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Speculation on traceless particles with charge and electromagnetic moments

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The general-relativistic equations of motion and subsidiary conditions are found for particles with spin whose energy-momentum tensor has zero trace (normally considered to be therefore massless) but which may have comoving internal charge currents. The procedure is an extension of that used in the pole-dipole model of traceless particles without charge currents.

I. INTRODUCTION

In two previous papers[1,2] we have presented a theory for the propagation of a traceless particle with momentum p^i and spin S^{ik}, that is, with pole and dipole moments of the energy-momentum tensor T^{ik}. The word "traceless" refers to the requirement

$$T^i{}_i = 0 \tag{1.1}$$

demanded of the energy-momentum tensor. We described such a particle as massless, but strictly speaking, tracelessness is its defining property.

To get the basic pole-dipole equations, we followed the procedure of Papapetrou,[3] which was to take moments of the equation of motion $T^{ik}{}_{;k} = 0$. We found the same equations as did Papapetrou (and others before and after him), except that the path parameter q could possibly represent a null geodesic trajectory. The new content of the theory were the corresponding moments of Eq. (1.1). Duval and Fliche[4] found these moments also, by a different method.

In the present paper, we wish to consider the equations governing the motion of a particle which is charged and has electromagnetic dipole moments, and still is traceless according to Eq. (1.1). This is to be regarded as a natural extension of the theory, but is not meant to correspond to any known particle at the present time. Consideration of particles of such a type go back at least as far as the series of articles in 1947 by Weyssenhoff and Raabe,[5] who at one point dress up a zero-mass particle with charge and electromagnetic moments. Their definitions and arguments are, however, quite different from ours.

We again use the method of Papapetrou,[3] extended to include charge and electromagnetic moments and fields. Other, more rigorous, thoroughly covariant approaches have been found[6,7] since Papapetrou. However, these methods invariably involve the definition of a momentum rest frame comoving with the particle in terms of which a center of mass and an invariant integration surface could be defined for use in the moment expressions.

We used the Papapetrou formalism not only for its simpler mathematical character, but also just because it does not refer to a rest frame, since for a traceless particle there may in fact not be a rest system. The momentum may not even be timelike. This feature is even more important when electromagnetic fields are involved. The Papapetrou approach uses constant *coordinate-time* surfaces in the moment expressions. Upon transformation of coordinates, the surfaces transform accordingly.

The situation is reminiscent of the classical problem of the transformation and conservation properties of the four-momentum of electromagnetic fields. For free fields, constant coordinate-time surfaces are appropriate, but for fields bound to a massive point particle (e.g., the electron) constant proper-time surfaces are appropriate.[8] In our case, the fields are in fact bound, but to a traceless particle. For such a bound field, the appropriate surface is, we suggest, again the constant coordinate-time surface.

The basic equations and moment definitions are given in Sec. II. Sections III and IV present the moment calculations, with the results for the spin and momentum equations given in Secs. V and VI. These two equations have the same form as those found by other authors (e.g., Dixon[6] in the general and DeGroot and Suttorp[7] in the special-relativistic case). The derivation is, nevertheless, briefly outlined here so as to provide the internal equations needed in the derivation of the side conditions. The arguments in any case are quite different.

The side conditions which are the moments of Eq. (1.1) are derived in Sec. VIII. In Sec. IX, we assemble the whole set of equations. Comparison with other work is contained in Sec. X. In Sec. XI, a discussion of the problems associated with radiation reaction is given. Appendix A gives the proofs of the transformation properties of the tensors of the theory. It is interesting to remark here that the Papapetrou approach leads directly to the polarization tensor $\bar{J}^{ik}$ as the basic and only electromagnetic dipolar object that transforms as a tensor. In Appendix B, it is shown that the assumption $\bar{J}^{ik}\sim S^{ik}$ leads to a rather simple trajectory equation.

II. BASIC EQUATIONS AND DEFINITIONS OF MOMENTS

The total energy-momentum tensor is taken to be the sum of a matter part T^{ik} and an electromagnetic part E^{ik}, with the equations of motion given by

$$T^{ik}{}_{;k}+E^{ik}{}_{;k}=0\,. \tag{2.1}$$

In addition, there is a current density J^i satisfying

$$J^i{}_{;i}=0\,. \tag{2.2}$$

These are the two basic equations of the problem The program is to take moments of these equations within the pole-dipole approximation.

To get the side conditions we use the definition of the traceless particle

$$T^i{}_i=0\,. \tag{2.3}$$

E^{ik} is separately traceless and its divergence is taken to have the usual form

$$E^{ik}{}_{;k}=-F^{ik}J_k\,, \tag{2.4}$$

where the electromagnetic field tensor F^{ik} is treated as a given function of position. Equations (2.1) and (2.2) then become

$$\partial(-g)^{1/2}T^{ik}/\partial x^k+(-g)^{1/2}\Gamma^i{}_{mk}T^{mk}-(-g)^{1/2}F^{ik}J_k=0\,, \tag{2.5}$$

$$\partial(-g)^{1/2}J^i/\partial x^i=0\,. \tag{2.6}$$

Equations (2.1)–(2.6) represent the *formal* definition of the particle. It is to be understood, however, that T^{ik} is already renormalized with an electromagnetic self-energy and that F^{ik} will depend on retarded and advanced integrals over J_k to be obtained ultimately in terms of the moments in the problem. See Sec. XI for a further discussion.

We assume the existence of a reference point $X^i(q)$ as a function of the path parameter q and about which moments are taken. The velocity of this point is $u^i=dX^i/dq$. Using

$$d\Omega=u^0(-g)^{1/2}d^3x \tag{2.7}$$

as the differential element and integrating on a constant time surface in whatever coordinate system is being used, we define the moments of T^{ik} as (these are the same as in Ref. 3)

$$M^{ik}=\int T^{ik}d\Omega\,,\quad M^i{}_k=\int T^i{}_k d\Omega\,, \tag{2.8}$$

$$M^{mik}=-\int[x^m-X^m(q)]T^{ik}\,d\Omega\,. \tag{2.9}$$

Moments of the J's are denoted by bars,

$$\bar{M}^i=\int J^i d\Omega\,, \tag{2.10}$$

$$\bar{M}^{ik}=-\int[x^i-X^i(q)]J^k d\Omega\,. \tag{2.11}$$

The antisymmetrized form is called $\bar{S}$,

$$\bar{S}^{ik}=\bar{M}^{ki}-\bar{M}^{ik};\quad \bar{S}^{i0}=-\bar{M}^{i0}\,. \tag{2.12}$$

These moments are not in general tensors. As shown in Appendix A, the combinations that do transform as tensors are

$$S^{ik}=(M^{ki0}-M^{ik0})/u^0\,, \tag{2.13}$$

$$\bar{J}^{ik}=\tfrac{1}{2}\bar{S}^{ik}+(1/2u^0)(u^k\bar{S}^{i0}-u^i\bar{S}^{k0});\ \bar{J}^{i0}=\bar{S}^{i0}\,. \tag{2.14}$$

It should be remarked that Eq. (2.14) is the "polarization tensor" which contains both the electric and magnetic dipole moments. In the present formulation we do not find that the parts (e.g., $\bar{S}^{ik}$) by themselves transform as tensors, only the combination. See, however, the discussion in Sec. X.

