}$ (see Ref. 9 if a more pedantic formulation is desired); then the above wave equation simplifies to

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\phi=0\,, \tag{6.3}$$

$$(y^\alpha\partial_\alpha+\tfrac{1}{2})\phi=0\,. \tag{6.4}$$

A set of solutions is given[15] by ($j=E-\frac{1}{2}=0,1,\ldots$)

$$\phi_{j,m}(y)=r^jY^{-E}Y_{j,m}(\Omega)\exp(-iEt)\,,\qquad Y\equiv(y^0y^0+y^5y^5)^{1/2},\quad y_0/y_5=\tan t\,, \tag{6.5}$$

where $Y_{j,m}$ are spherical harmonics. On the hyperboloid, $Y^2=(r^2+1/\rho)$, $r^2\equiv(y^1)^2+(y^2)^2+(y^3)^2$, so these solutions have the property that

$$\lim_{r\to\infty} r^{1/2}\phi_{j,m}(y)=Y_{j,m}(\Omega)\exp(-iEt)\,, \tag{6.6}$$

where Ω stands for the angular variables.

These solutions do not transform among themselves under the action of $L_{\alpha\beta}$. One finds, however, that

$$\lim_{r\to\infty} r^{1/2}L_{\alpha\beta}\phi_{j,m}(y) \tag{6.7}$$

is a finite linear combination of the $Y_{j,m}$, and that the matrices so defined are a realization of $D(\frac{1}{2},0)$ in the Hilbert space l^2. In other words, we introduce the space $\mathcal{V}$ of solutions of (6.3) and (6.4) satisfying the boundary conditions

$$\lim_{r\to\infty} r^{1/2}\phi(y)<\infty\,. \tag{6.8}$$

The subspace $\mathcal{V}_0$ of $\mathcal{V}$ on which this limit vanishes is an invariant subspace for the action of $L_{\alpha\beta}$ in $\mathcal{V}$, and the Rac is realized on the quotient.

Fields that satisfy $\lim r^{1/2}\phi(y)=0$ will be called gauge fields. If the degree of homogeneity has been fixed once and for all by (6.4), as we shall suppose from now on, then such fields are equivalently characterized by the statement that

$$\lim_{y^2\to 0}\phi(y)=0\,, \tag{6.9}$$

the limit being taken with $\vec{y},t$ fixed. The physical Rac states are thus given, not by the field ϕ, but by the restriction of ϕ to the cone $y^2=0$. Alternatively, one may adopt the language of Penrose,[19] and say that the physical states are given entirely by the value $\tilde{\phi}(t,\Omega)$ of $r^{1/2}\phi(y)$ at infinity.

The Di. Let ψ be a spinor field satisfying the wave equation[16]

$$(\kappa + \tfrac{3}{2})\psi = 0\,,\qquad \kappa \equiv 2i\Sigma^{\alpha\beta} y_\alpha \partial_\beta\,. \tag{6.10}$$

Here κ is the Dirac wave operator for de Sitter space, $\{\Sigma^{\alpha\beta}\}$, $\alpha, \beta = 0, 1, 2, 3, 5$, are the matrices of the four-dimensional, symplectic representation of so(3,2) = sp(2,R). The constant term has the value that is appropriate to the representation Di $= D(1, \frac{1}{2})$, thus the above is equivalent to (5.4) with $E_0 = 1$. Now we proceed as in the case of the Rac to fix the degree of homogeneity, replacing (6.10) by

$$\gamma^\alpha \partial_\alpha \psi = 0\,, \tag{6.11}$$

$$(y^\alpha \partial_\alpha + \tfrac{3}{2})\psi = 0\,. \tag{6.12}$$

A set of solutions is $(E = j + \frac{1}{2})$

$$\psi_{j,m} = \tilde{\gamma}^\alpha \partial_\alpha r^{E-1} Y^{-j} \mathcal{Y}^-_{j,m}(\Omega)\exp(-iEt)\,, \tag{6.13}$$

where $\mathcal{Y}^-_{j,m}$ are spinors of fixed angular momentum.[16] (The γ matrices used in this section are $\{\gamma^\alpha\} = \{\gamma^0, \gamma^1, \gamma^2, \gamma^3, i\}$, $\{\tilde{\gamma}^\alpha\} = \{\gamma^0, \gamma^1, \gamma^2, \gamma^3, -i\}$, where $\{\gamma^\mu\}$, $\mu = 0, 1, 2, 3$ are the usual, constant Dirac matrices.)

