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Pole-dipole model of massless particles. II

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The pole-dipole equations derived in a previous paper for massless particles (whose defining relation is that the energy-momentum tensor has zero trace) are examined. If the reference point X^i describing the motion is *not* somewhere on the disk perpendicular to the three-velocity through the energy center, then X^i describes a null geodesic with no assumptions. The property that X^i is the particle's energy center in *some* local reference frame (called a C-frame) is shown to be a constant of the motion. If the energy of the particle in the C-frames is not zero, then the trajectory is again a null geodesic. The conditions on the curvature needed to have the momentum parallel to the four-velocity are determined. Helicity properties are also discussed.

I. INTRODUCTION

In a previous paper,[1] the pole-dipole equations for a particle whose energy-momentum tensor T^{ik} satisfies $T^{ik}{}_{;k}=0$ and $T^i{}_i=0$ were derived by the method of Papapetrou.[2] Particles satisfying $T^i{}_i=0$ we called massless, but of course strictly speaking they are *traceless* particles. We shall use these words interchangeably, but traceless is the accurate one, in case any confusion about the words arises.

In the procedure of Papapetrou, a reference point X^i of the particle is chosen about which to form moments of T^{ik}. From these, certain combinations of the moments were shown by Papapetrou to transform as tensors, namely a momentum vector p^i and a spin tensor S^{ik}. Then the moments of $T^{ik}{}_{;k}=0$ yield

$$Dp^i/Dq=\tfrac{1}{2}R^i{}_{kmn}u^kS^{mn}\,, \qquad (1.1)$$

$$DS^{ik}/Dq=p^iu^k-p^ku^i\,, \qquad (1.2)$$

where $u^i=dX^i/dq$, q being a parameter along the curve. This form of the equations survives even if the curve happens to be a null geodesic, a possibility if the particle is massless.

In Ref. 1 we considered the corresponding moments of the equation $T^i{}_i=0$, and found

$$p^iu_i=da/dq\,, \qquad (1.3)$$

$$S^{ik}u_k=-au^i\,, \qquad (1.4)$$

where a is a scalar function of q.

Equations (1.1) and (1.2) are the same as for massive particles, but Eqs. (1.3) and (1.4) are not. For massive particles one can choose $q=s$, the path length, and has $u^iu_i=1$ and $p^ip_i=m^2$, which are candidates for replacing Eq. (1.3). As for Eq. (1.4), Mathisson[3] had in fact suggested this equation, with $a=0$, for the purpose of specifying X^i as the center of energy in the rest system. However, later authors[4] found difficulties with this definition, and suggested $S^{ik}p_k=0$ as preferable. In any case, for massive particles, Eq. (1.4) with $a=0$ or its alternative was *chosen* as appropriate, whereas for traceless particles, both Eqs. (1.3) and (1.4) follow directly from $T^i{}_i=0$.

Weyssenhoff and Raabe[5] long ago considered massless particles from the pole-dipole approach in flat space, and Mashhoon[6] recently considered the problem in general spaces. Duval and Fliche[7] derived Eqs. (1.3) and (1.4) independently of Ref. 1 by a completely different method.

In Ref. 1 it was argued that if $u^iu_i\neq 0$, then a must be zero; and if the trajectories are null geodesics, then a could be chosen to be zero as an initial condition. (The case $u^iu_i=0$ but the trajectory not a geodesic was not considered.) We therefore suggested that $a=0$ was always appropriate.

We did not see our way to proving generally that the trajectories were null geodesics, and contemplated the possibility that they were not. In flat space the solution could be obtained exactly. We had to assume that $\vec{p}\cdot\vec{S}\neq 0$ (using the notation of Sec. II below) however, to obtain null-geodesic trajectories. And even with this, we had to make a further assumption (that S^{ik} is bounded in time) to show that p^i is parallel to u^i. Thus even in flat space, traceless particles could be conceived (if say $\vec{p}\cdot\vec{S}=0$) that did not necessarily follow null geodesics. Other special cases were treated in Ref. 1.

The argument in Ref. 7 went rather differently. In that paper, two properties of S^{ik} were assumed to begin with: $\det S^{ik}=0$ and $\mathrm{Tr}\,S^2<0$. From these it was proved that $a=0$. Then it was argued that $\vec{p}\cdot\vec{S}$ had to be nonzero, whence the trajectories had to be null geodesics generally, not just in flat space.

The purpose of the present paper is to reexamine the consequences of Eqs. (1.1)–(1.4), and to analyze the conditions that enter the solutions. Thus although we can set $a=0$, we do not have to, and Sec. III examines general properties valid for $a\neq 0$. It is proved there that *null geodesics ensue without any assumptions at all, if $a\neq 0$*, and further that $da/dq=0$ so that Eq. (1.3) becomes

$$p^i u_i = 0 \tag{1.5}$$

generally.

In Sec. III, the consequences of assuming $a\neq 0$ are examined in detail, and in Sec. IV the consequences of $a=0$ are examined. The mathematical consequences derived do not rely on adopting a particular interpretation of the moments, etc. However, in order to make the results clearer from a physical point of view, we shall frequently describe them in terms of the extended-particle picture, that is, in terms of an energy center x^i_C and the particle structure. Following Papapetrou, we can always imagine letting the size of the particle become arbitrarily small.

The energy center appears in writing out $S^{\alpha 0}$ from Eq. (2.14) below, where α, β, etc. are space indices:

$$S^{\alpha 0} = M^0(x^\alpha_C - X^\alpha), \tag{1.6}$$

where

$$M^i = \int T^{0i}(-g)^{1/2} d^3x, \tag{1.7}$$

$$x^\alpha_C = (M^0)^{-1}\int x^\alpha T^{00}(-g)^{1/2} d^3x. \tag{1.8}$$

x^α_C is the energy center of the system considered and $S^{\alpha 0}=0$ means that X^i is at the energy center (unless of course $M^0=0$).

