Slowly rotating bodies with arbitrary charge in general relativity

Carey Briggs
Physics Department, Harvard University, Cambridge, Massachusetts 02138

Jeffrey M. Cohen
Physics Department, University of Pennsylvania, Philadelphia, Pennsylvania 19104
and Courant Institute of Mathematical Sciences, New York University, New York 10021

Gary L. DeWoolfson
Physics Department, University of Pennsylvania, Philadelphia, Pennsylvania 19104

Lawrence S. Kegeles
Physics Department, Stevens Institute of Technology, Hoboken, New Jersey 07030
(Received 19 May 1978; revised manuscript received 5 May 1980)

The Einstein-Maxwell equations are solved for a slowly rotating body with arbitrary charge. The solution is applied to a thin, rotating, charged shell. The angular momentum, gyromagnetic ratio, and other quantities of physical interest are computed. In particular, whenever the charge is less than the mass (but not necessarily small) the gyromagnetic ratio approaches 2 as the shell radius approaches the horizon. Under these conditions the rotational velocity of the inertial frames inside the shell approaches the rotational velocity of the shell. When the charge is greater than the mass there is no horizon and the gyromagnetic ratio can exceed 2. Furthermore, an example is given in which the inertial frames within the shell rotate in a direction opposite that of the shell.

I. INTRODUCTION

It is well known that the classical Maxwell equations can be generalized to include the effects of curved space-time. When this is done one finds electromagnetic behavior which differs from that expected from flat-space-time calculations. The introduction of rotation brings with it yet another possibility, this being the influence of rotation on the inertial frames and on the electromagnetic field.

In an effort to address these questions, the stationary, axially symmetric[1] metric[2,3]

$$ds^2 = -A^2dt^2 + B^2dr^2 + C^2d\theta^2 + E^2(d\phi - \Omega dt)^2 \qquad (1)$$

has been studied. Here A, B, C, E, and Ω are functions only of the variables r and θ. In the asymptotically flat regions of large r, r and θ are the usual spherical coordinates. For a discussion of symmetries and their implications, see Ref. 3. The metric contains the angular velocity of the inertial frames Ω explicitly. This angular velocity is measured relative to observers in the inertial frames of the asymptotically flat space-time at infinity.[3]

For the uncharged case Brill and Cohen[2,3] found the metric of this form [Eq. (1)] for slow rotation (to first order in Ω). Cohen then used these results to solve Maxwell's equations for this geometry in the same approximation.[4,5] Hence an exterior solution was obtained to the equations which result when terms quadratic in the charge and rotational velocity in the Einstein-Maxwell equations are neglected. This solution was used to analyze slowly rotating thin shells of arbitrary mass and small charge.[4,5] Cohen, Tiomno, and Wald calculated the gyromagnetic ratio of such a thin shell.

In this paper we work with slow rotation; that is, we ignore terms quadratic in the angular velocity, but now admit arbitrary charge. The exterior solution is found and applied to the thin-shell model. We calculate the gyromagnetic ratio and angular momentum of such a configuration with arbitrarily large charge.

II. EINSTEIN FIELD EQUATIONS

We use exterior calculus to derive the equations which determine the Einstein tensor for Eq. (1). A Cartan frame appropriate for our calculations is $\omega^0 = A\,dt$, $\omega^1 = B\,dr$, $\omega^2 = C\,d\theta$, $\omega^3 = E(d\varphi - \Omega\,dt)$. These forms are the basis dual to the orthonormal vector basis. The components of the Einstein tensor are determined by

$$8\pi T^{\mu\nu} = G^{\mu\nu},$$

where the $T^{\mu\nu}$ and $G^{\mu\nu}$ are the components of the stress-energy tensor and Einstein tensor relative to the orthonormal basis. One finds that the nontrivial field equations can be written as

$$-8\pi T^{00} = (BC)^{-1}[(C_r/B)_r + (B_\theta/C)_\theta + E^{-1}(CE_r/B)_r + E^{-1}(BE_\theta/C)_\theta] + [E\Omega_r/(2AB)]^2 + [E\Omega_\theta/(2AC)]^2,$$

$$-8\pi T^{03}=\tfrac{1}{2}(BCE^2)^{-1}\{[CE^3\Omega_r/(AB)]_r+[BE^3\Omega_\theta/(AC)]_\theta\}\,,$$

$$-8\pi T^{12}=(AC)^{-1}[(A_r/B)_\theta-A_\theta C_r/(BC)]+(CE)^{-1}[(E_r/B)_\theta-E_\theta C_r/(BC)]-[E^2\Omega_r\Omega_\theta/(2A^2BC)]\,,$$

$$8\pi T^{11}=[E\Omega_r/(2AB)]^2-[E\Omega_\theta/(2AC)]^2+(CB^2E)^{-1}E_rC_r+(AB^2E)^{-1}A_rE_r+(AB^2C)^{-1}C_rA_r$$
$$+(CE)^{-1}(E_\theta/C)_\theta+(AC)^{-1}(A_\theta/C)_\theta+(AC^2E)^{-1}A_\theta E_\theta\,,$$

$$8\pi T^{22}=[E\Omega_\theta/(2AC)]^2-[E\Omega_r/(2AB)]^2+(BC^2E)^{-1}E_\theta B_\theta+(ABC^2)^{-1}A_\theta B_\theta$$
$$+(AC^2E)^{-1}A_\theta E_\theta+(AB)^{-1}(A_r/B)_r+(BE)^{-1}(E_r/B)_r+(AB^2E)^{-1}A_rE_r\,,$$

$$8\pi T^{33}=(ABC)^{-1}(CA_r/B)_r+(ABC)^{-1}(BA_\theta/C)_\theta+(BC)^{-1}(C_r/B)_r+(BC)^{-1}(B_\theta/C)_\theta$$
$$-3[E\Omega_r/(2AB)]^2-3[E\Omega_\theta/(2AC)]^2\,.$$

These equations were first derived by Brill and Cohen and reported in a different form in Ref. 3 In these equations a subscript, r or θ, denotes partial differentiation. Generally, there are mechanical and electromagnetic contributions to the stress-energy tensor.