The solutions (6.13) have the property that the limit

$$\lim_{r\to\infty} r^{3/2}\psi_{j,m} \text{ exists}\,, \tag{6.14}$$

while "gauge fields" have the defining property

$$\lim_{r\to\infty} r^{3/2}\psi = 0 \tag{6.15}$$

or equivalently,

$$\lim_{y^2\to 0} \psi(y) = 0\,, \tag{6.16}$$

the limit being the same as in (6.9). The situation is in all respects the same as for the Rac, except that the scalar field ϕ has degree of homogeneity $-\frac{1}{2}$ while that of ψ is $-\frac{3}{2}$.

Because the solutions of singleton wave equations include so few that are physically relevant, difficulties akin to those encountered in more familiar gauge theories are to be expected. However, the fixing of the degrees of homogeneity, which permits us to define gauge fields by the limit $y^2 \to 0$ instead of doing it in terms of the limit $r \to \infty$, leads to an easy solution of these difficulties, or at least some of them.

VII. MORE ABOUT SINGLETONS

Singleton fields on the cone. A classical field theory of interacting singletons may perhaps be formulated as a variational principle and based on a Lagrangian density on de Sitter space, that is, on the twofold covering of the hyperboloid $y^2 = 1/\rho$ in R^5. The Lagrangian must be SO(3,2) invariant, but it should also be gauge invariant, that is, it must somehow be guaranteed that only physical singleton states propagate. A covariant propagator will contain many unphysical modes that must be suppressed through the choice of interactions.

The problem appears greatly simplified when it is remembered that the physical content of a singleton field is preserved by projection on the cone $y^2 = 0$, while unphysical modes are eliminated. Therefore, by restriction of the homogeneous fields of the preceding section to the cone, all complications seem to disappear. (Of course, it is not immediately clear how one should proceed when other fields are present as well, but one problem at a time.)

An integral over the cone of a Lagrangian density constructed from homogeneous fields cannot have any meaning. To obtain a finite action we must integrate over the three essential dimensions only. For $y^2 = 0$ let

$$y_\pm \equiv y_5 \pm y_3\,,\quad y_\mu = y_+ u_\mu\,,\quad \mu = 0, 1, 2\,. \tag{7.1}$$

The variable u may be regarded as the coordinates of a three-dimensional Minkowski space M_3. Fields on M_3 are defined by (see the Appendix)

$$\phi(y) = y_+^{-1/2}\tilde{\phi}(u)\,,\qquad \psi(y) = y_+^{-3/2} M^{-1}\tilde{\psi}(u)\,. \tag{7.2}$$

Now the action of SO(3,2) in R^5 induces an action in M_3 that is precisely that of the conformal group in M_3. Singleton fields may therefore be interpreted (locally) as massless fields in a three-dimensional spacetime (Penrose's infinity[19]). The problem is therefore to construct a Lagrangian density on M_3 that is invariant under the conformal group SO(3,2). The conformal degrees of $\tilde{\phi}$ and $\tilde{\psi}$ are $-\frac{1}{2}$ and -1, respectively. The action

$$\mathcal{L} = \int d^3u\, F(u) \tag{7.3}$$

is invariant if and only if the conformal degree of F is -3.

Proposition 4. Let F be a scalar field on M_3 of conformal degree -3; then the field L on R^5 defined by

$$L(y) = y_+^{-3} F(u) \tag{7.4}$$

is a scalar field. Conversely, if L is a scalar field on R^5, homogeneous of degree -3, then F is a scalar field on M_3 of conformal degree -3.