Thus for example, from Eq. (1.4), it follows that in a local tetrad in which the object is instantaneously traveling in the x direction,

$$S^{x0} = -au^x/u^0, \quad S^{y0} = S^{yx}u^x/u^0, \quad S^{z0} = S^{zx}u^x/u^0. \tag{1.9}$$

The mathematical significance of $a=0$ is from this equation that $S^{x0}=0$. In terms of the notation of Sec. II below, this means that the component of $\vec{R}$ $(=S^{\alpha 0})$ along the space direction of motion must be zero, $\vec{R}$ being the border column of S^{ik}. The extended-particle interpretation of this is much more illuminating. $S^{x0}=0$ means from (1.6) that the component of $X^\alpha - x^\alpha_C$ along the direction of motion is zero. That is, X^i lies on a disk perpendicular to $\vec{u}$ that goes through the energy center in the coordinates used. It is this type of appeal to an extended-particle picture that we shall use to make the significance of some of our results clearer.

For massless particles there is in general no invariant definition of a center of energy. However, in Sec. IV, we show that the conditions of Ref. 7 [repeated in Eq. (4.1) in a different notation] and $a=0$ are the necessary and sufficient conditions for the existence of (an infinite number of) frames, called C-frames, in which X^i is the energy center (or mathematically, in which $\vec{R}=0$).

Further, if (and only if) a C-frame exists initially will C-frames exist at every point of the trajectory. The succession of C-frames is the closest the theory of traceless particles comes to a comoving "rest" frame. (Of course, the traceless particle is not at rest in a C-frame.) The condition $\vec{p}\cdot\vec{S}\neq 0$ is shown to be equivalent to the assertion that the particle's mechanical energy p^0 in the C-frame is not zero.

In Sec. V we consider under what conditions p^i can be parallel to u^i.

Another exactly solvable problem is that of constant curvature. The solution is written down in Sec. VI and compared to the flat-space solution.

In Sec. VII we discuss the results and in particular the relation of Moller's theorem[8] to the massless case.

II. NOTATION

In this section we list the notation used. The antisymmetric spin tensor S^{ik} splits into two pieces:

$$S_\alpha = (S_1, S_2, S_3) = c^{-1}(S^{23}, S^{31}, S^{12}), \tag{2.1}$$

$$R^\alpha = (R^1, R^2, R^3) = (S^{10}, S^{20}, S^{30}). \tag{2.2}$$

In the local tetrad at a point in spacetime, these form ordinary space vectors, and dot and cross products are defined in the usual way. Next

$$S^*_{ik} = \tfrac{1}{2} e_{ikmn} S^{mn}, \quad S^{ik} = \tfrac{1}{2} e^{ikmn} S^*_{mn}, \tag{2.3}$$

where

$$e_{ikmn} = (-g)^{1/2}\epsilon_{ikmn}, \quad e^{ikmn} = (-g)^{-1/2}\epsilon^{ikmn}, \tag{2.4}$$

where the ϵ are the alternating symbols.

Helicity vectors are defined from momentum and velocity:

$$h_m = p^k S^*_{km}, \tag{2.5}$$

$$H_m = u^k S^*_{km}. \tag{2.6}$$

The components of H_m are

$$H_0 = -c(-g)^{1/2}u^\alpha S_\alpha ,\qquad H_\alpha = (-g)^{1/2}[u^0 c S_\alpha + (\vec{u}\times\vec{R})_\alpha] . \tag{2.7}$$

The quantities

$$h = -h_0/u_0 ,\quad H = -H_0/u_0 \tag{2.8}$$

are called helicity scalars; they will be shown to be scalars later.

General relations that can be verified by direct expansion are

$$(-g)\det(S^{ik}) = \tfrac{1}{16}(S^*_{ik}S^{ik})^2 = c^2(-g)(R^\alpha S_\alpha)^2 , \tag{2.9}$$

$$S^*_{ik}S^{jk} = -c\delta^j_i(-g)^{1/2}R^\alpha S_\alpha . \tag{2.10a}$$

If Eq. (1.4) is multiplied by S^*_{im} and Eq. (2.10a) used in the result, we get

$$S^*_{ik}S^{jk} = \delta^j_i aH , \tag{2.10b}$$

$$c(-g)^{1/2}R^\alpha S_\alpha u_m = aH_m . \tag{2.11}$$

Equations (2.10b) and (2.11) are true whether or not $a=0$; but if $a\neq 0$ then they show that

$$a = \text{scalar} , \tag{2.12}$$

$$H_m = -Hu_m ,\quad a\neq 0 . \tag{2.13}$$

So H_m is null and parallel to u_m.

Finally we recall the definitions of S^{ik} and p^i in the development of Ref. 1:

$$S^{ik} = \int_{t=\text{const}} [(x^i - X^i)T^{k0} - (x^k - X^k)T^{i0}](-g)^{1/2}d^3x , \tag{2.14}$$

$$p^i = \int_{t=\text{const}} T^{i0}(-g)^{1/2}d^3x + \Gamma^i{}_{jk}S^{k0}u^j/u^0 . \tag{2.15}$$

III. GENERAL RESULTS

In this section we develop the consequences of Eqs. (1.1)–(1.4) for general a. We show for $a\neq 0$ that

(a) the trajectories are null geodesics,

(b) a is a constant of the motion,

(c) H_m is parallel transported, $DH_m/Dq=0$, for q such that $Du_m/Dq=0$, and H is a constant of the motion.

For general a we show that

(d) $K = S_{ik}S^{ik} = 2(c^2S^2 - R^2)$ and $L = S^*_{ik}S^{ik} = -4c(-g)^{1/2}\vec{R}\cdot\vec{S}$ are constants of the motion.

And for $a=0$ we show that

(e) h_i is parallel to u_i or zero.