We first infer the mechanical contributions for an axially symmetric distribution of matter which is observed to rotate with a constant angular velocity ω about the z axis. All parts of the object rotate with the same angular velocity, i.e., the rotation is rigid. Let our first view be that of observers with respect to whom the object does not rotate. Let $(r,\theta,\bar\varphi,\bar t)$ be coordinates used by these observers. Then we may conclude that the local inertial frames are spanned by a Cartan frame $\bar\omega^0=A\,d\bar t$, $\bar\omega^1=B\,dr$, $\bar\omega^2=C\,d\theta$, $\bar\omega^3=E\,d\bar\varphi$. For if the ω^ν's represent a local Cartan frame, then the Lorentz transformation

$$\omega^0=\gamma(\zeta\bar\omega^3+\bar\omega^0)\,,\qquad \omega^3=\gamma(\bar\omega^3+\zeta\bar\omega^0) \tag{2}$$

with

$$\zeta=E(\omega-\Omega)/A\,,\qquad \gamma=(1-\zeta^2)^{-1/2}$$

establishes the connection between the Cartan

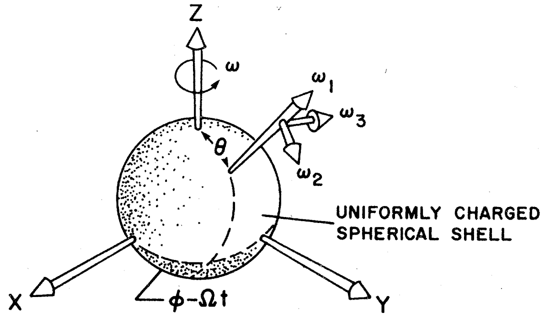

FIG. 1. Orientation of Cartan's moving orthonormal frames ω_ν relative to the uniformly charged shell which rotates with angular velocity ω.

frames. Hence we may infer

$$T_{\rm mech}=T^{\mu\nu}_{\rm mech}\,\omega_\mu\otimes\omega_\nu\,.$$

In the rest frame of the rotator $\bar T^{00}=\rho$, $\bar T^{0i}=0$, and $\bar T^{ij}=t^{ij}$, where ρ is the mass density and t^{ij} is the three-dimensional stress tensor. We conclude,

$$(T^{\mu\nu}_{\rm mech})=\begin{pmatrix}\gamma^2(\rho+\zeta^2t^{33}) & \gamma\zeta t^{31} & \gamma\zeta t^{32} & \gamma^2\zeta(\rho+t^{33})\\ \gamma\zeta t^{13} & t^{11} & t^{12} & \gamma t^{13}\\ \gamma\zeta t^{23} & t^{21} & t^{22} & \gamma t^{23}\\ \gamma^2\zeta(\rho+t^{33}) & \gamma t^{31} & \gamma t^{32} & \gamma^2(t^{33}+\zeta^2\rho)\end{pmatrix}.$$

The electromagnetic contribution to the stress-energy tensor is well known.[6] Relative to the orthonormal basis, the electric and magnetic vectors are written as

$$\vec{\rm E}=e^1\omega_1+e^2\omega_2+e^3\omega_3\,,$$
$$\vec{\rm H}=h^1\omega_1+h^2\omega_2+h^3\omega_3\,.$$

Then,

$$4\pi T^{00}_{E+M}=\tfrac{1}{2}(E^2+H^2)\,,$$
$$4\pi T^{0i}_{E+M}=\epsilon^i{}_{jk}e^jh^k\,,$$
$$4\pi T^{ij}_{E+M}=\tfrac{1}{2}(E^2+H^2)\delta^{ij}-e^ie^j-h^ih^j\,.$$

III. MAXWELL EQUATIONS

To write Maxwell's equations in curved space we express the electromagnetic-field tensor and the electric current vector as differential forms. In Cartan's notation[7,8] the field tensor is

$$f=\tfrac{1}{2}f_{\mu\nu}\omega^\mu\Lambda\,\omega^\nu\,,$$

where

$$(f_{\mu\nu})=\begin{pmatrix}0 & -e^1 & -e^2 & -e^3\\ e^1 & 0 & h^3 & -h^2\\ e^2 & -h^3 & 0 & h^1\\ e^3 & h^2 & -h^1 & 0\end{pmatrix}.$$

In the rest frame of our rotator the current four-vector is simply

$$\bar{I} = \rho_e \bar{\omega}_0 \,,$$

where ρ_e is the local charge density. Using (2) we obtain

$$\bar{I} = \rho_e \gamma \omega_0 + \rho_e \gamma \zeta \omega_3 \,,$$

or, in the dual representation,

$$\bar{I} = -\rho_e \gamma \omega^0 + \rho_e \gamma \zeta \omega^3 \,.$$

This dual representation leads to succinct statements of Maxwell's equations using the de Rham operators d and δ.[8,9] The homogeneous equations are

$$df = 0 \,,$$

from which we obtain

$$(ACe_2)_r - (ABe_1)_\theta - BCE[h_2(\Omega_\theta/C) + h_1(\Omega_r/B)] = 0 \,,$$
$$(AEe_3)_r = 0 \,,$$
$$(AEe_3)_\theta = 0 \,,$$
$$(BEh_2)_\theta + (CEh_1)_r = 0 \,.$$

The remaining four equations are

$$\delta f = {}^*d{}^*f = 4\pi I \,,$$

which gives

$$(ACh_2)_r - (ABh_1)_\theta + BCE[e_2(\Omega_\theta/C) + e_1(\Omega_r/B)] = 4\pi\rho_e\gamma\zeta ABC \,,$$
$$(Aeh_3)_r = 0 \,,$$
$$(AEh_3)_\theta = 0 \,,$$
$$(CEe_1)_r + (BEe_2)_\theta = +4\pi\rho_e\gamma BCE \,.$$

IV. FORMULATION AND SOLUTION

In this section we find an exterior solution to these Einstein-Maxwell field equations to first order in the angular velocity, but with an arbitrarily large charge. The solution generalizes the slow-rotation limit of the Kerr-Newmann[11] exterior solution. We work with the metric [Eq. (1)] (Ref. 3) in isotropic form,[3,10]

$$ds^2 = -V^2 dt^2 + \psi^4[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta (d\varphi - \Omega\, dt)^2] \,,$$

where V and ψ are functions only of r. Suitable forms for the electric and magnetic field vectors are[4]

$$E = e\omega_1 \,, \quad H = n\cos\theta\omega_1 + p\sin\theta\omega_2$$

where e, p, and n are functions only of r.