If F is local and polynomial then the only possible form for L is (see the Appendix)

$$L = -\tfrac{1}{2}(\partial_\alpha\phi)^2 + \bar{\psi}(\kappa + \tfrac{3}{2})\psi + g_1\phi^6 + g_2\phi^2\bar{\psi}(i\hat{\gamma}_5 \not{y})\psi\,. \tag{7.5}$$

The most general interaction depends on only two real coupling constants g_1 and g_2. For F one finds (see the Appendix)

$$F=-\tfrac{1}{2}(\partial\tilde{\phi}/\partial u^a)^2-i\bar{\tilde{\psi}}\gamma^a\partial_a\tilde{\psi}+g_1\tilde{\phi}^6+g_2\tilde{\phi}^2\bar{\tilde{\psi}}\hat{\gamma}_5\tilde{\psi}\,. \quad (7.6)$$

The summations are on $a=0,1,2$ and $\hat{\gamma}_5=\gamma_0\gamma_1\gamma_2\gamma_3$.

Global coordinates. Three-dimensional Minkowski space has a natural causal structure that is locally, but not globally, conformally invariant. The cone $y^2=0$, on the other hand, also has a natural causal structure that is globally SO(3,2) invariant. This latter causal structure is the physically relevant one, but it is awkward to describe it on M_3; this is due to the singularities in the transformations (7.1) and (7.2). Global coordinates on the twofold covering $\overline{C}$ of the cone are given by

$$y_5=r\cos t,\quad y_0=r\sin t,\quad -2\pi<t<2\pi\,,$$

$$y_3=r\cos\theta,\quad y_2=r\sin\theta\sin\phi,\ldots\,.$$

An invariant distribution on $\overline{C}\times\overline{C}$ depends on $y\cdot y'\equiv y^\alpha y'_\alpha$ and on $\tau\equiv t-t'$; in τ it is piecewise constant. Figure 1 shows the regions in which y is in the future, in the past and spacelike relative to y', with

$$\cos\psi\equiv\vec{y}\cdot\vec{y}'/rr',\quad y^\alpha y'_\alpha=rr'(\cos\tau-\cos\psi)\,.$$

Any invariant distribution is independent of τ (that is, it depends only on $y\cdot y'$) within each of the four regions.

Quantization on the cone. Here we limit ourselves to the case of the Rac fields. A basis is given by

$$\phi_{LM}(y)=(2L+1)^{-1/2}r^{-1/2}e^{-iEt}Y_{LM}(\hat{y})\,,$$

with $E=L+\frac{1}{2}$ and $\hat{y}=\vec{y}/r$; it is orthonormal with respect to the inner product

$$(\phi,\phi')=\int_{S^2}\overline{\phi}(y)i\overleftrightarrow{\partial}_t\phi'(y)\,d\Omega\,.$$

For each (L,M) in the range $M=-L,\ -L+1,\ldots,L$ and $L=0,1,\ldots,$ introduce creation and destruction operators $a_N=a_{LM}$ satisfying the usual commutation relations

$$[a_N,a^*_{N'}]=\delta_{NN'}\,,\quad [a_N,a_{N'}]=[a^*_N,a^*_{N'}]=0\,.$$

(This leads to conventional, Bose-Einstein statistics, although the possibility of other alternatives should not be excluded.) Defining the field operator

$$\phi(y)=\sum_N[\phi_N(y)a_N+\overline{\phi}_N(y)a^*_N]\,,$$

one finds

$$[\phi(y),\phi(y')]=\frac{1}{4\pi}(2rr')^{-1/2}[s(\tau+i\epsilon)-s(\tau-i\epsilon)]\,,$$

$$s(\tau)=(\cos\tau-\hat{y}\cdot\hat{y}')^{-1/2}\,.$$

Here $\tau=t-t'$ and $s(\tau)$ is defined as an analytic function of $\cos\tau$ in the complex plane with a cut along the real axis from +1 to $-\infty$, positive for $\cos\tau>1$. The commutator is thus zero for spacelike separation of y,y'; that is, for $y\cdot y'>0$. For equal times, if $\dot{\phi}(y)=(d/dt)\phi(y)$,

$$[\phi(y),\dot{\phi}(y')]_{t=t'}=i(rr')^{-1/2}\delta(\hat{y},\hat{y}')\,.$$

To complete the quantization of free Rac fields on the cone we define the Fock space in terms of a vacuum state on which a_N vanishes.