The proof of these statements all start from Eq. (1.4),

$$S^{ik}u_i = au^k \tag{3.1}$$

and its q derivative (we use dots to denote D/Dq)

$$S^{ik}\dot{u}_i = p^k u_i u^i + 2\dot{a}u^k + a\dot{u}^k . \tag{3.2}$$

Multiply (3.1) by u_k and sum

$$au^k u_k = 0 . \tag{3.3}$$

Multiply Eq. (3.2) by $\dot{u}_k$ and sum

$$(u_i u^i)p^k\dot{u}_k + 2\dot{a}u^k\dot{u}_k + a\dot{u}^k\dot{u}_k = 0 . \tag{3.4}$$

For $a=0$ these equations tell us nothing since $p^k\dot{u}_k=0$ follows from $p^k u_k = 0$ and Eq. (1.1). However, for $a\neq 0$, Eq. (3.3) tells us that u^k is null, $u^k u_k = 0$, whence $\dot{u}^k u_k = 0$. Then from Eq. (3.4), $\dot{u}^k\dot{u}_k = 0$. These last three equations prove that *the trajectory is a null geodesic*, i.e., that $\dot{u}^k = bu^k$ where b is a scalar (see Appendix A). We have thus uncovered our most important point.

If $\dot{u}^k = bu^k$ is used in Eq. (3.2), then we have that $\dot{a}=0$. Thus a is a constant of the motion.

Next take S^*_{mk} of (3.1) and use (2.10b). For $a=0$ we get no information, but for $a\neq 0$

$$S^*_{mk}u^k = Hu_m ,\quad a\neq 0 . \tag{3.5}$$

Thus H plays the same role for S^* that a plays for S [compare (3.5) with (3.1)].

Take now D/Dq of (3.5) using $\dot{u}^k = bu^k$:

$$\dot{S}^*_{mk}u^k = \dot{H}u_m ,\quad a\neq 0 . \tag{3.6}$$

The left-hand side is $\frac{1}{2}e_{mki\,j}\dot{S}^{ij}u^k$ which is zero by Eq. (1.2). Thus Eq. (3.6) gives

$$\dot{H} = 0 ,\quad a\neq 0 . \tag{3.7}$$

That is, H is a constant of the motion.

From (2.13) we get

$$\dot{H}_m = -H\dot{u}_m ,\quad a\neq 0 . \tag{3.8}$$

Thus for the special parameter q such that $\dot{u}_m = 0$, it follows that $\dot{H}_m = 0$, i.e., H_m is parallel transported.

Thus we have established statements (a), (b), and (c) at the head of this section.

To prove (d), just take the implied derivatives

$$\dot{K} = 2S_{ik}\dot{S}^{ik} = 4p^i u^k S_{ik} = -4ap^i u_i = -4a\dot{a} = 0 .$$

To prove that L is a constant, just take D/Dq of (2.10) and use the fact that $\dot{a}=0$ and $\dot{H}=0$. In fact, from Eq. (2.10) it follows that $S^*_{ik}S^{jk}$ is parallel transported.

Relationships between the constants of motion can be obtained from the determinantal equations associated with (1.4) and (3.5):

$$a^4 + a^2(c^2S^2 - R^2) - (-g)c^2(\vec{R}\cdot\vec{S})^2 = 0 , \tag{3.9}$$

$$H^4 - H^2(c^2S^2 - R^2) - (-g)c^2(\vec{R}\cdot\vec{S})^2 = 0 ,\quad a\neq 0 . \tag{3.10}$$

Subtraction gives

$$a^2 = H^2 - c^2S^2 + R^2\,, \quad a \neq 0\,. \tag{3.11}$$

The quantities S^2 and R^2 are not separately constants of the motion, by any arguments that we have found.

Finally we multiply[7] Eqs. (3.1) and (3.2) by $e_{jkmn}u^m$. Using

$$e_{jkmn}\, e^{ikfg} = \delta^{fgi}_{mnj}\,, \tag{3.12}$$

we get

$$H_n u_j - H_j u_n = S^*_{jn} u^m u_m\,, \tag{3.13}$$

$$H_n \dot{u}_j - H_j \dot{u}_n = -S^*_{nj}\dot{u}_m u^m - \dot{S}^*_{nj} u_m u^m\,. \tag{3.14}$$

If we multiply by $e_{jkmn}p^m$ and sum we get

$$h_n u_j - h_j u_n = -a\dot{S}^*_{jn}\,, \tag{3.15}$$

$$h_n \dot{u}_j - h_j \dot{u}_n = -a e_{jn\,mk} p^m \dot{u}^k\,. \tag{3.16}$$

For $a \neq 0$, the right-hand sides of Eqs. (3.13) and (3.14) are zero, and we recover Eqs. (2.13) and (4.8). However, we do not get the same type of relationship for h_m from Eqs. (4.15) and (4.16), since the right-hand sides there are not necessarily zero.

However, for $a = 0$, the right-hand sides of (3.15) and (3.16) *are* zero, whence

$$h_m = -hu_m\,, \quad a = 0\,. \tag{3.17}$$

Thus for $a = 0$, h_m is parallel to u_m. This is statement (e) at the head of this section.

IV. $A = 0$ AND THE CENTER-OF-ENERGY MOTION

In the previous section, the properties of the $a \neq 0$ points were discussed. In this section we consider the $a = 0$ points. From (2.11) these will be characterized by

$$L = -4c(-g)^{1/2}\vec{\mathrm{R}}\cdot\vec{\mathrm{S}} = 0\,. \tag{4.1}$$

The other constant of the motion has three possibilities:

$$K = 2(c^2S^2 - R^2) > 0\,, \tag{4.2a}$$

$$K < 0\,, \tag{4.2b}$$

$$K = 0\,. \tag{4.2c}$$

The discussion can then be separated into three cases: $L = 0$, $K > 0$, $L = 0$, $K < 0$, and $L = 0$, $K = 0$.

