Mirror properties of baryon magnetic moments

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A generalization of charge symmetry, called "flavor symmetry," is applied to the constituent-quark model to relate the sum of magnetic moments of mirror pairs of baryons to quark moments. With only the added assumption that the ratio of u- and d-quark moments is -2, the general sum rule, $\mu(\Sigma^+)+\mu(\Xi^0)-\mu(\Sigma^-)-\mu(\Xi^-)=2.640$ nuclear magnetons, is established. Relativistic corrections are estimated to be less than 4%. Larger deviations from the sum rule would indicate electromagnetic annihilation of highly correlated pairs in the quark sea.

The nonrelativistic constituent-quark model has been surprisingly successful[1] in accounting for some of the baryon magnetic moments. For example, it was shown[2] by Bég, Lee, and Pais that the ratio of neutron and proton magnetic moments is obtained with remarkable precision for the non-relativistic SU(6) model when the ratio of up- and down-quark moments is set equal to the electric charge ratio. Also De Rújula, Georgi, and Glashow[3] have pointed out that if the strange-quark moment is reduced by the ratio of quark masses, SU(6) yields excellent agreement with the measured moment[4] of the Λ.

The purpose of this article is to establish some quite general consequences of the nonrelativistic constituent-quark model of baryons, in particular, generalizations of the well-known mirror properties of nuclear magnetic moments.[5] The sum of the magnetic moments of mirror nuclei (i.e., those obtained from one another by exchanging neutrons and protons) depend only on the probability distributions of orbital and spin angular momenta in the approximation that both nuclei are described by the same wave function, that is, on the assumption of charge symmetry of the nuclear Hamiltonian.

A similar result was obtained[6] in 1952 for the sum of neutron and proton moments but, in this case, it was used to determine the distribution of pion angular momenta in the Fock wave function describing the pion field. The constituent-quark model of the nucleon permits a similar conclusion concerning the distribution of angular momenta of quarks. For this model, charge symmetry implies that the three-quark wave function of the nucleon is unchanged when u and d quarks are interchanged. A generalization of charge symmetry, which I shall call "flavor symmetry" applies to the interchange of quark flavors for any pair of baryons comprised of quarks having only two flavors. Among the members of the baryon octet, these include the Σ^+ and Ξ^0 (u and s quarks) and the Σ^- and Ξ^- (d and s quarks), and I shall show that flavor symmetry leads to sum rules for the moments of these pairs.

The assumption of flavor symmetry is implicit in other work[1] on this subject; this assumption states that this sextet of baryons is described by a single three-quark wave function. I shall label operators and variables appearing in the wave function by 1 and 2 for the two like quarks and 3 for the odd quark. Undesignated flavors will be denoted by α, β, γ. Then the magnetic-moment operator for a baryon made up of two quarks of flavor α and one of flavor β is

$$\vec{\mathfrak{M}}^{\alpha\beta}=\mu_\alpha(2\vec{S}+\vec{L})+(\mu_\beta-\mu_\alpha)(\vec{\sigma}_3+\vec{l}_3)\,, \qquad (1)$$

when it is assumed that μ_α, the intrinsic moment of the quark, is the Dirac moment. The total-baryon-spin operator is $\vec{S}$, the total-orbital-angular-momentum operator is $\vec{L}$, and $\vec{\sigma}_3$ and $\vec{l}_3$ are the Pauli spin operator and orbital-angular-momentum operator, respectively, of the odd quark,

The assumption of flavor symmetry means that the magnetic moments $\mu(\alpha^2,\beta)$ or $\mu(\beta^2,\alpha)$ of mirror baryons, including two α and one β quark or two β and one α quark, respectively, are obtained by using the same wave function to calculate the expectation values of $\mathfrak{M}_z^{\alpha\beta}$ or $\mathfrak{M}_z^{\beta\alpha}$. It is implicit in this assumption that the effect on the wave function of differences in quark masses is small.[3] The sum of the moments of mirror baryons is then immediately found to be

$$\mu(\alpha^2,\beta)+\mu(\beta^2,\alpha)=(\mu_\alpha+\mu_\beta)\langle 2S_z+L_z\rangle\,, \qquad (2)$$

where the expectation value is taken with respect to the common wave function.