The equations determining the solution follow from the previous sections and are obtained when terms quadratic in the angular velocities are neglected. The nontrivial equations are

$$8\pi[\rho_m + (1/8\pi)e^2] = (-1)(r\Psi^2)^{-2}\left\{ r\left[\frac{(r\Psi^2)_r}{\Psi^2}\right]_r + \Psi^{-2}[r(r\Psi^2)_r]_r - 1 \right\} \,, \tag{3}$$

$$8\pi[t^{11} - (1/8\pi)e^2] = (r\Psi^2)^{-2}\left\{ \left[\frac{(r\Psi^2)_r}{\Psi^2}\right]^2 - 1 \right\} + 2(rV\Psi^4)^{-1}\, \frac{(r\Psi^2)_r}{\Psi^2}\, V_r \,, \tag{4}$$

$$8\pi[t^{33} + (1/8\pi)e^2] = (V\Psi^2)^{-1}\left\{ (V_r/\Psi^2)_r + (r\Psi^2)^{-1}\left[\frac{V(r\Psi^2)_r}{\Psi^2}\right]_r \right\} , \tag{5}$$

$$8\pi[r\Psi^2V^{-1}(\omega - \Omega)(\rho_m + t^{33}) + (1/4\pi)ep] = (-1)\frac{r}{2(r\Psi^2)^4}\left[\frac{(r\Psi^2)^4\Omega_r}{V\Psi^2}\right]_r \,, \tag{6}$$

$$[(r\Psi^2)^2 e]_r = 4\pi\rho_e r^2\Psi^6 \,, \tag{7}$$

$$p = -\tfrac{1}{2}(r\Psi^4)^{-1}[(r\Psi^2)^2 n]_r \,, \tag{8}$$

$$[r\Psi^2 Vp]_r + V\Psi^2 n + (r\Psi^2)^2 e\Omega_r = 4\pi\rho_e(\omega - \Omega)r^2\Psi^6 \,. \tag{9}$$

Everything said for the remainder of this section pertains to the regions exterior to the charge and mass distribution.

From Eq. (7), we have,

$$e = q/(r\Psi^2)^2 \,. \tag{10}$$

Equations (3)–(5) are now satisfied by

$$\Psi^2 = (1 + \alpha r^{-1})^2 - (q/2r)^2 \,,$$
$$V\Psi^2 = (1 + \alpha r^{-1})(1 - \alpha r^{-1}) + (q/2r)^2 \,. \tag{11}$$

When there is no rotation ($\Omega = 0$), Eqs. (11) represent a spherically symmetric solution to the Einstein-Maxwell field equations, the Reissner-Nordström metric in isotropic form, which is asymptotically flat, and asymptotically becomes the Minkowski metric expressed in spherical coordinates. The metric describes the space-time geometry around a spherically symmetric charged body centered at the origin of the coordinates. The constant 2α is the mass as seen by observers at infinity.

Use of Eq. (8) in Eq. (6) produces

$$q[(r\Psi^2)^2 n]_r = \frac{1}{2}\left[\frac{(r\Psi^2)^4\Omega_r}{V\Psi^2}\right]_r ,$$
$$\Omega_r = \frac{V\Psi^2}{(r\Psi^2)^4}[2q(r\Psi^2)^2 n - \mathfrak{M}_0] , \qquad (12)$$

where $\mathfrak{M}_0$ is a constant of integration.

Using Eqs. (8) and (12) to eliminate p and Ω_r in Eq. (9), we get

$$-\frac{1}{2}\left\{\frac{V}{\Psi^2}[(r\Psi^2)^2 n]_r\right\}_r + nV\Psi^2\left(1+\frac{2q^2}{(r\Psi^2)^2}\right) = q\mathfrak{M}_0\frac{V\Psi^2}{(r\Psi^2)^4} . \qquad (13)$$

When $|q| \neq m$ the general solution of this equation is (see the Appendix)

$$n = \mathfrak{M}_0 q\beta^3/(6\alpha) + \mathfrak{M}_1(1 - 12\xi^2\alpha^2\beta^2 + 16\xi^4\alpha^3\beta^3) + \mathfrak{M}_2\left[16[\xi^2\alpha^2\beta^2 - \alpha\beta/4 - (1+2\xi^2)/(24\xi^2)] + b^{-1}(1 - 12\xi^2\alpha^2\beta^2 + 16\xi^4\alpha^3\beta^3)\ln\left(\frac{\beta-\beta_1}{\beta-\beta_2}\right)\right] , \qquad (14)$$

where

$$\xi = |q|/m ,$$
$$b = (1-\xi^2)^{1/2} ,$$
$$\beta_1 = (2\xi^2\alpha)^{-1}(1+b) ,$$
$$\beta_2 = (2\xi^2\alpha)^{-1}(1-b) ,$$
$$\beta = (r\Psi^2)^{-1} .$$

When $|q| = m$, we have

$$(r\Psi^2)^3 n = q\mathfrak{M}_0/(6\alpha) + \mathfrak{M}_1 r^3[1 + (6\alpha/r)] + \mathfrak{M}_2\{1 + (3\alpha/r) + [18\alpha^2/(5r^2)] + [8\alpha^3/(5r^3)]\} . \qquad (15)$$

