Duality for heavy-quark systems

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We give a proof of the duality relation $\langle\sigma_{\text{bound}}\rangle \simeq \langle\sigma_{\text{free}}\rangle$ for nonrelativistic potential models, using Feynman propagators. There are important and calculable corrections to the duality relation, both for smooth long-range potentials and for singular short-range potentials. We illustrate the corrections for the exactly solvable harmonic-oscillator, linear, and Hulthén potentials.

I. INTRODUCTION

The spectrum of hadrons in the process $e^+e^- \to$ hadrons is characterized in the ψ and Υ regions by a few narrow states and a continuum containing resonance structure. Duality is supposed to relate an appropriate average of this highly structured physical cross section to the smooth cross section for quark pair production, $e^+e^- \to q\bar{q}$, as calculated in perturbative quantum chromodynamics (QCD).[1-3] This idea has been used by a number of authors[2-9] to relate the ψ and Υ data to QCD cross sections, to estimate leptonic widths, and to determine quark masses. Duality is usually implemented by averaging the cross sections using reasonable smearing functions, or by comparing energy moments. For example, the moments method was used recently to obtain values of the heavy-quark masses with very small statistical errors, $m_c = 1.45 \pm 0.05$ GeV (Refs. 8 and 9) and $m_b = 4.58 \pm 0.08$ GeV.[9]

The statistical precision with which the heavy-quark masses can (apparently) be determined made us curious to know how well the duality relations work, and to see if there was a systematic procedure for calculating corrections. We have examined this problem in the context of potential scattering, since potential models provide a very successful description of the ψ and Υ bound states.[10-13] Several authors have previously given JWKB (Refs. 14–16) or Thomas-Fermi[17] derivations of a duality relation between averaged cross sections. This relation has been checked numerically[18,19] and seems to work well when the free and bound $q\bar{q}$ cross sections are averaged locally or smeared with Gaussians, except near the $q\bar{q}$ threshold. However, the accuracy of the moments method had not previously been checked and existing derivations of duality did not provide a method for estimating corrections.

We have constructed a new proof of the duality relations for both nonsingular and singular potentials. In Sec. II we give the background to the duality problem. Our derivation of duality for nonsingular potentials is given in Sec. III A and discussed in Sec. III B. The derivation is based on an argument mentioned briefly by Bell and Pasupathy,[17] and shows clearly the conditions under which the duality relations hold. It also allows us to calculate the corrections to those relations. This is illustrated for the linear and oscillator potentials in Secs. IV A and IV B.

The duality relations must be modified for singular potentials[10,16] or potentials which vary rapidly near the origin, by including the effects of the short-range interaction on the "free" $q\bar{q}$ cross section. This is discussed in Sec. III C and is illustrated for the Coulomb-plus-linear potential of QCD in Sec. IV C. The derivations of the duality relations suggested to us an extension of duality to general transition rates, which we give in Sec. III D.

II. BACKGROUND

The annihilation reaction $e^+e^- \to$ hadrons proceeds through a virtual photon state which couples to a $q\bar{q}$ pair, $e^+e^- \to \gamma \to q\bar{q} \to$ hadrons. At low energies (the ψ and Υ regions for $c\bar{c}$ and $b\bar{b}$, respectively) the $q\bar{q}$ forms a series of bound $^3S_1$ states. Above the threshold for production of mesons containing the heavy quark q, the $q\bar{q}$ states appear as broad resonances which merge at high energies into a smooth continuum. The continuum cross section is well described by the production of a "free" $q\bar{q}$ pair followed by fragmentation of the q and $\bar{q}$ into the observed hadrons.[10]

The "free" cross section for $e^+e^- \to q\bar{q}$ is given relativistically for quarks of mass m_q with three colors by

$$\sigma_{\text{free}} = 6\pi\alpha^2 e_q^2 v(1 - \tfrac{1}{3}v^2)/W^2 , \tag{1}$$

where W is the total energy and $v = (1 - 4m_q^2/W^2)^{1/2}$ is the velocity of either quark in the center-of-mass system. The corresponding nonrelativistic expression is

$$\sigma_{\text{free}} = 6\pi\alpha^2 e_q^2 v |\psi_E(0)|^2/W^2 , \tag{2}$$

where $\psi_E(r)$ is the $q\bar{q}$ wave function for energy $E=W-2m_q$, and $v=(E/m_q)^{1/2}$. With conventional plane-wave normalization, $|\psi_E(0)|^2=1$ for noninteracting particles. We will use the nonrelativistic expressions throughout this paper. These are expected to be valid for the heavy-quark systems in most of the bound-state and resonance region.[10-12]

Single-channel potential models describe the production of hadrons in e^+e^- annihilation as proceeding through the formation of $q\bar{q}$ bound states in a confining potential, followed by the decay of these states. The cross section for $e^+e^-\to$ bound states is given by

$$\sigma_{\text{bound}}=\sum_n 6\pi^2\delta(W-M_n)\Gamma_n(e^+e^-)/W^2 , \tag{3}$$

where $M_n=2m_q+E_n$ is the mass of the nth $q\bar{q}$ bound state, E_n is its nonrelativistic excitation energy, and $\Gamma_n(e^+e^-)$ is its leptonic width. The leptonic width is given in terms of the square of the bound $q\bar{q}$ wave function at the origin by the van Royen–Weisskopf formula,[20,21]

$$\Gamma_n(e^+e^-)=4\pi\alpha^2e_q^2|\psi_n(0)|^2/m_q^2 \tag{4}$$

and

$$\sigma_{\text{bound}}=\sum_n 24\pi^3\alpha^2e_q^2m_q^{-2}|\psi_n(0)|^2\delta(E-E_n)/W^2 . \tag{5}$$