Indefinite-metric quantization of Rac fields on the whole of de Sitter space will be set up in a future publication[20]; we also plan a similar construction for Di fields.

The quantization scheme developed here is the most conventional one possible. The implications of adopting it, for the dynamics of massless particles, are not completely clear at this time; therefore, it is important to keep alternative options in mind. One possibility is parastatistics or color, another is to reverse the normal association between spin and statistics. (This association is less strong in three dimensions than in four dimensions—a fact that may be relevant to singleton dynamics.)

VIII. INTERTWINING OPERATOR

Theorem 3 (in Sec. VI) shows that massless particles can be interpreted as two-singleton states. This will now be expressed in terms of (classical) massless fields and two-singleton fields. Again we limit ourselves to Racs and integer-spin massless fields.

As in Sec. V, let $h_{\mu_1\cdots\mu_s}$ stand for the components of a symmetric tensor field on de Sitter space. Let $x\to y$ be an embedding of de Sitter space as the hyperboloid $y^2=1/\rho$, let y^α_μ be the coefficients of the differential of this map. Let $k_{\alpha_1\cdots\alpha_s}$ be the components of a symmetric tensor field on R^5, completely determined by

$$h_{\mu_1\cdots\mu_s}(x)=y^{\alpha_1}_{\mu_1}\cdots y^{\alpha_s}_{\mu_s}k_{\alpha_1\cdots\alpha_s}(y)\,, \quad (8.1)$$

$$y^\alpha k_{\alpha\cdots}(y)=0\,, \quad (8.2)$$

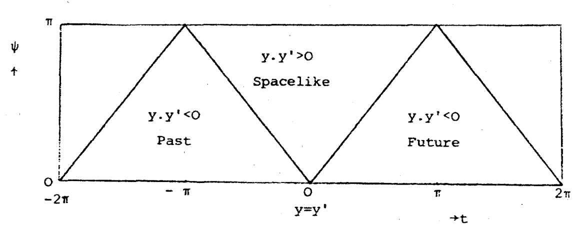

FIG. 1. Regions on the cone are described relative to a fixed y' on it. The variable ψ is given by $\cos\psi=y\cdot y'/rr'$. Points labeled $(\psi,2\pi)$ are to be identified with points labeled $(\psi,-2\pi)$.

$$(y^\alpha \partial_\alpha + s + 1) k_{\beta \cdots} = 0 . \tag{8.3}$$

The degree of homogeneity is chosen for later convenience. The subsidiary conditions (5.2) take the form

$$\eta^{\alpha\beta} k_{\alpha\beta} = 0 , \quad \eta^{\alpha\beta} \partial_\alpha k_{\beta \cdots} = 0 . \tag{8.4}$$

When (8.2)–(8.4) are satisfied, then the wave equation (5.1) reduces to

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta k_{\gamma \cdots} = 0 . \tag{8.5}$$

Now let us combine all these fields (with $s = 0, 1, \ldots$) into a single scalar field on $R^5 \times R^5$, defined by

$$K(y, z) \equiv \sum_s z^{\alpha_1} \cdots z^{\alpha_s} k_{\alpha_1 \cdots \alpha_s}(y) . \tag{8.6}$$

The problem is to relate this field to a two-singleton field Φ. This was done already in Ref. 9, but a much more convenient formula can be given.

Consider the (classical) field on $\overline{C} \times \overline{C}$ defined by

$$\Phi(p, q) = \frac{1}{\pi} \int_0^\infty \frac{d\lambda}{\lambda} K(\lambda p + q/\lambda, \lambda p - q/\lambda) . \tag{8.7}$$

It has the following properties.