We show also that there are quantities that transform as vectors,

$$P^i=(1/u^0)(M^{i0}+\Gamma^i{}_{jk}u^jS^{k0}-F^i{}_k\bar{J}^{k0})\,, \tag{2.15}$$

$$\bar{p}^i=\bar{M}^i-d(\bar{S}^{i0}/u^0)dq\,, \tag{2.16}$$

and one that transforms as a scalar

$$e=\bar{M}^0/u^0\,. \tag{2.17}$$

Finally, it is assumed that certain quantities can be expanded about X^i:

$$g_{ik}(x)=g_{ik}(X)+(\Gamma_{k,in}+\Gamma_{i,kn})\delta x^n+\cdots\,, \tag{2.18}$$

$$\Gamma^i{}_{jk}(x)=\Gamma^i{}_{jk}(X)+\Gamma^i{}_{jk,m}(X)\delta x^m+\cdots\,, \tag{2.19}$$

$$F^{ik}(x)=F^{ik}(X)+F^{ik}{}_{,n}(X)\delta x^n+\cdots\,, \tag{2.20}$$

where $\delta x^n=x^n-X^n$.

This completes the basic relations needed in the derivations below.

III. THE MOMENT EQUATIONS

We now take the moments of Eqs. (2.5) and (2.6). Multiply (2.5) successively by u^0d^3x, u^0 $u^0\delta x^r d^3x$, and $u^0\delta x^r\delta x^s d^3x$ and integrate. Then, just as in Ref. 3, we get

$$d(M^{i0}/u^0)/dq+\Gamma^i{}_{mk}M^{mk}-\Gamma^i{}_{mk,n}M^{nmk}-F^i{}_k\bar{M}^k+F^i{}_{k,n}\bar{M}^{nk}=0\,, \quad (3.1)$$

$$d(M^{ri0}/u^0)/dq-M^{i0}u^r/u^0+M^{ir}+\Gamma^i{}_{mk}M^{rmk}-F^i{}_j\bar{M}^{rj}=0\,, \quad (3.2)$$

$$u^rM^{si0}+u^sM^{ri0}-u^0(M^{sir}+M^{ris})=0=Z^{rsi}\,, \quad (3.3)$$

with electric field corrections in Eqs. (3.1) and (3.2). Taking the same moments of Eq. (2.6) we get

$$d(\bar{M}^0/u^0)/dq=0\,, \quad (3.4)$$

$$d(\bar{M}^{r0}/u^0)/dq-\bar{M}^0u^r/u^0+\bar{M}^r=0\,, \quad (3.5)$$

$$u^r\bar{M}^{s0}+u^s\bar{M}^{r0}-u^0\bar{M}^{sr}-u^0\bar{M}^{rs}=0\,. \quad (3.6)$$

The spin equation comes from Eq. (3.2), the momentum equation comes from Eq. (3.1). Equation (3.4) shows that charge is a constant; Eq. (3.5) relates moments of $\bar{J}^i$ to u^i.

IV. CONVERSION TO TENSOR NOTATION

One major problem is to convert the M's in these equations to the S^{ik} and $\bar{J}^{ik}$, etc. that transform as tensors [see Eqs. (2.13)–(2.18)]. The procedure for doing this is similar to how it was done in Ref. 3.

From (3.3), noting that $Z^{rsi}+Z^{sir}-Z^{irs}=0$, we get

$$2M^{sir}=u^rS^{is}+u^iS^{rs}-(u^s/u^0)(u^rS^{i0}+u^iS^{r0})\,. \quad (4.1)$$

From (3.2), after a tedious reduction modeled after that in Ref. 3 in Sec. III, we get

$$M^{ab}=\frac{u^au^b}{u^0}\left(\frac{M^{00}}{u^0}+\Gamma^0{}_{cd}S^{d0}\frac{u^c}{u^0}\right)+\frac{u^b}{u_0}\frac{DS^{a0}}{Dq}-\frac{d}{dq}\frac{u^aS^{0b}}{u^0}-u^d\Gamma^a{}_{cd}S^{cb}+(F^a{}_c+u^aF^0{}_c/u^0)\bar{M}^{bc}\,. \quad (4.2)$$

This last expression should be considered symmetrized, since $M^{ab}=M^{ba}$.

To get Eq. (4.2), the first step is to set $i=0$ in Eq. (3.2) to get

$$M^{0b}=dS^{b0}/dq+(u^b/u^0)(M^{00}+\Gamma^0{}_{cd}S^{d0}u^c)-\Gamma^0{}_{cd}u^dS^{cb}+F^0{}_c\bar{M}^{bc}\,. \quad (4.3)$$

We shall need this equation later. The rest of the derivation of (4.2) is to substitute this back into (3.2) and reduce the result.

The corresponding electromagnetic quantities are obtained from Eq. (3.6):

$$\bar{M}^{sr}=\tfrac{1}{2}\bar{S}^{rs}-(1/2u^0)(u^s\bar{S}^{r0}+u^r\bar{S}^{s0})=\bar{J}^{rs}-u^s\bar{J}^{r0}/u^0 \quad (4.4)$$

and from Eq. (3.5),

$$\bar{M}^r=u^r\bar{M}^0/u^0+d(\bar{S}^{r0}/u^0)/dq\,. \quad (4.5)$$

Finally, consider putting (4.3) into (2.15). We get

$$P^i=(1/u^0)[(u^i/u^0)(M^{00}+\Gamma^0{}_{cd}S^{d0}u^c)+DS^{i0}/Dq+F^0{}_c\bar{M}^{ic}-F^i{}_k\bar{J}^{k0}]\,. \quad (4.6)$$

Many of the equations obtained in this section contain $\bar{M}^{ik}$. This of course can be put in terms of $\bar{J}^{ik}$ by Eq. (4.4).