The wave function of the three-quark system having total angular momentum $\frac{1}{2}$ may consist of a linear combination of ^2S, ^2P, ^4P, and ^4D states. Denoting the real[7] amplitudes of these states by their term symbols, one obtains

$$\mu(\alpha^2,\beta)+\mu(\beta^2,\alpha)$$
$$=(\mu_\alpha+\mu_\beta)[1-\tfrac{2}{3}(^2P)^2+\tfrac{1}{3}(^4P)^2-(^4D)^2], \quad (3)$$

where use has been made of the normalization condition to eliminate the amplitude of the (presumably) dominant ^2S state.[8]

From Eq. (3) the following mirror relations follow immediately:

$$\mu(p)+\mu(n)=(\mu_u+\mu_d)A\,, \quad (4a)$$

$$\mu(\Sigma^+)+\mu(\Xi^0)=(\mu_u+\mu_s)A\,, \quad (4b)$$

$$\mu(\Sigma^-)+\mu(\Xi^-)=(\mu_d+\mu_s)A, \quad (4c)$$

and

$$A=[1-\tfrac{2}{3}(^2P)^2+\tfrac{1}{4}(^4P)^2-(^4D)^2]\,. \quad (5)$$

The ambiguities associated with the different configurations may be eliminated by taking ratios. Thus the two general results

$$\frac{\mu(\Sigma^+)+\mu(\Xi^0)}{\mu(p)+\mu(n)}=\frac{\mu_u+\mu_s}{\mu_u+\mu_d} \quad (6a)$$

and

$$\frac{\mu(\Sigma^-)+\mu(\Xi^-)}{\mu(p)+\mu(n)}=\frac{\mu_d+\mu_s}{\mu_u+\mu_d} \quad (6b)$$

depend only on the assumption that the baryons are states of three quarks having flavor symmetry.

By means of these two equations measured moments of the baryons could be used to calculate the two ratios of quark moments μ_u/μ_d and μ_s/μ_d. It is also possible to eliminate μ_s by combining the two equations to give the sum rule

$$(\mu_u+\mu_d)[\mu(\Sigma^+)+\mu(\Xi^0)-\mu(\Sigma^-)-\mu(\Xi^-)]$$
$$=(\mu_u-\mu_d)[\mu(p)+\mu(n)]\,. \quad (7)$$

The commonly made assumption[2] that μ_u and μ_d are in the ratios of the electric charges of the u and d quarks

$$\mu_u=-2\mu_d\,, \quad (8)$$

converts Eq. (7) into the particularly interesting sum rule

$$\mu(\Sigma^+)+\mu(\Xi^0)-\mu(\Sigma^-)-\mu(\Xi^-)=3[\mu(p)+\mu(n)]\,. \quad (9)$$

It should be noted that, since this sum rule is actually a statement about ratios, it does not contain assumptions about quark masses other than that implicit in Eq. (8).

Insertion of the value

$$\mu(p)+\mu(n)=0.880\ \mu_N \quad (10)$$

(μ_N = nuclear magneton) into Eq. (9) leads to the sum rule for hyperon moments

$$\mu(\Sigma^+)+\mu(\Xi^0)-\mu(\Sigma^-)-\mu(\Xi^-)=2.640\ \mu_N\,. \quad (11)$$

Available data[9] on the hyperon moments do not at this time provide a significant test of this sum rule because of the relatively large estimated errors for most of them. However, it is likely that more precise values will soon be forthcoming.[10] For the same reason, the use of Eq. (6) along with Eq. (8) to obtain a significant measure of μ_s/μ_d must await the new measurements.

The generality of Eqs. (7), (9), and (11), even within the context of the three-quark, flavor-symmetric model is marred by the effect of relativistic corrections. Such corrections may be estimated in a manner similar to that used for the triton magnetic moment.[11] Their relative importance can be expected to be considerably smaller than the relativistic contributions to the Hamiltonian ("hyperfine" splittings), which have been estimated by De Rujula, Georgi, and Glashow[3] (DGG) to be about 20%, because correction terms in the magnetic-moment operators depending only on a single quark will satisfy the mirror relations and will therefore not modify the sum rules, Eq. (7) *et seq*. However, they do lead to corrections in the moments of individual baryons and to modification of the coefficient A, Eq. (5). It follows that only the two-quark terms arising from the insertion of electromagnetic coupling[12] into the DGG "two-body Coulombic interaction" will lead to a change and all such terms couple states of different orbital angular momentum. Therefore their contribution to the moments will be proportional to the (small) amplitudes of the admixed P and D states.