(1) Homogeneity. Equation (8.3) gives $(y \cdot \partial_y + z \cdot \partial_z + 1) K = 0$ and (8.7) therefore implies that $(p \cdot \partial_p + q \cdot \partial_q + 1)\Phi = 0$. Further,

$$\begin{aligned}(p \cdot \partial_p - q \cdot \partial_q) \Phi(p, q) &\sim K\left(\lambda p + \frac{q}{\lambda}, \lambda p - \frac{q}{\lambda}\right)\Bigg|_{\lambda \to 0}^{\lambda \to \infty} \\ &= \lim_{\lambda \to \infty} \lambda^{-1} K(p + q/\lambda^2, p - q/\lambda^2) \\ &\quad - \lim_{\lambda \to 0} \lambda K(\lambda^2 p + q, \lambda^2 p - q) ,\end{aligned}$$

which vanishes provided only that $K(y, z)$ exists for $y = \pm s$. Thus

$$(p \cdot \partial_p + \tfrac{1}{2}) \Phi = (q \cdot \partial_q + \tfrac{1}{2}) \Phi = 0 . \tag{8.8}$$

(2) Transversality. Equation (8.2) requires that all components of K be "transverse." This can always be achieved by adding a suitable field of the form $y \cdot z \Lambda$ to K. Now (8.7) is to be evaluated at $p^2 = q^2 = 0$, so that the integral involves the values of K on the surface $y \cdot z = y^2 + z^2 = 0$ only. Longitudinal components of K (terms of the form $y \cdot z \Lambda$ with Λ well defined at $y \cdot z = 0$) are therefore irrelevant for the evaluation of Φ on $\overline{C} \times \overline{C}$. In other words, when one requires that K be transverse, namely, condition (8.2) or

$$y \cdot \partial_z K = 0 , \tag{8.9}$$

then one restricts the extrapolation of Φ off $\overline{C} \times \overline{C}$.

(3) Gauge invariance. If each $h_{\mu_1 \cdots \mu_s}$ is a gauge field, then K has the form

$$K = [z \cdot \partial_y + (z \cdot \partial_z - y \cdot \partial_y - 2) z \cdot y / y^2] \Lambda , \tag{8.10}$$

with $(z \cdot \partial_z + y \cdot \partial_y + 1)\Lambda = y \cdot \partial_z \Lambda = 0$. Thus, when $z \cdot y = 0$, $K(y, z) = (z \cdot \partial_y + y \cdot \partial_z)\Lambda$, and $K(\lambda p + q/\lambda, \lambda p - q/\lambda) = (\lambda \partial/\partial\lambda) \Lambda(\lambda p + q/\lambda, \lambda p - q/\lambda)$; substituting this into (8.9) one obtains $\Phi = 0$ by the argument used above to establish (8.10).

(4) Wave equations and the Lorentz condition. The trace condition and Lorentz condition (8.4), and the wave equation (8.5), are expressed by

$$\partial_z{}^2 K = \partial_y \cdot \partial_z K = \partial_y{}^2 K = 0 . \tag{8.11}$$

It is easy to see that this gives

$$\partial_p{}^2 \Phi = \partial_q{}^2 \Phi = \partial_p \cdot \partial_q \Phi = 0 . \tag{8.12}$$

Each of these conditions involves the values of Φ off $\overline{C} \times \overline{C}$. We have already restricted the extrapolation of Φ by the conditions (8.10) of homogeneity, and this is sufficient to give unambiguous meaning to the first two conditions. The third condition, $\partial_p \cdot \partial_q \Phi = 0$, restricts the extrapolation further; it is related to (8.9). This does not mean that the condition $\partial_p \cdot \partial_q \Phi = 0$ serves only to govern the extrapolation. In fact, it serves to eliminate all two-Rac fields constructed from two Racs with opposite energy sign.