V. THE SPIN EQUATION

To get the spin equation, we return to Eq. (3.2) and construct it using the q derivative of S^{ik} as defined in Eq. (2.13):

$$DS^{ab}/Dq-(u^b/u^0)DS^{a0}/Dq+(u^a/u^0)DS^{b0}/Dq=(u^bF^0{}_c-u^0F^b{}_c)\bar{M}^{ac}/u^0-(u^aF^0{}_c-u^0F^a{}_c)\bar{M}^{bc}/u^0\,. \quad (5.1)$$

For the factors involving DS^{i0}/Dq, use Eq. (4.6) and for $\bar{M}^{bc}$ use (4.4). Then we get

$$DS^{ab}/Dq=P^au^b-P^bu^a+F^a{}_m\bar{J}^{mb}-F^b{}_m\bar{J}^{ma}\,. \quad (5.2)$$

This is the spin equation.

VI. THE EQUATION FOR X^i

The basic equation is (3.1). In this equation, for M^{i0} use what Eq. (2.15) gives and for M^{mk} use what Eq. (3.2) gives [in both of these using (4.6) to get $(u^i/u^0)(M^{00}+\Gamma^0{}_{cd}S^{d0}u^c)$ in terms of P^i]. Continuing in Eq. (3.1), for M^{nmk} use what (4.1) gives, for $\bar{M}^{rj}$ use Eq. (4.4), for $\bar{M}^k$ use Eq. (4.5). Following this prescription, we get eventually

$$DP^i/Dq=\tfrac{1}{2}R^i{}_{jkn}u^jS^{kn}-F^i{}_{k;m}\bar{J}^{km}+\bar{p}^kF^i{}_k\,. \quad (6.1)$$

This is the desired equation.

VII. THE ELECTROMAGNETIC EQUATIONS

The nonelectromagnetic equations (3.1) and (3.2) have yielded equations of motion for spin and momentum. One might expect that the electromagnetic equations (3.4)–(3.6) might yield equations of motion for $\bar{J}^{ik}$ and $\bar{p}^i$. In fact, from the definition in (2.16) and Eq. (3.5), we find

$$\bar{p}^i=eu^i\,, \quad (7.1)$$

where e is given by Eq. (2.18). This is a solution for $\bar{p}^i$.

However, there emerges no equation for $\bar{J}^{ik}$. To go further one must assume, or derive on some microscopic model, how $\bar{J}^{ik}$ may be related to other quantities for which we do have an equation, such as S^{ik}. For example, the usual assumption for massive particles is $\bar{J}^{ik}=GS^{ik}$, where G is some scalar. This type of assumption is pursued in Appendix B.

Our ignorance of $\bar{J}^{ik}$ in this theory is a special case of the general feature of General Relativity that systems characterized by charge and mass densities with no "equation of state" connecting them are indeterminate. A possible approach[9] toward bettering the situation is to imagine that the system contains two types of particles with number densities, say n_1, n_2, with an energy density and pressure (or whatever) a function of them. From this one could hope to obtain an equation of motion for each of the n_i, and this would ultimately be equivalent to determining $\bar{J}^{ik}$.

This type of procedure would seem, however, to be inappropriate here, since the object is already characterized by its moments, not by subsidiary number densities. The equation of state should, in the context of the theory, be in terms of the moments. Thus something like $\bar{J}^{ik}=GS^{ik}$ is what we would desire and it should be regarded as the moment of an equation of state. (If the theory went beyond the pole-dipole approximation, it would need supplementary relations of this type for each higher moment.) However, there is no basic reason to believe that $\bar{J}^{ik}=GS^{ik}$ is adequate for the problem here.

VIII. THE AUXILIARY CONDITIONS

The traceless particle is defined such that $T^i{}_i + E^i{}_i = 0$. Further, we assume that $E^i{}_i = 0$ by itself, so that we are left with just the condition in Ref. 1, $T^i{}_i = 0$. Then we must consider moments of this equation. The argument of Appendix A of Ref. 1 can be taken over here without alteration since Eqs. (A10) and (A11) of that appendix are unaltered. Thus $M^i{}_i$ and $M^{ji}{}_i$ will be zero in all systems of coordinates if they are zero in one.

Similarly Eq. (B1) of Ref. 1 is still valid, is in fact Eq. (4.1) above. This leads directly to Eq. (B4) of Ref. 1 which reads

$$S^{ik}u_k = -au^i, \tag{8.1}$$

where a is a scalar equal to $-S^{0k}u_k/u^0$. Thus the first of the auxiliary conditions is unaltered by the presence of charge or a magnetic moment so long as the tracelessness of T^{ik} is preserved.

It is different, however, with the other condition. From Eq. (A9) of Ref. 1,

$$M^i{}_i = 0 = g_{ik}M^{ik} - (g_{nj}\Gamma^j{}_{km} + g_{kj}\Gamma^j{}_{nm})M^{mkn}. \tag{8.2}$$

For M^{ik} use Eq. (4.2), which has electromagnetic effects and for M^{mkn} use (4.1). After substitution, all the quantities must be in terms only of well-identified tensors (e.g., p^i). After a laborious calculation in which much cancellation takes place, we find

$$M^i{}_i = 0 = P^iu_i + da/dq + F_{ik}(\bar{M}^{ik} + \bar{J}^{k0}u^i/u^0). \tag{8.3}$$