Integration of Eq. (12) yields Ω. For $|q| \neq m$,

$$\Omega = \Omega_0 + \tfrac{1}{3}\mathfrak{M}_0\beta^3(1-\xi^2\alpha\beta) - 2q\mathfrak{M}_1\beta(1 - 4\xi^2\alpha^2\beta^2 + 4\xi^4\alpha^3\beta^3) - 2q\mathfrak{M}_2\left[[4\xi^2\alpha^2\beta^3 - 2\alpha\beta^2 - \tfrac{2}{3}\xi^{-2}(1+\tfrac{1}{2}\xi^2)\beta] + (4\alpha b)^{-1}(4\alpha\beta - 16\xi^2\alpha^3\beta^3 + 16\xi^4\alpha^4\beta^4 - 1)\ln\left(\frac{\beta-\beta_1}{\beta-\beta_2}\right)\right] . \qquad (16)$$

for $|q| = m$,

$$(r\Psi^2)^4(\Omega - \Omega_0) = \tfrac{1}{3}\mathfrak{M}_0 r[1 + (\alpha/r)] - 2q\mathfrak{M}_1 r^3[1 + (6\alpha/r) + (8\alpha^2/r^2) + (4\alpha^3/r^3)] - (q/2)\mathfrak{M}_2\{1 + [12\alpha/(5r)] + [8\alpha^2/(5r^2)]\} . \qquad (17)$$

V. THIN SHELL; BOUNDARY CONDITIONS

In this section we use our results to study a particular model. The configuration considered is that of a massive, charged rotating shell of coordinate radius r_0. The shell rotates rigidly about the z axis with an angular velocity ω. The distribution of matter is specified by

$$\rho_m = K\delta(r - r_0), \quad \rho_e = \sigma\delta(r - r_0),$$
$$1 = \int \delta(r - r_0)\omega^1 \wedge \omega^2 \wedge \omega^3 .$$

From Eq. (7) we conclude that regularity at the origin requires

$$e = 0 \quad \text{for } r < r_0 .$$

Integration of Eq. (7) across the shell identifies the constant q as the total charge:

$$q = \int_{t=\text{const}} \sigma\delta\omega^1 \wedge \omega^2 \wedge \omega^3 ,$$

We conclude from Eqs. (3)–(5) that Ψ and V are continuous across the shell. This is shown as follows: Eq. (3) yields

$$-2\pi r\Psi^5 T^{00} = (r\Psi)_{,r,r} .$$

We use Eqs. (3) and (4) in Eq. (5) to obtain the relation

$$8\pi r V\Psi^5(T^{33} + \tfrac{1}{4}T^{00} + \tfrac{1}{2}T^{11}) = (rV\Psi)_{,r,r} .$$

But if F is a function such that $F_{,r,r} \sim \delta(r - r_0)$, then F is continuous at r_0. Hence V and Ψ are continuous across the shell. Regularity at the origin implies V and Ψ are constant in the interior. Therefore,

$$\Psi^2 = (1 + \alpha r_0^{-1})^2 - (q/2r_0)^2$$
$$V\Psi^2 = (1 + \alpha r_0^{-1})(1 - \alpha r_0^{-1}) + (q/2r_0)^2 \quad \text{for } r \leq r_0 .$$

Integration of Eqs. (3)–(5) across the shell determines the mass density and the remaining components of the stress supporting the shell:

$$\begin{aligned} K &= 2\alpha(1+\alpha r_0^{-1}) - q^2/(2r_0)\,, \\ t^{11} &= 0,\quad t^{22}=t^{33}=S\delta(r-r_0)\,, \\ S &= \alpha^2(r_0V_0)^{-1}(1-\xi^2) \\ &= \alpha^2(r_0V_0)^{-1}(1-q^2m^{-2})\,. \end{aligned} \tag{18}$$

Equations (6), (8), and (9) may now be solved for n, p, and Ω. We find

$$\Omega = \Omega_0 + \Omega_1/r^3\,,$$

$$n = n_0 + n_1/r^3\,,$$

$$2rp = -(r^2n)_r \quad \text{for } r<r_0\,.$$

Regularity at the origin requires $\Omega=\Omega_0$, $n=-p=n_0$.

Asymptotic flatness at infinity, together with the stipulation that the observers at infinity be in an inertial frame, require Ω to vanish at large r. Also, the magnetic fields vanish at infinity. From the solutions of the previous section, we obtain

$$n(r) = \mathfrak{N}_0 q\beta^3/(6\alpha) + \mathfrak{N}_1\mathfrak{R}(r)\,, \tag{19}$$

where

$$\mathfrak{R}(r) = -\tfrac{3}{32}\alpha^{-3}(1-\xi^2)^{-2}\left[2\alpha\beta + 4\alpha^2\beta^2 - (\tfrac{16}{3})\xi^2(1+2\xi^2)\alpha^3\beta^3 + (2b)^{-1}(1-12\xi^2\alpha^2\beta^2 + 16\xi^4\alpha^3\beta^3)\ln\left(\frac{1-2\alpha\beta(1+b)}{1-2\alpha\beta(1-b)}\right)\right]\,,$$

$$\mathfrak{R}(r) = \beta^3[1+O(\beta)] \quad (r\to\infty)\,,$$

$$\Omega(r) = \tfrac{1}{3}\mathfrak{N}_0\beta^3(1-\xi^2\alpha\beta) + 2q\mathfrak{N}_1\mathfrak{I}(r)\,, \tag{20}$$

where

$$\mathfrak{I}(r) = \tfrac{-3}{64}\alpha^{-3}(1-\xi^2)^{-2}\Big[\beta - 2\alpha\beta^2 - \tfrac{8}{3}(1+\tfrac{1}{2}\xi^2)\alpha^2\beta^3 + \tfrac{8}{3}\xi^2(1+2\xi^2)\alpha^3\beta^4$$
$$-(4\alpha b)^{-1}(4\alpha\beta - 16\xi^2\alpha^3\beta^3 + 16\xi^4\alpha^4\beta^4 - 1)\ln\left(\frac{1-2\alpha\beta(1+b)}{1-2\alpha\beta(1-b)}\right)\Big]\,,$$

$$\mathfrak{I}(r) = -\tfrac{1}{4}\beta^4[1+O(\beta)] \quad (r\to\infty)\,.$$

All that remains is to satisfy the junction conditions of Eqs. (6) and (9). In what follows we assume that p is a regular distribution—it is not a δ function or a derivative of a δ function. Then Eqs. (6), (8), and (9) imply that Ω and n are continuous across the shell.