One sees, then, that when K in (8.9) describes massless fields with positive (negative) energy, then Φ describes two Racs with positive (negative) energy. In fact, this mapping is a unitary bijection, though we do not exhibit an explicit expression for the inverse mapping.

IX. CONTRACTION OF MASSLESS REPRESENTATIONS

(a) The contraction from a Lie algebra $\mathcal{A}_1$ to the Lie algebra $\mathcal{A}_0$ will be realized in the common underlying vector space $\mathcal{A}$ by a family $\{S_\lambda\} 0 < \lambda \leq 1$, of vector-space isomorphisms, such that $\lambda \to S_\lambda$ is a continuous map into $GL(\mathcal{A})$ with S_1 the identity. One defines a family $\{\mathcal{A}_\lambda\}$ of Lie algebras, each isomorphic to $\mathcal{A}_1$, with the bracket

$$[x, y]_\lambda = S_\lambda{}^{-1} [S_\lambda x, S_\lambda y] ,$$

where [,] is the bracket of $\mathcal{A}_1$. The contracted Lie algebra $\mathcal{A}_0$ is defined by the bracket $[x, y]_0 = \lim_{\lambda \to 0} [x, y]_\lambda$. This will not be equivalent to $\mathcal{A}_1$ unless $\lim_{\lambda \to 0} S_\lambda$ is invertible.

For representations on a Hilbert space $\mathcal{H}$ we shall realize this contraction with a domain $\mathcal{E}$, dense in $\mathcal{H}$ and endowed with a complete Hausdorff, locally convex topology, and a family $\{Z_\lambda\}$ of closed one-to-one linear operators. Each Z_λ is densely defined in $\mathcal{H}$ with domain containing $\mathcal{E}$, Z_1 is the identity operator on $\mathcal{H}$, and the restriction of Z_λ to $\mathcal{E}$ is continuous (for the topology of $\mathcal{E}$). Furthermore, the map defined by $\lambda \to Z_\lambda \phi$ from

(0, 1) to $\mathcal{H}$ is continuous and $\lim\| Z_\lambda \phi\|$ as $\lambda \to 0$ exists. We denote by $\mathcal{K}$ the closed subspace of $\mathcal{E}$ that consists of all ϕ such that $\lim\| Z_\lambda\phi\| = 0$ and we shall realize the contracted representation on the completion $\mathcal{H}^0$ of the locally convex space $\mathcal{E}/\mathcal{K}$ with respect to the Hilbert space norm $\lim\| Z_\lambda \phi\|$. If U^1 denotes the original representation of $\mathfrak{G}_1$ in $\mathcal{H}$ and if we suppose that its domain of definition contains $Z_\lambda \mathcal{E}$ for $0<\lambda \leq 1$, then we can define a representation U^λ of $\mathfrak{G}_\lambda$ by

$$Z_\lambda U^\lambda(x) = U^1(S_\lambda x) Z_\lambda .$$

These representations will also be defined on a domain containing $\mathcal{E}$. Now for all x in $\mathfrak{G}_1$ and ϕ in $\mathcal{E}$, $\tilde{U}^0(x)\phi \equiv \lim_{\lambda\to 0} U^\lambda(x)\phi$ will be defined and belong to $\mathcal{E}$, and $\mathcal{K}$ will be invariant under $\tilde{U}^0(x)$. We can thus finally define a representation U^0 of $\mathfrak{G}_0$ in $\mathcal{H}^0$ as the quotient of $\tilde{U}^0$ on $\mathcal{E}/\mathcal{K}$.

Notice that this type of contraction involves only one equivalence class of representations of $\mathfrak{G}_1$, in contrast with the other usual type of (Wigner-Inönü) contractions that utilize an infinite family of inequivalent representations.