In Ref. 7, the relations $L = 0$, $K > 0$ were assumed as properties of S^{ik}, and from them it was proved that $a = 0$. Once $a = 0$, it was argued there that

$$h \sim \vec{\mathrm{p}}\cdot\vec{\mathrm{S}} \neq 0 \tag{4.3}$$

had to be true, and that this led to null geodesics.

Having assumed $a = 0$ to begin with in this discussion, we shall show that $L = 0$, $K > 0$ implies the existence of frames in which $\vec{\mathrm{R}} = 0$, not simply $a = 0$, and therefore in which X^i is the energy center. We consider the other two cases also, and list the results as a set of theorems, followed by a list of proofs.

(f) $L = 0$, $K > 0$ are the necessary and sufficient conditions for the existence of rest frames of inertia, called C-frames, in which $\vec{\mathrm{R}} = 0$. In such frames $\vec{\mathrm{S}} \parallel \vec{\mathrm{u}}$, unless $\vec{\mathrm{u}} = 0$. Also in such frames if $p^0 \neq 0$ (the mechanical energy not zero) then there exists an energy center in them, and $\vec{X}$ is at the center.

(f′) $L = 0$, $K < 0$ leads to a contradiction, and cannot therefore be satisfied.

(f″) $L = 0$, $K = 0$ correspond to a pole particle. We present however only a plausibility argument, depending on the assumption that u^i is a physical velocity, not spacelike.

(g) If (and only if) a C-frame exists initially will a C-frame exist at *every* point of the trajectory. If an object is a pole particle at one point, by (f″), then it is a pole particle at all points of the trajectory.

(h) In a C-frame, p^0 is the mechanical energy, and on physical grounds is taken not to be zero.[9] The condition $p^0 \neq 0$ in the C-frame is the necessary and sufficient condition for Eq. (4.3). The proof assumes $u^0 \neq 0$.

(i) If $\vec{\mathrm{p}}\cdot\vec{\mathrm{S}} \neq 0$, or equivalently $p^0 \neq 0$ in the C-frame, then the trajectories satisfying (4.2a) are null geodesics; conversely, if the trajectory is a null geodesic, then $\vec{\mathrm{p}}\cdot\vec{\mathrm{S}} \neq 0$ everywhere on the trajectory if $\vec{\mathrm{p}}\cdot\vec{\mathrm{S}} \neq 0$ initially. If, however, $\vec{\mathrm{p}}\cdot\vec{\mathrm{S}} = 0$ (or $p^0 = 0$), then[9] the trajectory need not be a null geodesic.

(j) H is a constant of the motion, H_m is parallel to u_m, and H_m is parallel transported along the trajectory if u_m is.

We shall now prove these assertions.

Proof of (f). The proof here will be made by analogy with the corresponding problem in the relativistic theory of electromagnetism, in which the field tensor F^{ik} contains the electric field E^α as border elements and the magnetic field B_α as interior elements in the same way that S^{ik} contains R^α and cS_α. It is well known[10] in electromagnetism that $\vec{\mathrm{E}}\cdot\vec{\mathrm{B}} = 0$ and $E^2 - B^2 < 0$ are the necessary and sufficient conditions for the existence of reference frames in which $\vec{\mathrm{E}} = 0$. In our case, if $L = 0$, $K > 0$ are satisfied at some point P of the trajectory, then since K and L are scalars, they are satisfied in a rest frame of inertia. Thus by considering Lorentz boosts to other rest frames of inertia, we can make the same argument as is made in electromagnetism, to find that

$R^\alpha = 0$ in some frame. The velocity of transformation is

$$\vec{v} = \vec{R} \times \vec{S}/S^2 .$$

Note that if $R = cS$, then v becomes c. So Eq. (4.2a) must remain an inequality. Further, from (1.9) it follows in this C-frame where $\vec{R} = 0$ that $\vec{u}$ is parallel to $\vec{S}$.

And finally, if $p^0 = M^0 = \int T^{00}$ is not zero in the C-frame, then Eq. (1.8) defines the energy center in that frame, and $R^\alpha = 0$ and Eq. (1.6) show that $\vec{X}$ is at that center.

The proof of (f′) goes the same way. The same theorem in electromagnetism[10] says that $\vec{E} \cdot \vec{B} = 0$ and $E^2 - B^2 > 0$ are the necessary and sufficient conditions for the existence of frames in which $\vec{B} = 0$. So by analogy, if $L = 0$, $K < 0$, then there exist frames in which $\vec{S} = 0$. But then Eq. (1.4) says that $\vec{R} = 0$ in such a frame, which contradicts the premise $K < 0$.

There is now the one other possibility mentioned in (f″), namely $K = L = 0$, i.e., $\vec{S} \cdot \vec{R} = 0$ and $cS = R$. Clearly the special case $cS = R = 0$ belongs to this class of solutions, and this case is the pole particle with X^i the center of energy. We shall make a plausibility argument that $cS = R \neq 0$ also corresponds to a pole particle, but with X^i not the center of energy.

To do this we show that $\vec{S}$ has the structure of an orbital angular momentum viewed from $\vec{X}$, and also that in flat space as a special case, S relative to the energy center, S_C, is zero.

To show the first of these things, we go to an arbitrary rest system of inertia in which therefore $M^i = p^i = \int T^{i0}$. Equation (1.4) for $a = 0$ then can be written

$$\vec{R} = c\vec{U} x \vec{S} \quad (\vec{U} = \vec{u}/u^0 = d\vec{r}/dct) . \tag{4.4}$$

We shall assume that u^i is a physical velocity, not spacelike. Equation (4.4) is compatible then with $cS = R$ only if $U = 1$ and $\vec{R}$, $\vec{U}$, and $\vec{S}$ are mutually orthogonal. Then crossing $\vec{U}$ into (4.4) gives

$$\vec{S} = \vec{R} \times \vec{U}/c = (\vec{x}_C - \vec{X}) \times M^0 \vec{U}/c . \tag{4.5}$$

This has the structure of an orbital angular momentum, seen from X^i, since $\vec{x}_C - \vec{X}$ is the radius vector from $\vec{X}$ to $\vec{x}_C$, the energy center, since M^0/c^2 can be interpreted in this equation as a mass concentrated at x_C, and since $c\vec{U}$ has the significance of a velocity, the only well-defined velocity in the problem characteristic of the particle. In this sense $\vec{S}$ may be interpreted as an orbital effect.