Integration of Eqs. (6) and (9) gives

$$\Omega_r\big|_{r_-}^{r_+} = -4(K+S)(\omega-\Omega_0)(r_0\Psi_0)^{-2}\,,$$

$$[(r\Psi^2)^2 n]_r\big|_{r_-}^{r_+} = -2q(r_0\Psi_0^2)^2(r_0^2V_0\Psi_0^2)^{-1}(\omega-\Omega_0)\,,$$

where r_- and r_+ denote the limit as we approach r_0 from below and above, respectively. A subscript indicates that the functions are to be evaluated at $r=r_0$.

From these conditions we obtain the following equations:

$$(r_0\Psi_0^2)^4\Omega_0 = \tfrac{1}{3}\mathfrak{N}_0(r_0\Psi_0^2 - \xi^2\alpha) + 2q(r_0\Psi_0^2)^4\mathfrak{N}_1\mathfrak{I}_0\,, \tag{21}$$

$$(r_0\Psi_0^2)^3 n_0 = q\mathfrak{N}_0/(6\alpha) + (r_0\Psi_0^2)^3\mathfrak{N}_1\mathfrak{R}_0\,, \tag{22}$$

$$\mathfrak{N}_0(r_0\Psi_0^2 - \tfrac{4}{3}\xi^2\alpha) - 2q(r_0\Psi_0^2)^3\mathfrak{N}_1\mathfrak{R}_0 = 4(r_0\Psi_0^2)^4(r_0V_0\Psi_0^2)^{-1}(K+S)(\omega-\Omega_0)\,, \tag{23}$$

$$\mathfrak{N}_1(r_0\Psi_0^2)^2[2(r_0\Psi_0^2)\mathfrak{R}_0 - \mathfrak{R}_0'] - q\mathfrak{N}_0/(6\alpha) - 2(r_0V_0\Psi_0^2)^{-1}(r_0\Psi_0^2)^4 n_0 = -2q(r_0\Psi_0^2)^4(r_0V_0\Psi_0^2)^{-2}(\omega-\Omega_0)\,. \tag{24}$$

Here $\mathfrak{R}'$ is the derivative of $\mathfrak{R}$ with respect to β.

We conclude,

$$\mathfrak{N}_1 = \mathfrak{F}q\mathfrak{N}_0/(12\alpha)\,, \tag{25}$$

where

$$\mathfrak{F} = \frac{(-1)\beta_0^2(3K+4S)}{\{\mathfrak{R}_0[2K(K+S)+q^2] + V_0(K+S)\mathfrak{R}'\}} \tag{26}$$

and

$$12\alpha(K+S)\omega = \mathfrak{N}_0\{\beta_0^2[(K+S)+\tfrac{1}{2}V_0K] + q^2\mathfrak{F}[2(K+S)\mathfrak{I}_0 - \tfrac{1}{2}V_0\mathfrak{R}_0]\}\,. \tag{27}$$

Equations (21) and (22) now give Ω_0 and n_0.

To obtain the expression for the gyromagnetic ratio of the shell, we introduce the total angular momentum of the shell:

$$J = \int_{t=\text{const}} E\,T^{03}\omega^1\wedge\omega^2\wedge\omega^3\,,$$

Insertion of

$$T^{03} = \xi(\rho_m + t^{33}) + (4\pi)^{-1} ep \sin\theta$$

establishes the connection between our constants and the angular momentum to first order in the angular momentum. Substituting for e and using Eq. (8),

$$J = 2\pi\left(\int_0^{\pi} \sin^3\theta\, d\theta\right)(4\pi)^{-1}\left\{(r_0\Psi_0^2)^2 V_0^{-1}(\omega - \Omega_0)(K+S) - \tfrac{1}{2} q \int_{r_0}^{\infty} dr[(r\Psi^2)^2 n]_{,r}\right\}$$

$$= \tfrac{2}{3}(r_0\Psi_0^2)^2 V_0^{-1}(\omega - \Omega_0)(K+S) + \tfrac{1}{3} q (r_0\Psi_0^2)^2 n_0 .$$

Using Eq. (23) to eliminate the term in $(\omega - \Omega_0)$ and Eq. (22) to eliminate the term in n_0 we find

$$6J = \mathfrak{N}_0 . \qquad (28)$$

Therefore, by Eq. (25),

$$\mathfrak{N}_1 = \mathfrak{F} qJ/m$$

and

$$n = (2 + \mathfrak{F})\beta^2 qJ/m \quad \text{for } (r \to \infty) .$$

From this we identify the g factor of the shell [defined as $2\mu(qJ/m)^{-1}$, where μ is the magnetic dipole moment]:

$$g = 2 + \mathfrak{F} .$$

VI. REDUCTION IN SPECIAL CASES

In this section we take a closer look at our results in some special cases. Throughout the section we restrict our analysis to configurations for which $V \geq 0$ and $K \geq 0$.

A. $\xi \ll 1$ ($\xi = |q|/m$)

When $\xi \ll 1$ and $V \geq 0$, our present results reduce to those obtained by Cohen, Tiomno, and Wald.[5] One obtains from Eqs. (20), (21), (27), and (28) $\Omega(r) = 2J/(r\Psi^2)^3$, where

$$J = \frac{m(r_0\Psi_0^2)^2\omega}{1 + V_0^2\Psi_0/(2-\epsilon)} = \tfrac{1}{2}(r_0\Psi_0^2)^3\Omega_0 .$$

Here $\epsilon = \alpha r_0^{-1}$ and for $V \geq 0$, $0 \leq \epsilon \leq 1$. When $\epsilon = 1$, $V = (1-\epsilon)(1+\epsilon)^{-1} = 0$. Then

$$J = M(r_0\Psi_0^2)^2\omega$$

and

$$\Omega_0 = \omega .$$

So when $r_0 \to \alpha$ (the Schwarzschild radius of the shell) inertial observers at the surface of the shell see no rotation of the shell.