(b) As earlier, let $\{L_{\alpha\beta}\}$, $\alpha, \beta = 0, 1, 2, 3, 5$, be the basis for so(3, 2). We define S_λ by

$$S_\lambda L_{\mu\nu} = L_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3$$

$$S_\lambda L_{\mu 5} = \lambda^2(2-\lambda) L_{\mu 5} .$$

Our "massless" representation U^1 [that is, $D(s+1, s)$ with $s \neq 0$ or else $D(1,0)\oplus D(2,0)$] is realized in $\mathcal{H} = L^2(\mathrm{R}^3, d^3p/|\vec{\mathrm{p}}|)$ as specified in Sec. II [that is, by extension to $\mathfrak{C}$ of either V_s^+ or V_{-s}^+ and restriction to SO(3, 2)]. Next,

$$Z_\lambda \phi(p) = \lambda\phi(\lambda p), \quad 0<\lambda \leq 1$$

defines a family of unitary operators that we shall restrict to the dense-invariant subspace $\mathcal{E}$ of differentiable vectors for the representation of $\mathfrak{C}$. Each Z_λ evidently commutes with the Lorentz subalgebra generated by $\{L_{\mu\nu}\}$, $\mu, \nu = 0, 1, 2, 3$; while

$$Z_\lambda^{-1} L_{05} Z_\lambda = \tfrac{1}{2}[\lambda^{-2} p_0 + p_0^{-1}(J^2 + D^2 + iD)] .$$

In the limit $\lambda \to 0$, the Lorentz subalgebra remains unchanged while

$$L_{05}^0 \equiv \lim_{\lambda\to 0} Z_\lambda^{-1} U^1(S_\lambda L_{05}) Z_\lambda = p_0 = P_0 .$$

Therefore, $L_{\mu 5}^0$, defined in a similar way for $\mu = 1, 2, 3$, are identical to P_μ on $\mathcal{E}$, and $U^0 = \lim_{\lambda\to 0} U^\lambda$ is just the representation V_s^+ or V_{-s}^+ of $L(\mathcal{P})$. Since the Z_λ are unitary on $\mathcal{H}$, the kernel subspace $\mathcal{K}$ is zero.

We have just obtained V_s^+ and V_{-s}^+ by contraction of $D(s+1, s)$ or of $D(1,0)\oplus D(2,0)$, making use of the two equivalent realizations of these latter representations in $\mathcal{H} = L^2(\mathrm{R}^3, d^3p/|\vec{\mathrm{p}}|)$. This shows that each massless representation of SO(3, 2) can be contracted (in our strict sense and with kernel subspace $= 0$) to either V_s^+ or V_{-s}^+. It is doubtful whether any other representations of SO(3, 2) have this property. Contractability thus furnishes yet another justification for the term massless as applied to the massless representations of SO(3, 2). [Strictly, only the direct sum $D(1,0)\oplus D(2,0)$ deserves the designation, though we have sometimes applied it to each irreducible part.]

(c) Contractions to nonfaithful or nonintegrable representations. If we choose $Z_\lambda \equiv I$, then the $D(s+1, s)$ will contract to their restriction to the Lorentz subgroup, which are reducible into a direct integral of UIR $D_L(s, \sigma)$ of the principal series, σ varying over the spectrum (which is R and simple) of the dilation D. Since $D_L(0, \sigma)$ and $D_L(0, -\sigma)$ are equivalent and the dilatation D exchanges the $\mathcal{H}_j^\tau$ $(\tau = 1, 2)$, the two contracted representations of $D(\tau, 0)$ will be equivalent one to the other and the integral will be taken over the spectrum of D^2. By suitable choices of the Z_λ one can obtain an integral (or sum) over any Borel subset of the spectrum R of D.

Now let $\mathcal{E}$ be the subspace of differentiable vectors for a representation of $\mathfrak{C}$ with helicity $\pm s \neq 0$, say $\mathcal{T}_s^+$ to fix ideas, on $\mathcal{H} = L^2(\mathrm{R}^3, d^3p/|\vec{\mathrm{p}}|)$, and θ an involution transforming $\mathcal{T}_s^+$ into $\mathcal{T}_{-s}^+$.