On the assumption that these admixtures are themselves generated by the hyperfine terms, the relativistic corrections to the sum rules may then be estimated to be about 4%. This is consistent with the conclusion of DGG that deviations of quark magnetic moments from the Dirac moment can be expected to arise only in second order of the color coupling.

Within this limitation of about 4%, the extent to which the sum rule, Eq. (11), is violated by measured hyperon moments can be interpreted as a measure of deviations from the constituent three-quark model, that is, as a measure of the quark sea. A relatively small amplitude for the quark sea may lead to rather large effects due to electromagentic annihilation of correlated quark pairs.

These effects may be illustrated by a model in which the valence-quark state $|\Psi_0\rangle$ of the three-quark system is supplemented by a state including additional quark pairs. Thus the baryon state is written as

$$|\Psi\rangle = |\Psi_0\rangle + |\Psi_1\rangle\,, \tag{12}$$

where $|\Psi_1\rangle$ is a state of three valence quarks and an additional quark pair. In its simplest form, $|\Psi_1\rangle$ is a product of a core of valence quarks in the state $|\Psi_0\rangle$ with a pair state. This form is assumed because excitation of the core would cost energy in amounts corresponding to the splitting between SU (6) supermultiplets. States $|\Psi_0\rangle$ and $|\Psi_1\rangle$ may be described as Fock wave functions in terms of a distribution of occupation numbers.[13]

If the Dirac spinor field operator for a quark of flavor f and electric charge q_f is $\Psi_f(\vec{x})$, the magnetic-moment operator is

$$\vec{\mathfrak{M}} = \frac{i}{2}\sum_f q_f \int d^3x (\overline{\psi}_f \vec{\gamma}\psi_f \times \vec{x})\,. \tag{13}$$

The magnetic moment is obtained by taking the expectation value of $\mathfrak{M}_z$ in the state $|\Psi\rangle$. Since $\mathfrak{M}_z$ contains pair-annihilation terms, this expectation value includes a cross term between $|\Psi_0\rangle$ and $|\Psi_1\rangle$ as well as the usual contributions of the spin and orbital angular momenta of the core and sea. It should be remarked incidentally that the kinematical relativistic corrections to these latter contributions are automatically included when the moments are calculated from the operator Eq. (13).

I assume that the total state $|\Psi\rangle$ is flavor-symmetric, that is, each member of the baryon sextet is described by the same[14] $|\Psi\rangle$ with appropriate permutations of flavors. Thus $|\Psi\rangle$ may be expressed in terms of fixed occupation numbers $N_1, N_2, N_3 \cdots$ for each flavor and the flavor assignments among the states of type 1, 2, 3 $\cdots$ permuted. For example, the three-quark core states have $N_1 = 2$, $N_2 = 1$ where, for the proton, type 1 is a u quark, type 2 a d quark, while for the neutron the roles are reversed.

With this understanding, the Dirac spinor field operators are written $\psi_1(\vec{x})$, $\psi_2(\vec{x})$, etc., with the types 1, 2, etc., to be assigned appropriate flavors for each case in turn. Then the magnetic moment for a member of the sextet takes the form

$$\mu(\alpha^2,\beta) = \frac{i}{2}\left\langle \Psi^{\alpha\beta}\right| q_\alpha \int d^3x(\overline{\psi}_1\vec{\gamma}\psi_1 \times \vec{x})_z + q_\beta \int d^3x(\overline{\psi}_2\vec{\gamma}\psi_2 \times \vec{x})_z + q_\gamma \int d^3x(\overline{\psi}_3\vec{\gamma}\psi_3 \times \vec{x})_z + \cdots \left|\Psi^{\alpha\beta}\right\rangle\,. \tag{14}$$

It can easily be shown by means of an expansion of the ψ_i in spinor plane waves that each term in Eq. (14) is inversely proportional to the quark mass in the nonrelativistic limit. Therefore

$$\mu(\alpha^2,\beta) = \langle \Psi | \mu_\alpha O_1 + \mu_\beta O_2 + \mu_\gamma O_3 + \cdots |\Psi\rangle\,, \tag{15}$$

where the O_i are "universal" operators, i.e., independent of flavor in the nonrelativistic limit. The terms in Eq. (15) of the form $\langle\Psi_0|\mu_\alpha O_1 + \mu_\beta O_2|\Psi_0\rangle$ are just those obtained by using Eq. (1) plus the kinematical relativistic corrections. The mirror property, now including the cross terms between $|\Psi_0\rangle$ and $|\Psi_1\rangle$ due to pair annihilation, obtained from Eq. (15) is

$$\mu(\alpha^2,\beta) + \mu(\beta^2,\alpha) = (\mu_\alpha + \mu_\beta)(A + X_1 + X_2) + 2\mu_\gamma X_3\,. \tag{16}$$