The last term is written in terms of $\bar{J}^{ik}$ by means of Eq. (4.4). The final result is then

$$P^iu_i = -da/dq + F_{ik}\bar{J}^{ik}. \tag{8.4}$$

IX. THE COMPLETE SET OF EQUATIONS

The four basic equations are Eqs. (6.1), (5.2), (8.1), and (8.4). We collect them together here in one place,

$$DP^i/Dq = \tfrac{1}{2}R^i{}_{jkn}u^jS^{kn} - F^i{}_{k;m}\bar{J}^{km} + eu^kF^i{}_k, \tag{9.1}$$

$$DS^{ik}/Dq = P^iu^k - P^ku^i + F^i{}_m\bar{J}^{mk} - F^k{}_m\bar{J}^{mi}, \tag{9.2}$$

$$P^iu_i = -da/dq + F_{ik}\bar{J}^{ik}, \tag{9.3}$$

$$S^{ik}u_k = -au^i. \tag{9.4}$$

As shown in Appendix A, it is possible to use another momentum p^i related to P^i by

$$P^i = p^i - F^i{}_kw_m\bar{J}^{km} \tag{9.5}$$

in terms of a vector w_m which is general except for its normalization

$$w^iu_i = 1. \tag{9.6}$$

It is useful to have equations in terms of the p^i since then they are more easily compared with the results of previous authors,

$$Dp^i/Dq = \tfrac{1}{2}R^i{}_{jkn}u^jS^{kn} + (D/Dq)F^i{}_kw_n\bar{J}^{kn} - F^i{}_{k;n}\bar{J}^{kn} + F^i{}_keu^k, \tag{9.7}$$

$$DS^{ik}/Dq = p^iu^k - p^ku^i + F^i{}_m(\bar{J}^{mk} - u^kw_m\bar{J}^{mn}) - F^k{}_m(\bar{J}^{mi} - u^iw_n\bar{J}^{mn}), \tag{9.8}$$

$$p^iu_i = -da/dq + F_{ik}(\bar{J}^{ik} - u^kw_n\bar{J}^{in}), \tag{9.9}$$

$$S^{ik}u_k = -au^i. \tag{9.10}$$

Equations (9.1)–(9.4) are 15 equations in the 15 unknowns P^i, S^{ik}, u^i, and a. The six $\bar{J}^{ik}$ must be obtained by some other theory.

X. COMPARISON WITH OTHER WORK

Equations (9.7) and (9.8) are the same whether or not the particle is traceless and therefore can be compared to results that others have obtained for charged massive particles. Dixon[6] has worked the problem in covariant general relativistic form for charged massive particles. Our Eqs. (9.7) and (9.8) correspond to his Eqs. (6.32) and (6.31). There is complete agreement, with the following remarks.

Dixon separates out a four-vector in the rest (or rather, momentum) system which is the electric-dipole moment. He calls this q^k and it, in our terminology, plays the role of

$$q^k = -\bar{J}^{km}w_m. \tag{10.1}$$

By the same token he has a tensor which in the momentum system is the magnetic-moment tensor and in our terminology plays the role of

$$m^{ik}=\bar{J}^{ki}-u^i w_m J^{km}+u^k w_m \bar{J}^{im}. \qquad (10.2)$$

If there exists a rest system in which $u^i=(1,0,0,0)$, then for the space indices α and β, and with w^i chosen to be u^i, Eqs. (10.1) and (10.2) give in this system

$$q^\alpha \to \bar{J}^{\alpha 0}=\bar{S}^{\alpha 0}\equiv \int d\Omega\, \delta x^\alpha J^0, \quad q^0=0 \qquad (10.3)$$

$$m^{\alpha\beta}\to \bar{J}^{\alpha\beta}\to \bar{S}^{\alpha\beta}\equiv \tfrac{1}{2}\int d\Omega(\delta x^\alpha J^\beta - \delta x^\beta J^\alpha), \quad m^{i0}=0 \qquad (10.4)$$

the last forms coming from the definitions in Sec. II. These expressions do have the character of electric and magnetic moments.

In the case of flat space, we can compare our results to those of Ref. 7. In particular, our Eqs. (9.1) and (9.2) correspond to DeGroot and Suttorp's Eqs. (96) and (102), (111) and (112) in their Chap. IV. Their Eqs. (115) and (116) correspond to our (9.7) and (9.8) if we identify w^i with

$$w^i=p^i/p^k u_k. \qquad (10.5)$$

Although this satisfies Eq. (9.6), we can not be certain in the case of traceless particles, that $p^i u_i$ is not zero. Thus Eq. (10.5) may not be generally applicable.

It is interesting that the Papapetrou approach has led directly to $\bar{J}^{ik}$ as the basic electromagnetic tensor and not, say, to q^k or m^{ik} (which require w^i for their definition). This $\bar{J}^{ik}$ is precisely the moment combination that DeGroot and Suttorp use as the basic tensor in their work.

It should be emphasized in comparing our results with other work that in principle we cannot assume at the outset that there is a realizable rest or momentum system for a traceless particle. And, as stated in the Introduction we do not therefore define the integration surfaces in the moments as constant proper-time, but rather as constant coordinate-time surfaces.

XI. RADIATION REACTION

In Eq. (2.4), $F^{ik}J_k$ is the electromagnetic force acting on the particle. Ever since Dirac,[10] this force has been understood to contain an external part $F_{\text{ext}}{}^{ik}J_k$ and a radiation reaction part $\tfrac{1}{2}(F^{ik}_{\text{ret}}-F^{ik}_{\text{adv}})J_k$ from the retarded and advanced fields of the charges themselves. The field from the charge also contributes to a renormalized self-energy that we anticipate will be included in our T^{ik}. The theory of Dirac has its roots in the work of Lorentz and has been elaborated by many others.[11,12,13,14]

It must also be remembered that the external field F^{ik}_{ext} not only contributes to the force on the particle, but also helps to curve the space: it enters the Einstein equations.

There are then two problems: First, how can the original energy-momentum tensor be conceived to give the self-energy and radiation reaction effects in a consistent way, and second, how can F^{ik}_{ret} and F^{ik}_{adv} be written in terms of the moments $\bar{p}^k$ and $\bar{J}^{ik}$ that the rest of the theory contains?

The first is one of the central questions of classical radiation theory, even in flat space, and lies essentially beyond the scope of the present article. We shall simply assume that such a consistent formulation is possible so that the force in Eq. (2.4) is the usual radiation reaction plus external forces:

$$E^{ik}{}_{;k}=-(\tfrac{1}{2}F^{ik}_{\text{ret}}-\tfrac{1}{2}F^{ik}_{\text{adv}}+E^{ik}_{\text{ext}})J_k \qquad (11.1)$$

and that the T^{ik} of Eq. (2.1) contains a (localized) renormalized electromagnetic self-energy. For work along these lines in flat space, see Refs. 11–14. Notice that in the pole-dipole theory, an explicit form for T^{ik} is not needed [Eqs. (9.1)–(9.4) determine its moments], but the form on the right in Eq. (11.1) is needed for an explicit solution to the problem.