We can decompose $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3$ according to the spectrum of P_0, in such a way that on the closure of $\mathcal{E}_1$ in $\mathcal{H}$ the latter is $[0, p_0^1]$, on that of $\mathcal{E}_2$ it is $[p_0^1, p_0^2]$ and $[p_0^2, +\infty)$ for $\mathcal{E}_3$; these subspaces will be invariant under the Lie algebra action. We then can choose Z_λ' to coincide with θZ_λ on $\mathcal{E}_1$, with I on $\mathcal{E}_2$ and with Z_λ on $\mathcal{E}_3$. Then the restriction of $L_{\mu 5}$ to $\mathcal{E}_1$ contract to the P_μ of V_{-s}^+, that to $\mathcal{E}_2$ will contract to 0, and that to $\mathcal{E}_3$ will contract to the P_μ of V_s^+. The spectrum of the limit of L_{05} will be $[0, p_0^1]$ on the component defined by $\mathcal{E}_1$, 0 on the one defined by $\mathcal{E}_2$, and R^+ on the one defined by $\mathcal{E}_3$. However, due to the boundary conditions that appear, this limit will be symmetric, not essentially self-adjoint. Thus the contracted Poincaré Lie algebra representation, where both helicities ($+s$ and $-s$) appear, will not be integrable to a Poincaré group representation. If for simplicity we take $p_0^1 = p_0^2$, i.e., $\mathcal{E}_2 = \{0\}$, we thus see that an irreducible integrable representation of so(3, 2), which does not distinguish between the helicity signs, can be contracted into a nonintegrable representation of the Poincaré Lie algebra $L(\mathcal{P})$, decomposable into a sum of two Schur-irreducible[21] representations of $L(\mathcal{P})$, one for each helicity sign.

It is conjectured that no contraction procedure of the type here considered will allow the contraction from an irreducible representation $D(s+1, s)$ of SO(3, 2) to a representation $V_s^+ \oplus V_{-s}^+$ of the Poincaré group $\mathcal{P}$, representing a massless particle with both helicity signs. Clearly the procedure used above requires the passage to nonintegrable Poincaré representations, with the mentioned difficulty regarding the non-self-adjointness of the energy operator, if we want both helicity signs to be present. This could be related to a similar difficulty in the localizability[22] of massless particles, such as the photon. (The problem in proving completely this conjecture is that one needs to prove the nonexistence of a contracting family Z_λ of operators, which is difficult to handle.) On the other hand, for fully polarized particles such as the neutrino ($|n| = \frac{1}{2}$) the impossibility to obtain both helicity signs from the same $D(s+1, s)$ within the framework of unitary group representations is in full accordance with the experimental situation.

APPENDIX

There is an interesting complication in accounting for the Di on M_3. The matrix M in Eq. (7.2) is required in order to reduce the action of the three-dimensional Poincaré subgroup $\mathcal{P}_3$ of SO(3, 2) to the familiar form. The translation generators P_μ and their action on four-dimensional de Sitter spinors are

$$P_\mu = L_{\mu 3} + L_{\mu 5} = i y_\mu \partial_- - i y_+ \partial_\mu - \tfrac{1}{2}\gamma_\mu(1 - i\gamma_3),$$

$$y_\pm = y_5 \pm y_3, \quad \partial_\pm = \partial_5 \mp \partial_3, \quad \mu = 0, 1, 2, .$$

We require that the action of P_μ on $\tilde{\psi} = y_+^{3/2} M \psi$ reduce to $P_\mu = -i\partial/\partial u^\mu$, and find that

$$M = 1 + (i/2)\gamma^\mu u_\mu (1 - i\gamma_3).$$

The dilatation operator $D = L_{35}$ now becomes

$$D = -i(1 + u\partial_u) - (i/2)(1 - i\gamma_3)$$

and the wave operator $\tilde{\not{y}}\not{\delta}$ is reduced to

$$\tfrac{1}{2}(1 - i\gamma_3)(-i\gamma^\mu \partial/\partial u^\mu).$$

Restriction to the chirality subspace $i\gamma_3 = 1$ thus gives a three-dimensional chiral field with conformal degree −1.

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