It is now assumed that only up-, down-, and strange-quark pairs need be considered. The annihilation terms are given by

$$X_i = 2\langle\Psi_0|O_i|\Psi_1\rangle \tag{17}$$

and the coefficient A by Eq. (5). Contributions proportional to $\langle\Psi_1|\Psi_1\rangle$ have been neglected since the overall probability for the occurrence of the sea is assumed to be small.

The generalization of the important sum rule Eq. (11) is immediate. By direct calculation from Eq. (16) it is found to be

$$\mu(\Sigma^+) + \mu(\Xi^0) - \mu(\Sigma^-) - \mu(\Xi^-) = 3[\mu(p) + \mu(n) - 2(\mu_s - \mu_d)X_3]\,. \tag{18}$$

Thus deviations from Eq. (11) provide a measure of the contribution from annihilation of the "third-quark" pairs to the moment.

Some idea of the magnitude of this deviation may be obtained by assuming that the deviation of $\mu(p)+\mu(n)$ from its canonical value of unity ($\mu_d = -1\mu_N$, $A=1$) is due entirely to the contributions of pair-annihilation terms. If it is also assumed that $X_1 = X_2 = X_3 = X$, Eq. (16) yields

$$\mu(p) + \mu(n) = 1 + 2(\mu_s - \mu_d)X \tag{19}$$

and, from Eq. (10),

$$2(\mu_s - \mu_d)X = -0.12\,. \tag{20}$$

Thus the corresponding deviation of Eq. (18) from the basic sum rule Eq. (11) is 12%. The possibility that configuration mixing and relativistic corrections are partly responsible for the difference $\mu(p)+\mu(n)-1$, makes it appear likely that this is an upper limit on the correction to Eq. (11).

If Eq. (20) is inserted into Eq. (18), the result is the "canonical sum rule"

$$\mu(\Sigma^+) + \mu(\Xi^0) - \mu(\Sigma^-) - \mu(\Xi^-) = 3\mu_N\,, \tag{21}$$

which is just what would be expected on the basis of the simplest imaginable interpretation of the constituent-quark model.

Although the overall probability amplitude for the state $|\Psi_1\rangle$ is expected to be small, it does not place a bound on the X_i so that a large value, as suggested by Eq. (20), is conceivable. The X_i are proportional to $\int d^3k\, K(\vec{k}) f(\vec{k}, -\vec{k})$, where $K(\vec{k})$ is a kinematical factor and $f(\vec{k}_1, \vec{k}_2)$ is the normalized (two-particle) amplitude for a quark pair of momenta $\vec{k}_1$ and $\vec{k}_2$. Large values of the X_i would indicate that quark pairs of equal and opposite momenta are highly correlated or, alternatively, that the pair is closely correlated in configuration space.

Finally, a general sum rule following from Eq. (16) is

$$\mu(p)+\mu(n)+\mu(\Sigma^+)+\mu(\Xi^0)+\mu(\Sigma^-)+\mu(\Xi^-) = 2(A+X_1+X_2+X_3)(\mu_n+\mu_d+\mu_s)\,, \quad (22)$$

which is of interest because of the proportionally to the sum of the three-quark moments, a quantity that vanishes in unbroken SU(3). Thus the notion that SU(3) leads to the vanishing of the sum of the six moments is quite general, including configuration mixing and quark sea contributions.

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[12]See R. G. Sachs, Phys. Rev. 72, 91 (1947), for the corresponding determination of relativistic corrections to the magnetic moment of the deuteron in a gauge theory of strong interactions.

[13]Compare the contributions of the "pion sea" due to pion-pair annihilation as described in Ref. 6. In this case, the annihilation term cancels in the sum of the moments because the charge-symmetry operation interchanges pions of equal and opposite charge.

[14]Although the form of $|\psi\rangle$ is the same, the mass to be associated with a quark of given type will depend on its flavor.