To show that relative to the center of energy, the spin is zero in flat space, we go back to Eq. (2.14) and srite $x^i - X^i$ as $x^i - x^i_C + x^i_C - X^i$. Then

$$\vec{S} = \vec{S}_C + \vec{R} \times \vec{M}/M^0 . \tag{4.6}$$

Dot this by $\vec{S}$, use $p^i = M^i$ in the rest system, reorder the dot cross product, and use $\vec{p} \cdot \vec{U} = p^0$ from (1.5) to find $\vec{S} \cdot \vec{S}_C = 0$. Then dot (4.6) by $\vec{R}$ to find $\vec{R} \cdot \vec{S}_C = 0$ [since $\vec{R} \cdot \vec{S} = 0$ from (4.4)]. Thus if $cS = R$, then we have found that *$\vec{S}_C$ is parallel to $\vec{U}$*. If we now dot (4.6) into $\vec{M}$, we get

$$\vec{M} \cdot \vec{S} = M^0 S_C . \tag{4.7}$$

This is a general relation in the rest system of inertia. In flat space where $\vec{p}$ (i.e., $\vec{M}$) is parallel to $\vec{U}$ [and hence from (4.5) perpendicular to $\vec{S}$], the left-hand side is zero. Thus in flat space

$$S_C = 0 . \tag{4.8}$$

To prove that $S_C = 0$ in curved spaces directly, we would have to prove that $\vec{M}$ is parallel to $\vec{U}$ generally, but we have not been able to do this.

This completes the plausibility argument. To sum up: if $cS = R$, then (1) the special case $cS = R = 0$ corresponds to a pole particle viewed from the center of energy, (2) the structure of (4.5) is that of an orbital angular momentum, and (3) at least in the case of flat space, $S_C = 0$. For these reasons we suggest that $cS = R$ generally refers to a pole particle.

Proof of (g). K and L are constants of the motion.

Proof of (h). From Eq. (2.15) in a rest frame of inertia, $p^i = M^i = \int T^{i0}$. Thus p^0 in a C-frame corresponds to the mechanical energy.

Now if $h \neq 0$, then by Eq. (4.10) below, u^i is null and must have a space part. In a C-frame, this space part $\vec{u}$ must be parallel to $\vec{S}$ according to (f), so that $\vec{p} \cdot \vec{S} \neq 0$ is equivalent to $\vec{p} \cdot \vec{u} \neq 0$. But by (1.5) this means $p^0 u^0 \neq 0$. Thus $h \neq 0$ implies $p^0 \neq 0$.

On the other hand, if $p^0 \neq 0$ and $u^0 \neq 0$ in a C-frame (the latter meaning that the particle proceeds into the future in this system), then $p^0 u^0 \neq 0$ and by (1.5) $\vec{p} \cdot \vec{u} \neq 0$. But in a C-frame, $\vec{u}$ is parallel to $\vec{S}$ by (f), so that $p^0 \neq 0$ has led to $h \sim \vec{p} \cdot \vec{S} \sim \vec{p} \cdot \vec{u} \neq 0$.

Proof of (i). Consider Eq. (3.2) in a C-frame with $k = 0$ and $a = 0$:

$$p^0 u^i u_i = 0 . \tag{4.9}$$

Using from (h) that $p^0 = 0$ is unphysical[9] we get

$$u^i u_i = 0 , \tag{4.10}$$

$$\dot{u}^i u_i = 0 . \tag{4.11}$$

Equation (4.10) is valid in all frames since $u^i u_i$ is a scalar.

Now $\dot{u}^i$ is parallel to u^i since now the right-hand sides of Eqs. (3.14) and (3.13) are zero, and hence both are proportional to H^i, which cannot be zero. It cannot be zero since it would mean in a C-frame that both $\vec{R}$ and $\vec{S}$ are zero [apply Eq. (2.7) to a C-frame]. That is, the particle is a pole particle. But the particle cannot become a pole particle according to theorem (g). Thus H^m cannot be zero. And $\dot{u}^i$ is parallel to u^i, whence the trajectory is a (null) geodesic.

The argument in Ref. 7 used Eq. (4.2). Multiply Eq. (3.2) by S^*_{mk} and sum over k:

$$h_m u^i u_i = 0 . \tag{4.12}$$

Since by assumption of (4.2) at least one component of h_m is not zero, it follows that u^i is null. Then Eqs. (3.15) and (3.16) show that $\dot{u}^i$ is parallel to u^i giving null geodesics.

These arguments show that if p^0 or $\vec{p}\cdot\vec{S}$ are not zero along the trajectory, then the trajectory is a null geodesic. We now show that if the trajectory is a null geodesic, then $\vec{p}\cdot\vec{S}$ and p^0 cannot be zero, provided they are not zero initially.

If the trajectory is a null geodesic then

$$\dot{u}_i = b u_i , \tag{4.13}$$

$$\dot{h}_i = e h_i , \tag{4.14}$$

where b and e are scalars. Equation (4.13) corresponds to the definition of a null geodesic, and Eq. (4.14) follows from differentiating (3.15) and subtracting (3.16). Of course, b and/or e could be zero, but we assume that they do not diverge on the trajectory.