However, even with these identifications, we are still faced with the second question, of how to write the fields F^{ik}_{ret}, etc. in terms of the moments that we have defined in Sec. II. This can be answered directly. Write as usual F_{ik} in terms of the four-potential A_k,

$$F_{ik}=A_{k,i}-A_{i,k},$$

where the subscripts ret and adv will be omitted from now on. The fields that are generated by the source currents may be taken from the work of DeWitt and Brehme,[15]

$$A_i(x)=(4\pi/c)\int d^4x'(-g')^{1/2}J^k(x')G_{ik}(x,x'), \qquad (11.2)$$

in which $G_{ik}(x,x')$ appears as a bitensor. The integrand of (11.2) transforms as a scalar at x'^j (the index k having been contracted) and as a vector at x^j according to the index i. The argument we make is independent of whether G_{ik} is the retarded, advanced or whatever function, and makes no use of its explicit form.

We now break up Eq. (11.2) into a time integral over x'^0 and a space integral,

$$d^4x'=dx'^0 d^3x'=dq' d^3x' u'^0, \qquad (11.3)$$

where $u'^0=dx'^0/dq'$. Here q' is an integration variable having the same significance as the q of previous sections. The space integral is over a constant t'^0 surface. It thus corresponds to the type of integral used in the preceding sections.

The reference point of the particle is denoted $X'^i = X^i(q')$ as a function of q'. For each value of q' in the integrand of Eq. (11.2), we expand $G_{ik}(x,x')$ about the point $x'^i = X'^i(q')$ on the surface of constant q'

$$G_{ik}(x,x') = G_{ik}(x,X') + (x'^m - X'^m)\partial G_{ik}(x,X')/\partial X'^m + \cdots . \quad (11.4)$$

This is analogous to the expansions in Eqs. (2.19)–(2.21). (Notice the *partial* derivative on G_{ik}. This is not the tensor expansion in the style of Ruse.[16])

Putting (11.3) and (11.4) into (11.2), we may write the latter as

$$A_i(x) = (4\pi/c)\int dq'\Big(\int d\Omega'\, J^k(x')G_{ik}(x,X') + \tfrac{1}{2}\int d\Omega'[J^k(x')(x'^m - X'^m) - J^m(x')(x'^k - X'^k)][\partial G_{ik}(x,X')/\partial X'^m] + \tfrac{1}{2}\int d\Omega'[J^k(x')(x'^m - X'^m) + J^m(x')(x'^k - X'^k)][\partial G_{ik}(x,X')/\partial X'^m]\Big), \quad (11.5)$$

where $d\Omega'$ is as in Eq. (2.7).

The last integral in (11.5) is transformed in the usual way[17] using

$$J^k(x'^m - X'^m) + J^m(x'^k - X'^k) = J^r\frac{\partial}{\partial x'^r}[(x'^k - X'^k)(x'^m - X'^m)] + \frac{J^0}{u'^0}[u'^k(x'^m - X'^m) - u'^m(x'^k - X'^k)] + \frac{2J^0}{u'^0}u'^m(x'^k - X'^k). \quad (11.6)$$

The terms proportional to J^0 arose because $X'^i(q')$ responds to an x'^0 derivative. The first term on the right in Eq. (11.6) integrates to zero when placed in (11.5). The rest gives

$$A_i(x) = (4\pi/c)\int dq'\{\overline{M}^k(q')G_{ik}(x,X') + \tfrac{1}{2}\overline{S}^{mk}(q')[\partial G_{ik}(x,X')/\partial X'^m] + [\tfrac{1}{2}(u'^k\overline{S}'^{m0}/u'^0 - u'^m\overline{S}'^{k0}/u'^0) + u'^m\overline{S}^{k0}/u'^0][\partial G_{ik}(x,X')/\partial X'^m]\}. \quad (11.7)$$

The terms proportional to $\frac{1}{2}$ in the curly brackets combine into $\overline{J}^{mk}(q')$. The last term in the second square brackets gives a contribution to A_i equal to

$$(4\pi/c)\int dq'(\overline{S}^{k0}/u'^0)dG_{ik}/dq' = -(4\pi/c)\int dq' G_{ik}(x,X')d(\overline{S}^{k0}/u'^0)/dq' . \quad (11.8)$$

This term, combined with the first term of (11.7) gives a factor $\overline{p}^k$ as defined by Eq. (2.16).

With all this, Eq. (11.7) may be written

$$A_i(x) = (4\pi/c)\int dq'[\overline{p}^k(q')G_{ik}(x,X') + \overline{J}^{mk}(q')DG_{ik}(x,X')/DX'^m] . \quad (11.9)$$

In the last term, one finds at first $\overline{J}^{mk}\partial G_{ik}/\partial x'^m$. But it must be remembered that the k index on $G_{ik}(x,X')$ goes only with the X' dependence so that

$$\overline{J}^{mk}(q')DG_{ik}(x,X')/DX'^m = \overline{J}^{mk}\partial G_{ik}(x,X')/\partial X'^m - \overline{J}^{mk}\Gamma^r{}_{km}G_{ir}(x,X'). \quad (11.10)$$

The last term is, however, zero by symmetry. Thus we can replace the partial derivative by the covariant derivative in this term. But then we have the expression in Eq. (11.9) completely in tensor form.

This is as far as we take the discussion. Equation (11.9) shows how A_i emerges as a function of the $\overline{p}^k$ and $\overline{J}^{mk}$. One must then evaluate $A_i(x)$ at $X^i(q)$ in the manner of Dirac or DeWitt and Brehme, a calculation we do not include here.

XII. SUMMARY

In this paper we have endowed a traceless particle with comoving internal currents and sought to find a consistent set of equations determining the propagation of the pole and dipole moments through space-time. The results were written in Eqs. (9.1)–(9.10). The problem of electromagnetic radiation reaction was considered for the purpose of writing such corrections in terms of the same moments as appear in the rest of the problem.

The resulting equations are quite formidable and the methods used in Ref. 2 to elicit some simple properties of a traceless particle do not seem to be useful here, except in the limiting case where a charge exists but no electric or magnetic moments, i.e., where $\bar{J}^{ik}=0$. For then Eqs. (9.2)–(9.4) have the same form as without charge. Thus all the conclusions of Ref. 2 can be applied here directly. In particular, if $a \neq 0$ then the trajectories are null geodesics and a is a constant. And if $a=0$, and the particle is not a pole particle, then there exist "C-frames" in which X^i is the center of energy. If the energy in the C-frames is not zero, or equivalently if $\vec{p}\cdot\vec{S} \neq 0$, then with $a=0$ the trajectories are again null geodesics.