From Eqs. (19),

$$\mathfrak{R}(r) = \tfrac{-3}{32}\alpha^{-3}\{[2\alpha/(r\Psi^2)] + [4\alpha^2/(r\Psi^2)^2] + \ln V\} ,$$

while Eq. (26) becomes

$$\mathfrak{F} = \frac{16\epsilon^2(3-\epsilon)V_0}{3(2-\epsilon)\Psi_0\{2\epsilon[1-\epsilon(2+\epsilon)] + V_0\Psi_0^4 \ln V_0\}} .$$

The resulting gyromagnetic ratio is displayed in Fig. 2. As the shell radius approaches its Schwarzschild radius ($\epsilon \to 1$), $g \to 2$ the same values as for a Dirac particle. As the shell radius becomes large, $g \to 1$ the value to be expected when relativistic effects are negligible.[5]

Note finally from Eqs. (18) that $S \to \infty$ as $V_0 \to 0$. So when $\xi \ll 1$ and the radius of the shell approaches the Schwarzschild radius, we find that

$$S \to \infty,\ \Omega_0 \to \omega,\ g \to 2,\ J \to m(r_0\Psi_0^2)^2\omega .$$

We show next in what way these findings are not unique to the $\xi \ll 1$ condition.

B. The Horizon radius

From Eq. (11),

$$V\Psi^2 = (1 + \alpha r^{-1})(1 - \alpha r^{-1}) + (q/2r)^2 = 1 - \alpha^2 b^2 r^{-2} ,$$

where $b = (1-\xi^2)^{1/2}$. We note that for all $\xi < 1$, we have $V \to 0$ when $r_0 \to \alpha b$, the horizon radius. We conclude from Eqs. (18) that as the radius of the shell approaches the horizon radius,

$$(r_0 \to \alpha b),\ S \to \infty .$$

We see from Eq. (26) that as $S \to \infty$,

$$\mathfrak{F} \to \frac{(-1)4\beta_0^2}{2KR_0 + V_0 R_0'} .$$

Calculating R_0' and noting that

$$\ln\left[\frac{1 - 2\alpha\beta(1+b)}{1 - 2\alpha\beta(1-b)}\right] = \ln\left[\frac{1 - \alpha b r^{-1}}{1 + \alpha b r^{-1}}\right]^2 = \ln\left[\frac{V\Psi^2}{(1 + \alpha b r^{-1})^2}\right]^2 ,$$

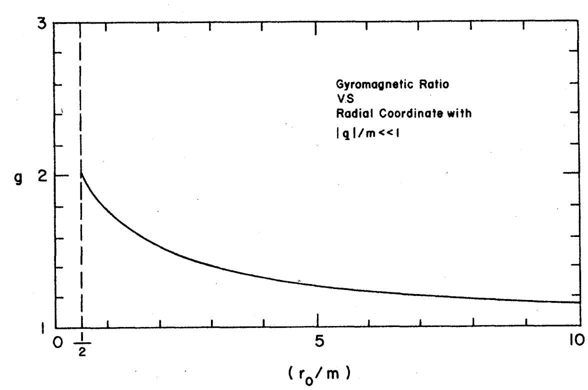

FIG. 2. Gyromagnetic ratio g vs radius for small charge-to-mass ratio $\xi = |q|/m \ll 1$. As the shell radius approaches the gravitational radius, $g \to 2$. All charge and mass are concentrated in the shell.

we obtain the expression

$$\mathfrak{F} \to \frac{(-1)4\beta_0^2 V_0}{y_0 + (V_0 y_1 + V_0^2 y_2)\ln\left[\frac{V_0 \Psi_0^2}{(1+\alpha b r_0^{-1})^2}\right]}.$$

All the y_i's are well behaved at $r_0 = \alpha b$ and $y_0(\alpha b) \neq 0$. Since $\lim_{V\to 0} V \ln V = 0$, we have the result that $\mathfrak{F} \to 0$ as $r_0 \to \alpha b$.

We find in a similar way from Eqs. (21) and (27) that as $V_0 \to 0$,

$$\Omega_0 \to \tfrac{1}{3}\mathfrak{N}_0 \beta_0^3 (1 - \xi^2 \alpha \beta_0)$$

and

$$\mathfrak{N}_0 \to 12\alpha\omega(r_0\Psi_0^2)^2 .$$

Using Eq. (28) and observing that

$$2m(1-\xi^2\alpha\beta_0)_{r=\alpha b} = (r_0\Psi_0^2)_{r0=\alpha b} ,$$

we conclude that $J \to m(r_0\Psi_0^2)\omega$ and $\Omega_0 \to \omega$ when $r_0 \to \alpha b$.

So whenever the charge-to-mass ratio of the shell is less than 1 and the shell radius approaches the horizon radius, we have

$$\Omega_0 \to \omega, \quad J \to m(r_0\Psi_0^2)^2\omega, \quad \text{and } g \to 2 .$$

C. $\xi = 1$

When $|q| \to m$, Eqs. (19) and (20) supply the desired expressions for $\mathfrak{R}$ and $\mathfrak{G}$. However, these functions are obtained more easily from Eqs. (15) and (17) by applying the boundary conditions at infinity, while keeping in mind the definition of $\mathfrak{G}$ as established by Eq. (20). In either case,

$$\mathfrak{R}(r) = (r\Psi^2)^{-3}\{1 + (3\alpha/r) + [18\alpha^2/(5r^2)] + [8\alpha^3/(5r^3)]\}$$

and

$$\mathfrak{G}(r) = -\tfrac{1}{4}(r\Psi^2)^{-4}\{1 + [12\alpha/(5r)] + [8\alpha^2/(5r^2)]\} .$$

When $|q|$ is set equal to m, our constants reduce to a simple form. From Eqs. (18),

$$K = m \text{ and } S = 0 .$$

Equations (26) and (27) are used to determine $\mathfrak{N}_0 = 6J$ and $\mathfrak{F}$:

$$\mathfrak{F} = (-1)(1 + 4\epsilon + 6\epsilon^2 + \tfrac{16}{5}\epsilon^3)^{-1} ,$$

$$J = \frac{m(r_0\Psi_0^2)^2\omega}{1 - \frac{1}{2}\mathfrak{F}\Psi_0^4} .$$

Again $\epsilon = \alpha r_0^{-1}$, but here the conditions $K \geq 0$ and $V \geq 0$ give rise to no restrictions and $0 \leq \epsilon \leq \infty$. The resulting gyromagnetic ratio is shown in Fig. 3. Again $g \to 2$ for the smallest permissible radius and $g \to 1$ as $r \to \infty$.