Consider now

$$d(-h)/dq = D(h_0/u_0)/Dq = (e-b)h_0/u_0 . \tag{4.15}$$

We wish to show that a parameter $q'(q)$ can be defined for which the corresponding $h' = h_0/u'_0 = h\, dq/dq'$ satisfies $dh'/dq' = 0$. If we compute dh'/dq' and use (4.15) we get

$$d(-h')/dq' = (e'-b')h' , \tag{4.16}$$

where

$$e' = e\, dq/dq' , \tag{4.17}$$

$$b' = b\, dq/dq' + (d^2q/dq'^2) dq'/dq . \tag{4.18}$$

Setting $e' = b'$ we get

$$d^2q/dq'^2 = (e-b)(dq/dq')^2 , \tag{4.19}$$

which can be integrated

$$dq/dq' = F \exp\left[\int_0^q (e-b)dq\right] . \tag{4.20}$$

Notice that dq/dq' is always positive (negative) if the constant F is positive (negative). Thus if we start at 0 with q' positive (negative), it will never change sign, go through a zero, or approach infinity over a finite segment of the trajectory (since we assume that e and b remain finite everywhere). Thus q is a monotonic function of q'.

Thus with q' defined by (4.20), the right-hand side of Eq. (4.16) is zero and h' is a constant of the motion. Since $h = h'u'_0$, this means that if h is not zero initially and if the trajectory is a null geodesic, then h will remain nonzero throughout the trajectory.

Proof of (j). Once the trajectories are null geodesics, direct differentiation of $H = -S^*_{k0}u^k/u^0$ gives

$$\dot{H} = 0 , \tag{4.21}$$

which says that H is a constant of the motion.

Next, Eq. (3.13) gives

$$H_m = -H u_m , \tag{4.22}$$

$$\dot{H}_m = -H \dot{u}_m . \tag{4.23}$$

Equations (4.9) and (4.10) exhibit the rest of assertion (j).

V. IS p^i PARALLEL TO u^i?

Perhaps the most interesting question after whether or not the trajectory is a null geodesic is whether or not p^i is parallel to u^i. Mashhoon[6] emphasized that p^i is an effective or canonical momentum and as such need not be parallel to u^i. Nevertheless, a massless particle is quite special, and one can ask whether or not p^i is parallel to u^i anyway.

In flat space (Ref. 1) and in constant curvature space (Sec. VI below), it turns out that p^i may be parallel to u^i and will be if initial conditions are chosen appropriately. In flat space, if this is not done then $S^{\alpha 0}$ will in fact diverge.

It is interesting to note that p^i is parallel to u^i in flat space for still another reason. In flat space only the first term of Eq. (2.15) remains, and also

$$S^{\alpha 0} = -p^0(X^\alpha - x^\alpha_C) \tag{5.1}$$

from Eq. (2.14). Take D/Dq of this, denoting dx^α_C/dq by u^α_C:

$$\dot{S}^{\alpha 0} = p^0(u^\alpha_C - u^\alpha) \tag{5.2}$$

since $\dot{p}^0=0$ in flat space. Compare this with Eq. (1.2): we get

$$p^\alpha/p^0=u_C^\alpha/u^0. \tag{5.3}$$

Now $u^0=u_C^0$ both being equal to dct/dq. Further, in flat space x_C^i is a fixed point of the particle, whence u_C^i cannot be spacelike. But from (5.3) p^i is parallel to u_C^i, so p^i is not spacelike. But u^i is already known to be null. Therefore, from Appendix A it follows that p^i is parallel to u^i.

This argument cannot be applied to the C-frame of Sec. IV since one cannot allow u_C^i to be not spacelike in general, since x_C^i need not be a fixed point of the particle.

Let us now suppose that

$$p^i=mu^i \tag{5.4}$$

in general spaces, where m is a scalar, and see if it leads to contradictions. We also assume null-geodesic trajectories so that

$$\dot{u}^i=bu^i. \tag{5.5}$$

Thus

$$\dot{p}^i=-nu^i, \tag{5.6}$$

where $n=-\dot{m}-mb$. Notice that if p^i is parallel to u^i, Eq. (1.5) shows that both are null. From this it follows that the path is a null geodesic without further ado.

It can now be seen that Eq. (5.5) determines u^i; Eq. (1.4) determines three of the S^{ik}'s in terms of the other three, u^i and a [see, for example, Eq. (1.9)]; Eq. (1.2) tells us that the S^{ik} are parallel transported, whence the remaining three S^{ik} are determined from initial conditions; and *one* of the three independent equations in (1.1) gives n in terms of S^{ik}, u^i, and $R^i{}_{kmn}$, once Eq. (5.6) is used. Thus everything is determined without use of two of the equations in (1.1). For consistency, these remaining equations must be satisfied automatically with these solutions.

If we define

$$Z^{ik}=\tfrac{1}{2}R^{ik}{}_{mn}S^{mn}, \tag{5.7}$$

then Eq. (1.1) becomes

$$Z^{ik}u_k=-nu^i \tag{5.8}$$

which has the same form as Eq. (1.4). Consistency can therefore be assured if, say,

$$S^i{}_mZ^{mn}-Z^i{}_mS^{mn}=0. \tag{5.9}$$

Or if we adopt a system of coordinates at some point P of the trajectory that has x the space direction of the particle, then (5.8) may be written in the form of (1.9):

$$Z^{x0}=-n,\quad Z^{y0}=Z^{yx},\quad Z^{z0}=Z^{zx}. \tag{5.10}$$

The first of these defines n: the other two must be satisfied with the solutions for S^{ik} already obtained from the other equations. If we go into a C-frame at the point P, in which $S^{y0}=S^{z0}=S^{x0}=0$, then the last two equations of (5.10) become

$$S^{yz}(R_{y0yz}+R_{xyyz})=0, \tag{5.11}$$

$$S^{yz}(R_{z0zy}+R_{zxzy})=0. \tag{5.12}$$

Now S^{yz} cannot be zero in this frame, otherwise S^{ik} is the zero tensor at P, which we reject as unphysical. (S^{yz} is a constant of the motion in any case, since H is.) Thus the circular brackets must be zero if the pole-dipole equations are to be consistent with p^i parallel to u^i. Thus Eq. (5.11) and (5.12) are necessary conditions for p^i to be parallel to u^i. They are not, however, sufficient conditions. One can (as in flat space) choose initial conditions to thwart this. But if Eqs. (5.11) and (5.12) are satisfied, then there is the option available for p^i to be parallel to u^i.