If there are dipolar moments $\bar{J}^{ik}$, however, then the equations (9.1)–(9.4) become far more difficult. The first and most important question is, what *are* the $\bar{J}^{ik}$? The equations do not tell us anything about them. As discussed in Sec. VII, this lack of information may be regarded as equivalent to a lack of an equation of state, which is always necessary to complete the definition of a general relativistic system. In the present problem if the equation of state is contained entirely within T^{ik}, then one does not need to supplement the basic equation $T^{ik}{}_{;k}=0$. However, when charge is added, with an explicit vector J^i that lies outside of T^{ik}, then an equation of state connecting J^i with T^{ik} becomes a necessity for a determinate problem.

If we adopt, as is customary for massive particles,

$$\bar{J}^{ik}=GS^{ik} \qquad (12.1)$$

as some approximate moment of an equation of state, then we expect, by comparing Eqs. (2.9), (2.11), (2.13), and (2.14) and to preserve q-reversal symmetry, a form such as

$$G=G_0\mu^0 . \qquad (12.2)$$

In Appendix B we show that (12.1) leads, for $a \neq 0$, to the trajectory equation

$$Du^i/Dq=GF^i{}_k u^k . \qquad (12.3)$$

If $a=0$, however, we have found no simple equation.

Even if we accept (12.3), there are still formidable obstacles to getting explicit solutions. First among these is radiation reaction, which is contained implicitly in $F^i{}_k$. The electromagnetic moment effect in the radiation reaction is extremely complicated. Second, an external electromagnetic field must be regarded as helping to curve the space. Thus taking the electromagnetic field into account is another very difficult (and separate) problem. A test of Eq. (12.1) would need somehow to solve Eq. (12.3), taking into account all these difficulties. We have not succeeded in doing this. If Eq. (12.1) is not suitable, then one must find some other way of handling $\bar{J}^{ik}$.

In any case, Eqs. (9.1)–(9.4) are the basic equations of propagation for a traceless particle with dipolar electromagnetic moments and Eq. (12.3) is a consequence of (12.1) if $a \neq 0$.

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APPENDIX A. TENSOR PROPERTIES

The procedure here rests on the arguments in Ref. 3 and we shall refer to that article for most of the basic equations. The transformation equation for the quantities M^{abc} and M^{ab} are the same as Eqs. (4.3) and (4.4) of Ref. 3. If, as in Ref. 1, Appendix A, we use

$$L^i{}_a=\partial x^i/\partial x'^a , \qquad (A1)$$

$$L^i{}_{ab}=\partial^2 x^i/\partial x'^a \partial x'^b , \qquad (A2)$$

we have for Eqs. (4.3) and (4.4) of Ref. 3,

$$M^{jik}=L^i{}_a L^k{}_b (L^j{}_c - u^j L^0{}_c/u^0) M'^{cab} , \qquad (A3)$$

$$M^{ik}=L^i{}_a L^k{}_b M'^{ab} - (L^i{}_{ac} L^k{}_b + L^i{}_a L^k{}_{bc}) M'^{cab} + (d/dq) L^i{}_a L^k{}_b L^0{}_c M'^{cab}/u^0 . \qquad (A4)$$

We need the corresponding transformations for the electromagnetic barred quantities. They have the same form as the quantities in (A3) and (A4) except for J^i having one index instead of the two that appear on T^{ik}. Thus a slight modification of Eqs. (A3) and (A4) is all that is needed. It can be verified by calculation that

$$\bar{M}^{ji}=L^i{}_a (L^j{}_c - u^j L^0{}_c/u^0) \bar{M}'^{ca} , \qquad (A5)$$

$$\bar{M}^i = L^i{}_a \bar{M}'^a - L^i{}_{ac} \bar{M}'^{ca} + (d/dq) L^i{}_a L^0{}_c \bar{M}'^{ca}/u^0 . \qquad (A6)$$

We can now derive the transformation equations of the tensors in the problem. First from (A3) we can construct the transformation of S^{ik} as defined in (2.13), with the M'^{cab} on the right replaced by what Eq. (4.1) gives. Using the fact that

$$L^i{}_a u'^a = u^i \;, \tag{A7}$$

we find eventually that

$$S^{ik} = L^i{}_a L^k{}_b S'^{ab} \;, \tag{A8}$$

which proves that S^{ik} transforms as a tensor.

However, for momentum we get an electromagnetic contribution. We start with Eq. (A4). For the M'^{ab} in the first term on the right we use what Eq. (3.2)—in primes—gives. By use of the equations just preceding Eq. (4.7) of Ref. 3, we get

$$M^{i0}/u^0 + \Gamma^i{}_{jk} u^j S^{k0}/u^0 = L^i{}_a [M'^{a0}/u'^0 + \Gamma'^a{}_{bc} u'^b S'^{c0}/u'^0] + L^i{}_a L^0{}_b F'^a{}_c \bar{M}'^{bc}/u^0 \,. \tag{A9}$$

The quantity on the left would transform as a tensor if it were not for the last term on the right. To deal with this last term, notice from (A5) that

$$F^i{}_k \bar{M}^{jk} = L^i{}_a L^j{}_b F'^a{}_c \bar{M}'^{bc} - u^j L^i{}_a L^0{}_b F'^a{}_c \bar{M}'^{bc}/u^0 \;, \tag{A10}$$

where it was assumed that F^{ik} transforms as a tensor. The last term of (A10) is, except for the factor u^j, the same as the last term of Eq. (A9). If we could get rid of this factor, we could just add Eqs. (A9) and (A10) and obtain a quantity transforming as a tensor.

The simplest method for eliminating the factor u^j is to define a *vector* w^n such that

$$w^n u_n = 1 \,. \tag{A11}$$

Other than that this must be satisfied, w_n is perfectly general at this point. Multiply Eq. (A10) by w_j and add to Eq. (A9). The result is

$$p^i = L^i{}_a p'^a \;, \tag{A12}$$

where

$$p^i = (M^{i0} + \Gamma^i{}_{jk} u^j S^{k0})/u^0 + F^i{}_k \bar{M}^{jk} w_j \,. \tag{A13}$$

The quantity p^i so defined, for w_n arbitrary except as restricted by Eq. (A11), transforms as a tensor according to Eq. (A12).