The dependence of the rotation of the inertial frames on the rotation of the massive body can

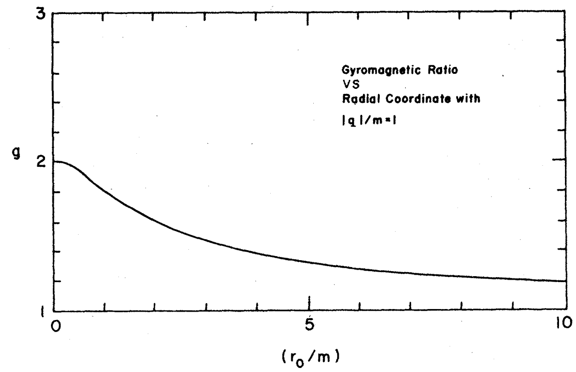

FIG. 3. Gyromagnetic ratio g vs radius for charge-to-mass ratio $\xi = |q|/m = 1$. As the shell radius approaches the gravitational radius, $g \to 2$. The shell surface area is given by $4\pi R_0^2$, where $R_0 = r_0\Psi_0^2$. In this case $R_0 = r_0 + m$.

be obtained from Eq. (21):

$$\frac{\omega - \Omega_0}{\omega} = \frac{2 + \frac{1}{2}\Psi_0^6\mathfrak{F}}{\Psi_0^2 - \frac{1}{2}\Psi_0^6\mathfrak{F}} .$$

D. $\xi \gg 1$

Our reduction in this case follows from the conditions $\xi \gg 1$ and $K \geq 0$. With $\xi \gg 1$ it follows that

$$K \geq 0 \Longrightarrow m \geq q^2/(2r_0) \text{ or } r_0/|q| \geq |q|/(2m) \gg 1 .$$

Therefore,

$$|q|/r_0 \ll 1 \text{ and } m/r_0 \ll 1 .$$

We find then that

$$\mathfrak{R}(r) = (1/r^3)[1 + O(m/r)]$$

and

$$\mathfrak{G}(r) = [-1/(4r^4)][1 + O(m/r)] .$$

With the above functions we get simply

$$n(r) = gqJ/(mr^3)$$

and

$$\Omega(r) = (2J/r^3)[1 - g\xi^2 m/(4r)]$$
$$= \tfrac{2}{3}q^2 r_0 r^{-3}(\nu - \tfrac{5}{6} - \tfrac{1}{2}r_0 r^{-1})\omega . \quad (29)$$

Our constants also have a simple form. To express them we define ν by $r_0 \equiv \nu q^2/(2m)$, where $K \geq 0 \Longrightarrow \nu \geq 1$. We obtain

$$g = 6\nu/(6\nu - 5) ,$$

$$J = [2/(3g)]mr_0^2\omega = \tfrac{1}{3}q^2 r_0\omega(\nu - \tfrac{5}{6}) ,$$

$$\Omega_0/\omega = \tfrac{8}{9}(3\nu - 4)/(\nu^2\xi^2) = \tfrac{2}{3}q^2 r_0^{-2}(\nu - \tfrac{4}{3}) .$$

In the range $1 \leq \nu \leq \frac{4}{3}$, we have $\Omega_0 \leq 0$. This behavior derives from the influence of the negative elastic stress on the inertial frames. It is interesting that within the shell the inertial frames rotate in a direction opposite to that of the shell.

The gyromagnetic ratio is shown in Fig. 4. Again we find the expected asymptotic approach to the nonrelativistic value of 1 for large shell radii. Note that here the value of g may exceed 2. In the limit that the local mass density vanishes ($\nu = 1$), the g factor for the shell approaches 6. The ascent toward the value of 6 rises sharply in the range $1 \leq \nu \leq \frac{4}{3}$, the range where $\Omega_0 \leq 0$.

For the electron and proton, $\xi \gg 1$. It is interesting that for the electron and proton values of charge and mass we obtain

$$g_e = 2.002$$

and

$$g_p = 5.586\ ,$$

if the corresponding shell radii are, respectively,

$$r_e = 2.346 \times 10^{-13}\ \text{cm}$$

and

$$r_p = 7.789 \times 10^{-17}\ \text{cm}\ .$$

VII. CONCLUSIONS

Although charged particles found in nature have $|q|/m \gg 1$, this case is not normally treated in the context of the Einstein-Maxwell equations. This is because in solving such problems difficulties arise. Among these: The large background electric field gives rise to a stress-energy tensor which is linear in electromagnetic perturbations. This gives a gravitational perturbation of the same order as the electromagnetic perturbation. Hence, both perturbations must be determined simultaneously. Such a case was treated in this communication.

When the charge-to-mass ratio of the rotating body is less than one ($|q|/m < 1$) and the shell radius approaches the event horizon, the gyromagnetic ratio g approaches two ($g \to 2$) and the angular velocity of inertial frames Ω approaches that of the shell. Our result is valid for all values of $|q|/m < 1$, finite as well as small. In fact, for $|q|/m = 1$, the same upper limit ($g \to 2$) results. (For $|q|/m \ll 1$, our result is in agreement with previous results.[5])

The region $|q|/m \gg 1$ is of interest, since all charged particles known in nature satisfy this condition. In this region, there is no event horizon and the gyromagnetic ratio g can exceed two. In fact, values of g up to 6 can be obtained from our general relativistic model with charge and mass concentrated together in a single shell ($1 \ll g \ll 6$). Furthermore, for some models the angular velocity of inertial frames within the shell can vanish or even become retrograde.