VI. CONSTANT CURVATURE

In Ref. 1 the flat-space case was solved completely. We here set down the solution for the metric of constant curvature (de Sitter):

$$ds^2=c^2dt^2-e^{2wt}(dx^2+dy^2+dz^2) \tag{6.1}$$

(where w is the square root of the curvature constant) for which the Riemann tensor is

$$R_{ikmn}=w^2(g_{in}g_{km}-g_{im}g_{kn}). \tag{6.2}$$

We shall not go through the details of the solution, but we write down the answer in order to show that some of the flat-space features cannot be generalized.

The four-velocity gives

$$u^0=\text{const}\times e^{-wt}, \tag{6.3}$$

$$u^\alpha/u^0=A^\alpha e^{-wt},\quad \sum(A^\alpha)^2=1 \tag{6.4}$$

where the A^α are constants. The four-momentum comes out

$$p^0=aw+Be^{-wt}, \tag{6.5}$$

$$p^\alpha=BA^\alpha e^{-2wt}+B^\alpha e^{-wt},\quad \sum A^\alpha B^\alpha=aw \tag{6.6}$$

where the B^α are constants. Next

$$S^{\alpha 0}=(w^{-1}B^{\alpha}-2aA^{\alpha})\,e^{-wt}+w^{-1}F^{\alpha}\,e^{-2wt}\,,$$
$$\sum F^{\alpha}A^{\alpha}=0 \tag{6.7}$$

and finally,

$$S^{\alpha\beta}=-w^{-1}\,e^{-3wt}(A^{\alpha}F^{\beta}-A^{\beta}F^{\alpha})+G^{\alpha\beta}\,e^{-2wt}/w\,,$$
$$\sum A^{\alpha}G^{\alpha\beta}=aA^{\beta}-w^{-1}B^{\beta}. \tag{6.8}$$

We see from Eqs. (6.4) and (6.6) that in general u^i is not parallel to p^i. In the flat-space case, this was also true, but $S^{\alpha 0}$ there diverged unless p^i was in fact parallel to u^i, which could be arrived at by a choice of constants. But in the constant-curvature case, $S^{\alpha 0}$ does not diverge, even if projected onto a local tetrad. There is no inconsistency in taking the limit $w=0$ since in this limit Eq. (6.7) shows that $S^{\alpha 0}\to w^{-1}(B^{\alpha}+F^{\alpha}) - t(B^{\alpha}+2F^{\alpha})$. Thus, to take the limit we need $B^{\alpha}+F^{\alpha}=0$, and then to avoid the divergence we must have $B^{\alpha}+2F^{\alpha}=0$, which makes $B^{\alpha}=F^{\alpha}=0$ and therefore p^i parallel to u^i.

But if $w\neq 0$, there is no divergence that requires p^i to be parallel to u^i. However, if p^i is parallel to u^i initially, then it is so always. From the above equations the general condition that $p^{\alpha}/p^0=u^{\alpha}/u^0$ is

$$B^{\alpha}=awA^{\alpha}. \tag{6.9}$$

This is a relation among constants which, if satisfied initially, will obviously be satisfied always.

By the same token, if $S^{\alpha 0}=0$ at some point t of the trajectory, then it follows that

$$B^{\alpha}+F^{\alpha}\,e^{-wt}=0 \text{ and } a=0. \tag{6.10}$$

Thus if $S^{\alpha 0}=0$ at $t=0$, $F^{\alpha}=-B^{\alpha}$, whence $S^{\alpha 0}$ is *not* zero at other points of the trajectory in general. $S^{\alpha 0}=0$ means that $X^{\alpha}=X^{\alpha}_C$. Thus if $X^{\alpha}=X^{\alpha}_C$ initially, it does not necessarily follow that $X^{\alpha}=X^{\alpha}_C$ afterward.

However, if *both* p^i is parallel to u^i and $S^{\alpha 0}=0$ initially, then from Eqs. (6.9) and (6.10) we have

$$F^{\alpha}=B^{\alpha}=a=0. \tag{6.11}$$

From this it follows that p^i will be parallel to u^i, and $S^{\alpha 0}=0$ always.

VII. DISCUSSION

In this paper, we have gone into some of the mathematical consequences and physical significance of the basic pole-dipole equations (1.1)–(1.4) of a massless particle. The most directly provable consequence was that if $a\neq 0$, the trajectories are null geodesics (Sec. III). If, however, $a=0$, then certain additional assumptions had to be made (Sec. IV) to prove the null geodesics. The assumptions were physically quite plausible ($p^0\neq 0$ in a frame in which X^i is the center of energy, or $\vec{p}\cdot\vec{S}\neq 0$), but nevertheless they had to be made.

One could argue, however, that the $a\neq 0$ solutions should carry over also for the $a=0$ points. The reasoning could be simply mathematical, that the solutions ought to be continuous functions of a. Or it could be physical, that the massless object in a pole-dipole approximation should move rigidly, and the path obtained for one point should be the path for neighboring points. Thus the motion of points for vanishingly small a should be the same as for $a=0$ points. (The physical and mathematical arguments are essentially the same.) If one does not accept such arguments, then the calculations in Sec. IV can be made.

For massless particles with spin, in any given coordinate system, one does not isolate the energy center and follow its motion. What can be done is to ensure that at the initial point there exists a frame (called a C-frame) in which the point X^i under discussion is the energy center. Then it was shown in Sec. IV that there exists a C-frame for X^i at every point of the trajectory, as X^i and the particle move through four-space according to the pole-dipole equations. If X^i is the energy center in some system at the start, then one could contemplate a coordinate system, built up from the string of C-frames, in which X^i remains always the energy center.