It turns out (see Sec. X) that this w^i is the counterpart of a vector used in the theory of massive particles, so that the p^i in Eq. (A13) is useful in comparing our results with other work. For traceless particles, we cannot in general use the expression in Eq. (10.5) since the denominator may be zero. However, we can always find a w^i to satisfy (A11). For example, suppose in some system of coordinates, $u^i = (1,1,0,0)$. Then we can choose $w^i = (1,0,0,0)$ in this system.

But it would be desirable to have a theory which did not rely on the addition of this arbitrary vector w^i. In fact, this can be accomplished without much trouble. The problem is still to get rid of the last term of Eq. (A9) and the method is basically still to use Eq. (A10). If we substitute Eq. (4.4) for $\bar{M}^{jk}$ on the left and the right, and anticipate that $\bar{J}^{ik}$ is a tensor (see Eq. (A22) below: The proof does not depend on anything we do here), then (A10) reduces to

$$-F^i{}_k \bar{J}^{k0} u^j/u^0 = -L^i{}_a F'^a{}_c \bar{J}^{c0} u^j/u'^0 - L^i{}_a L^0{}_b F'^a{}_c \bar{M}'^{bc} u^j/u^0 \tag{A14}$$

(using $L^j{}_b u'^b = u^j$ in the first term on the right). Cancel the factor u^j.

Add now (A9) and (A14) divided by u^j. We get

$$P^i = L^i{}_a P'^a \;, \tag{A15}$$

where P^i is given by Eq. (2.15) and does not depend on w_i. So as far as vector transformation properties go, we can use either p^i or P^i.

Equations (A8) and either (A12) or (A15) give the transformation properties of basically nonelectromagnetic tensors. As for the basically electromagnetic tensors, we first consider $\bar{J}^{ik}$. If we define

$$\bar{D}^{ik} = (u^k \bar{S}^{i0} - u^i \bar{S}^{k0})/u^0 \;, \tag{A16}$$

then

$$\bar{J}^{ik} = \bar{S}^{ik} + \bar{D}^{ik} \,. \tag{A17}$$

We need first to know how $\bar{S}^{ik}$ transforms. If, for simplicity, we define

$$W^{ik} = L^i{}_a L^k{}_b \bar{S}'^{ab} \;, \tag{A18}$$

$$V^{ik} = L^i{}_a L^k{}_b \bar{D}'^{ab} \;, \tag{A19}$$

then Eq. (A5) used in the definition of $\bar{S}^{ik}$ in (2.12) yields using (4.4)

$$\bar{S}^{ik} = W^{ik} - (u^k W^{i0} - u^i W^{k0})/2u^0 + \tfrac{1}{2} V^{ik} \,. \tag{A20}$$

Next we need $\bar{D}^{ik}$. To get this use (A5) with $\bar{S}^{i0}$ from (2.12). We get then, using (4.4),

$$\bar{D}^{ik} = (u^k W^{i0} - u^i W^{k0})/2u^0 + \tfrac{1}{2} V^{ik} \,. \tag{A21}$$

Thus we see that neither $\bar{S}^{ik}$ nor $\bar{D}^{ik}$ by themselves transform as tensors. But their sum does:

$$\bar{S}^{ik} + \bar{D}^{ik} = W^{ik} + V^{ik} = L^i{}_a L^k{}_b (\bar{S}'^{ab} + \bar{D}'^{ab}) \,. \tag{A22}$$

This establishes the tensor property of $\bar{J}^{ik}$. Next we must find a corresponding vector property of an electromagnetic "momentum", $\bar{p}^k$ as defined in Eq. (2.16). To do this, start with Eq. (A6). In this, the $\bar{M}'^{ca}$ in the last term is evaluated using Eq. (4.4). The result is

$$\bar{p}^i = L^i{}_a \bar{p}'^a \;, \tag{A23}$$

where $\bar{p}^i$ is defined as in Eq. (2.16).

Finally we wish to show that $\bar{M}^0/u^0$ transforms as a scalar. This is easily done by rewriting Eq. (2.16) using (4.5),

$$\bar{p}^i = eu^i\,, \tag{A24}$$

where e means $\bar{M}^0/u^0$. Now since $\bar{p}^i$ and u^i transform as tensors, Eq. (A24) shows that e must transform as a scalar.

APPENDIX B. REDUCTION OF THE EQUATIONS

In this appendix we assume Eq. (12.1) and also that $a \neq 0$, and shall derive Eq. (12.3) and a number of other consequences.

To start we need Eq. (9.4)

$$S^{ik}u_k = -au^i \tag{B1}$$

and its q derivative (denoted by overdots) which, after use of (9.2) and (9.3), can be written

$$S^{ik}\tilde{u}_k = -2\dot{a}u^i - a\tilde{u}^i + F^{mn}\bar{J}_{mn}u^i - p^i u^k u_k\,. \tag{B2}$$

Here we have introduced the notation

$$\tilde{u}^i = \dot{u}^i - GF^i{}_k u^k\,, \qquad \tilde{u}_i = \dot{u}_i - GF_i{}^k u_k\,, \tag{B3}$$

for convenience.

Multiply (B1) by u_i and sum,

$$0 = -au^i u_i\,. \tag{B4}$$

Here is where we make use of the assumption that $a \neq 0$. For with it we get

$$u^i u_i = 0\,, \tag{B5}$$

whence

$$u^i \dot{u}_i = 0\,, \tag{B6}$$

which leads with (B3) also to

$$u^i \tilde{u}_i = 0\,. \tag{B7}$$

Now multiply (B2) by $\tilde{u}_i$ and sum. The left-hand side goes out. If Eqs. (B5) and (B7) are used, we get

$$\tilde{u}^i \tilde{u}_i = 0\,. \tag{B8}$$

It is generally true for any two vectors a^i, b^i, that if $a^i a_i = a^i b_i = b^i b_i = 0$, then a^i is parallel to b^i. The proof is contained in Appendix A of Ref. 2, but it is simple to repeat. Go into a local tetrad in which $a^i = a^0(1,1,0,0)$. Assume $b^i = (b^0, b^1, b^2, b^3)$. The equation $a^i b_i = 0$ then gives the result that $b^0 = b^1$. Finally, the equation $b^i b_i = 0$ gives then

$$(b^2)^2 + (b^3)^2 = 0\,,$$

which means that $b^2 = b^3 = 0$. Thus $a^i = \text{const}\, b^i$ in this tetrad. But this is a tensor equation and therefore must be true in all systems.