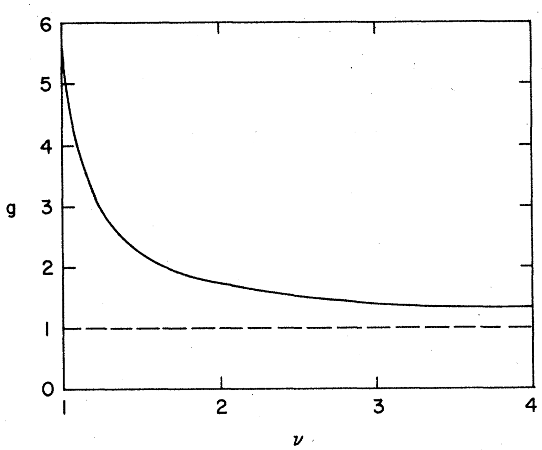

FIG. 4. Gyromagnetic ratio g vs radial parameter $\nu = 2mrq^{-2}$ for large charge-to-mass ratio, $\xi = |q|/m \gg 1$. Note that g can exceed 2.

ACKNOWLEDGMENT

The work of J. M. C. was supported in part by the U. S. National Science Foundation.

APPENDIX: SOLUTION OF EQUATION (13)

Herein we describe briefly how the solution to Eq. (13) may be found:

$$-\frac{1}{2}\left[\frac{V}{\Psi^2}[(r\Psi^2)^2 n]_{,r}\right]_{,r} + nV\Psi^2\left(1 + \frac{2q^2}{(r\Psi^2)^2}\right) = q\mathfrak{M}_0 \frac{V\Psi^2}{(r\Psi^2)^4}\ . \tag{13}$$

Expanding the derivative, one finds

$$\left[\frac{V}{\Psi^2}[(r\Psi^2)^2 n]_{,r}\right]_{,r} = 2nV\Psi^2[1 - q^2/(r\Psi^2)^2] + 2n_{,r}(r + rV^2\Psi^2) + r^2 V\Psi^2 n_{,r,r}\ .$$

Equation 13 becomes

$$r^2 V\Psi^2 n_{,r,r} + 2n_{,r}(r + rV^2\Psi^2) - 6nV\Psi^2 \frac{q^2}{(r\Psi^2)^2} = -2q\mathfrak{M}_0 \frac{V\Psi^2}{(r\Psi^2)^4}\ .$$

Multiplying this equation by $(r\Psi^2)^2$ and noting

$$[r^2 V\Psi^2 (r\Psi^2)^2]_{,r} = 2r(r\Psi^2)^2 + 2r^2 V^2\Psi^4 (r\Psi^2)\ ,$$

we get

$$[r^2 (r\Psi^2)^2 V\Psi^2 n_{,r}]_{,r} - 6nV\Psi^2 q^2 = -2q\mathfrak{M}_0 \frac{V\Psi^2}{(r\Psi^2)^2}\ .$$

The particular solution is given by

$$n_p = \frac{\mathfrak{M}_0 q}{6\alpha (r\Psi^2)^3} = \mathfrak{M}_0 q \beta^3/(6\alpha)\,.$$

In verifying this, observe that

$$(r\Psi^2)_{,r} = V\Psi^2$$

and

$$r^2(V\Psi^2)^2 = (r\Psi^2)^2 - 4\alpha(r\Psi^2) + q^2\,.$$

This leaves the homogeneous equation. To pursue it, make the substitution

$$n \equiv AB\,.$$

Then

$$[r^2(r\Psi^2)^2 V\Psi^2 A]B_{,r,r} + \{2r^2(r\Psi^2)^2 V\Psi^2 A_{,r} + [r^2(r\Psi^2)^2 V\Psi^2]_{,r} A\}B_{,r} + \{[r^2(r\Psi^2)^2 V\Psi^2 A_{,r}]_{,r} - 6AV\Psi^2 q^2\}B = 0\,.$$

Multiplying this equation by A, one obtains

$$[r^2(r\Psi^2)^2 V\Psi^2 A^2 B_{,r}]_{,r} + AB\{[r^2(r\Psi^2)^2 V\Psi^2 A_{,r}]_{,r} - 6AV\Psi^2 q^2\} = 0\,.$$

With the selection $A = 1 - 12\xi^2\alpha^2\beta^2 + 16\xi^4\alpha^3\beta^3$, we reduce the equation to

$$[r^2(r\Psi^2)^2 V\Psi^2 A^2 B_{,r}]_{,r} = 0\,.$$

The first integral is immediate and the second may be done by partial fractions.

---

[1]The Killing vectors associated with stationarity and axial symmetry are $\partial/\partial t$ and $\partial/\partial\varphi$, respectively. The forms dual to these are $-A^2dt - E^2\Omega(d\varphi - \Omega dt)$ and $E^2(d\varphi - \Omega dt)$.

[2]D. Brill and J. M. Cohen, Phys. Rev. 143, 1011 (1966).

[3]J. M. Cohen and D. R. Brill, Nuovo Cimento 56B, 209 (1968).

[4]J. M. Cohen, Phys. Rev. 148, 1264 (1966).

[5]J. M. Cohen, J. Tiomno, and R. Wald, Phys. Rev. D 7, 998 (1973).

[6]See, e.g., L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1971), p. 82.

[7]E. Cartan, *Les Systèmes Différentials Extérieurs* (Hermann, Paris, 1945); see also, e.g., D. Brill and J. Cohen, J. Math. Phys. 7, 238 (1966) and the references cited therein.

[8]H. Flanders, *Differential Forms* (Academic, New York, 1963).

[9]G. de Rham, *Variété's Différentiables* (Hermann, Paris, 1960).

[10]When there is no rotation ($\Omega = 0$) the metric is spherically symmetric. That is, for $\Omega = 0$, Killing's equations are satisfied by $C(\sin\varphi\omega^2 + \cos\varphi\cos\theta\omega^3)$, $C(\cos\varphi\omega^2 - \sin\varphi\cos\theta\omega^3)$, and $C\sin\theta\omega^3$. The vectors dual to these are $C(\sin\varphi\omega_2 + \cos\varphi\cos\theta\omega_3)$, $C(\cos\varphi\omega_2 - \sin\varphi\cos\theta\omega_3)$, and $C\sin\theta\omega_3$.

[11]E. T. Newman, E. Couch, R. Chinnapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. 6, 918 (1965).