Moller[8] showed that there is no unique energy center for spinning massive particles. In fact, the energy centers occupy a disk of radius cS_0/E_0 perpendicular to the spin $\vec{S}_0$ in the rest system (denoted by subscript 0). For massless just as for massive particles, there are many points that correspond to energy centers in various reference frames.

Instead of going into the rest frame, the best we can do for massless particles is to go into a C-frame, denoted by primes. If we go from primes to unprimes (in which the energy center is X^i) by the velocity $\vec{v}$, then[11]

$$\vec{R}=-E(\vec{X}'-\vec{X})=-\vec{v}\times\vec{S}'/(1-v^2/c^2)^{1/2}, \tag{7.1}$$

where $\vec{X}'$ is the energy center in the primed C-frame. E is the energy of the particle in the unprimes. If we transform E as the fourth component of a four-vector under Lorentz boosts between rest systems of inertia, then $E=(E'-\vec{p}'\cdot\vec{v})/(1-v^2/c^2)^{1/2}$, and (7.1) becomes

$$\vec{X}'-\vec{X}=\frac{\vec{v}\times\vec{S}'}{E'-\vec{p}'\cdot\vec{v}}. \tag{7.2}$$

Now either $\vec{p}'$ is parallel to $\vec{S}'$ or it is not. If it is parallel, then we can choose $\vec{v}$ to span all directions perpendicular to $\vec{S}'$, and all values up to $v=c$. Then $\vec{X}'-\vec{X}$ will cover a disk of radius cS'/E', which is just Moller's result.

If, however, $\vec{p}'$ is not parallel to $\vec{S}'$, Eq. (7.2) tells us that the range of $\vec{X}'-\vec{X}$ values is not circularly symmetric. In the direction perpendicular to both $\vec{S}'$ and $\vec{p}'$, the range of $\vec{X}'-\vec{X}$ goes up to cS'/E'. But in other directions it will not go this far, or it will go farther. Thus a modified form of Moller's theorem is possible for massless particles.

Finally, we come to the physically interesting question: In a given reference frame in a particular space, how does the center of mass X^i_C of the massless object move? Up to now what we have considered is how the point X^i moves. But X^i is not a body-fixed point of the particle in general; it is only a reference point. By the arguments of Sec. IV, it is a point (if $a=0$) that is the energy center in *some* reference frame, but not necessarily the observer frame.

The solution for X^i_C can be obtained directly from Eq. (1.6), with X^i known from the geodesic equation, S^{i0} known from the solution of the pole-dipole equations, and M^0 known in terms of other quantities from Eq. (2.15) with $i=0$. This is a complicated procedure, and about all one can say in general is that there is no reason why X^i_C must travel along a null geodesic, before taking the limit, if we do, of the point particle.

For constant curvature, Sec. VI, we have the complete solution. We have verified from the procedure of the preceding paragraph that the motion for X^i_C may be along a null geodesic if the constants of integration (i.e., the initial conditions) are chosen appropriately. But if the initial conditions are not so chosen, then the path will not be a null geodesic. Thus as far as the pole-dipole equations are concerned in this special case, the motion of X^i_C can be either a geodesic or not. Of course, there might be some physical reasons why initially the constants must satisfy certain relations, but these reasons are beyond the pole-dipole formulation if they exist at all.

A more easily answered question: Does X^i_C perform a trajectory that is the same as that of a particular X^i? To answer this, all one needs to do is add $S^{i0}=0$ to the pole-dipole equations, and see if a contradiction results. We have done this for the exterior Schwarzschild metric, and found that $S^{i0}=0$ is in fact not generally consistent with the pole-dipole equations. Thus an observer at rest in the gravitational field of the sun for example need not in general see the energy center of a pulse of light describe the trajectory of one of the solutions X^i of the pole-dipole equations. X^i_C will stick close to X^i (since both are "inside" the particle), but need not follow the same path. In the point-particle limit, however, they must of course coincide.

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APPENDIX A

We wish to prove here that two vectors which are not spacelike can be orthogonal only if both are null and parallel to each other. That is, if m^i and n^i satisfy

$$m^i m_i \geq 0\,,\quad n^i n_i \geq 0\,,\quad m^i n_i = 0\,, \tag{A1}$$

then

$$m^i m_i = 0 \text{ and } m^i = kn^i\,, \tag{A2}$$

where k is a scalar.

To prove this, go into a local orthogonal tetrad and let x be the space direction of m^i so that $m^i=m^0(1,f,0,0)$, where $f\leq 1$. Then $m^i n_i=0$ reads

$$m^0(n^0-fn^x)=0\,, \tag{A3}$$

i.e., $n^0=fn^x$, since m^0 cannot be zero if m^i is timelike or null. But then $n^i n_i \geq 0$ reads

$$(f^2-1)(n^x)^2-(n^y)^2-(n^z)^2\geq 0\,. \tag{A4}$$

This can be satisfied only be setting $n^y=n^z=0$ and $f=1$, which proves the assertion.

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[8]C. Moller, *The Theory of Relativity* (Oxford University Press, New York, 1972), 2nd ed., Section 6.3.

[9]If $p^0 = 0$, then T^{00} is either everywhere zero or is positive in some places, negative in others. If T^{00} is everywhere zero, the particle has little meaning. However, negative-energy density has been considered by many authors at least as far back as Einstein in early treatments of cosmology. We rule out $p^0 = 0$ in this paper, on physical grounds, but more exotic theories in which negative-energy density appears could be contemplated.

[10]See, for example, R. Resnick, *Introduction to Special Relativity* (Wiley, New York, 1968), problem 9, p. 184.

[11]Reference 8, Eq. (4.84′).