In our case, we see from Eqs. (B5), (B7), and (B8) that u^i and $\tilde{u}^i$ obey all these orthogonality properties. Therefore they must be parallel

$$\tilde{u}^i = bu^i\,, \tag{B9}$$

where b is some scalar function. (Notice that the same argument does *not* work for $\dot{u}^i$ and u^i, since $\dot{u}^i\dot{u}_i$ cannot in general be proved to be equal to zero.)

Equation (B9) is the trajectory equation for general q. Just as with geodesics, it is always possible to find *special parameters* q' for which the term in b in (B9) does not appear. We shall not go through the argument: it is unaltered in detail from the usual proof. The result is then that

$$\tilde{u}^i = 0\,, \quad \text{i.e.,} \quad Du^i/Dq = GF^i{}_k u^k\,, \tag{B10}$$

when these special parameters are used.

[It is interesting to notice that when the transformation $q = q(q')$ to the special parameter q' is made, the factor G becomes $G\,dq/dq'$. This reinforces the observation in Eq. (12.2), since $G\,dq/dq' = G_0(dx^0/dq)(dq/dq') = G_0\,dx^0/dq'$. That is, the form in (12.2) is preserved under such a transformation.]

The result in (B10) enables a number of other results to be proved. First of all, if (B9) is substituted into (B2), we get

$$\dot{a} = \tfrac{1}{2}F^{mn}\bar{J}_{mn}\,. \tag{B11}$$

This substituted into (9.3) gives

$$P^i u_i = \dot{a} = \tfrac{1}{2}F_{ik}\bar{J}^{ik}\,. \tag{B12}$$

Thus in general, a is not a constant and $P^i u_i$ is not zero. $P^i u_i$ being zero is what we might wish to reserve for the name "massless", since $\frac{1}{2}F_{ik}\bar{J}^{ik}$ has the significance of the dipolar energy. (That is, in an electric $\vec{E}$ and magnetic $\vec{B}$ field, it is $\vec{q}\cdot\vec{E} + \vec{m}\cdot\vec{B}$, where $\vec{q}$ is the electric and $\vec{m}$ the magnetic moment.)

However, a further calculation shows that the solutions break up into two branches, one of which is massless by this definition. To see this, notice that in Eq. (9.1) the second term on the right may be written $\frac{1}{2}F_{km}{}^{;i}\bar{J}^{km}$ upon use of the Maxwell equations $F_{[ik;m]} = 0$. Then, multiplying (9.1) by u_i, we get

$$\dot{P}^i u_i = \tfrac{1}{2}\dot{F}_{ik}\bar{J}^{ik}\,. \tag{B13}$$

Now from (B10) we get, after using (9.2),

$$P^i \dot{u}_i = GF_{ik}u^k P^i = \tfrac{1}{2}G\dot{S}^{ik}F_{ik}\,. \tag{B14}$$

To get this we used the fact that

$$F^i{}_k S^{km}S_{mi} = 0\,, \tag{B15}$$

which is true since $S^{km}S_m{}^i$ is symmetric in ik.

Equations (B12)–(B14) yield the following result. Take D/Dq of (B12) and subtract from this the sum of (B13) and (B14):

$$\dot{G}F_{ik}S^{ik} = 0\,. \tag{B16}$$

If we use (B11) this may be written

$$\dot{G}\dot{a} = 0 . \qquad \text{(B17)}$$

This tells us then that there are two branches to the solutions,

$$\begin{aligned} &\text{branch A: } G = \text{const} , \\ &\text{branch B: } a = \text{const} . \end{aligned} \qquad \text{(B18)}$$

It is not impossible that both G and a could be constant simultaneously.

Branch B is what we might call the massless branch, for the dipolar energy $F^{ik}\bar{J}_{ik}$ would be zero as well as $P^i u_i$. Another feature of the massless branch is that a is a constant, and can be set as small as desired. We can anticipate that this branch merges continuously onto the $a=0$ solutions, although we have not solved that case yet.

The last thing we do here is show generally for both branches that

$$K = S_{ik}S^{ik} + 2a^2 = \text{const} , \qquad \text{(B19)}$$

$$L = S^*{}_{ik}S^{ik}/a = \text{const} , \qquad \text{(B20)}$$

where

$$S^*{}_{ik} = \tfrac{1}{2} e_{ikmn} S^{mn} . \qquad \text{(B21)}$$

For branch B we see that $S_{ik}S^{ik}$ and $S^*{}_{ik}S^{ik}$ are separately constants of the motion.

To prove (B19), take d/dq of $S_{ik}S^{ik}$ and use Eq. (9.2) and in the result use Eqs. (9.3), (9.4), (B11), and (B15). To prove (B20), take d/dq of $S^*{}_{ik}S^{ik}$, use (B21) and (9.2). One needs also the equation

$$S^*{}_{mi}u^i = Hu_m \qquad \text{(B22)}$$

where

$$H = -S^*{}_{k0}u^k/u_0 \qquad \text{(B23)}$$

is the helicity associated with velocity. Equation (B22) can be obtained from Eq. (9.4) using

$$S^*{}_{ik}S^{im} = \delta_k{}^m aH , \qquad \text{(B24)}$$

which itself can be proved from the definition of $S^*{}_{mi}$ and (9.4). Another relation needed in the proof of (B20) is

$$F^i{}_k S^{km} S^*{}_{mi} = 0 , \qquad \text{(B25)}$$

which follows from (B24).

The two constants K and L are related, as can be seen from the determinantal equation resulting from (9.4):

$$a^4 + \tfrac{1}{2}a^2 S_{ik}S^{ik} = \tfrac{1}{16}(S^{ik}S^*{}_{ik})^2 = 0 . \qquad \text{(B26)}$$

In fact it can be shown that K is twice the helicity squared,

$$S_{ik}S^{ik} + 2a^2 = 2H^2 . \qquad \text{(B27)}$$

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The invariant-surface definition does not require the surface to be the proper-time surface; any surface picked arbitrarily but once and for all will do. But the procedure seems somewhat artificial unless the surface plays some special role.

One could try a proper-time surface even for a traceless particle defined as the surface generated by the vectors a^i satisfying $a^i u_i = 0$. If u^i is null, then the a^i are all spacelike vectors except the one which is u^i itself. In an *xyt* diagram, such a surface would be the two-dimensional plane disk tangent at some point to the light cone, the line of tangency being u^i. Such a surface cannot be used, however, since the particle will be unbounded in extension in directions arbitrarily close to u^i.

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