Duality requires that the cross sections (2) and (3) be equal when suitably averaged over energy, that is,

$$\langle v|\psi_E(0)|^2\rangle\sim\left\langle\sum_n(\pi/\alpha^2e_q^2)\Gamma_n(e^+e^-)\delta(E-E_n)\right\rangle=\left\langle\sum_n 4\pi^2|\psi_n(0)|^2m_q^{-2}\delta(E-E_n)\right\rangle . \tag{6}$$

[Note that $3e_q^2\langle v|\psi_E(0)|^2\rangle$ is just the average of the nonrelativistic ratio $R_q=\sigma(e^+e^-\to\text{hadrons})/\sigma(e^+e^-\to\mu^+\mu^-)$ for a single quark flavor.] Our objectives in Sec. III are to establish this relation and a procedure for calculating corrections to it.

III. DERIVATION OF THE DUALITY RELATION

Our derivation of the duality relation uses the Feynman propagator $K(\vec{\mathbf{r}}',\vec{\mathbf{r}},t)$. For bound states,

$$K(\vec{\mathbf{r}}',\vec{\mathbf{r}},t)=\sum_n\psi_n(\vec{\mathbf{r}}')e^{-(i/\hbar)E_nt}\psi_n^*(\vec{\mathbf{r}})$$
$$=e^{-(i/\hbar)H(\vec{\mathbf{r}}')t}\delta(\vec{\mathbf{r}}'-\vec{\mathbf{r}}) , \tag{7}$$

where ψ_n and E_n are the wave functions and energies for the Hamiltonian H. We first show that the energy-averaged cross sections are related by a Fourier transform to the short-time behavior of K. We then relate the short-time propagator K for a confining potential $V(\vec{\mathbf{r}}')$, regular at the origin, to the free propagator K_0. We find that $K=K_0$ plus correction terms which are nonnegligible for the potential used to fit the $c\bar{c}$ and $b\bar{b}$ systems. We discuss the correction terms, then generalize our proof to the case of singular potentials. We end the section with a statement of duality for general transition rates.

A. Nonsingular potentials

The cross sections for e^+e^- annihilation into free or bound $q\bar{q}$ systems are given in Eqs. (2) and (5). These cross sections can be expressed in terms of the Fourier transforms of the free propagator $K_0(\vec{\mathbf{r}}',\vec{\mathbf{r}},t)$ and the propagator $K(\vec{\mathbf{r}}',\vec{\mathbf{r}},t)$ for the confining potential by

$$\sigma_{\text{free}}=12\pi^2\alpha^2e_q^2m_q^{-2}\tilde{K}_0(0,0,E)/W^2 \tag{8}$$

and

$$\sigma_{\text{bound}}=12\pi^2\alpha^2e_q^2m_q^{-2}\tilde{K}(0,0,E)/W^2 , \tag{9}$$

where (with $\hbar$ now inserted explicitly)

$$\tilde{K}_0(\vec{\mathbf{r}}',\vec{\mathbf{r}},E)=\int_{-\infty}^{\infty}dt\ e^{(i/\hbar)Et}K_0(\vec{\mathbf{r}}',\vec{\mathbf{r}},t)$$
$$=\frac{1}{(2\pi\hbar)^2}\int d^3p\ \delta\left(E-\frac{p^2}{m_q}\right)e^{(i/\hbar)\vec{\mathbf{p}}\cdot(\vec{\mathbf{r}}'-\vec{\mathbf{r}})}$$
$$=(m_q^2v/2\pi\hbar^2)e^{(i/\hbar)\vec{\mathbf{p}}\cdot(\vec{\mathbf{r}}'-\vec{\mathbf{r}})} \tag{10}$$

and

$$\tilde{K}(\vec{\mathbf{r}}',\vec{\mathbf{r}},E)=\int_{-\infty}^{\infty}dt\ e^{(i/\hbar)Et}K(\vec{\mathbf{r}}',\vec{\mathbf{r}},t)$$
$$=2\pi\hbar\sum_n\delta(E-E_n)\psi_n(\vec{\mathbf{r}}')\psi_n^*(\vec{\mathbf{r}}) . \tag{11}$$

Since duality relates the energy averages $\langle\sigma_{\text{bound}}\rangle$ to $\langle\sigma_{\text{free}}\rangle$, we wish to relate appropriate energy averages of $\tilde{K}$ and $\tilde{K}_0$.

We average $\tilde{K}$ by convoluting with a smooth function $f(E'-E)$, and define

$$\langle\tilde{K}(E)\rangle=\int_{-\infty}^{\infty}dE'\ f(E'-E)\tilde{K}(0,0,E') . \tag{12}$$

From Eq. (11), we see that $\langle\tilde{K}(E)\rangle$ is given by

$$\langle\tilde{K}(E)\rangle=\int_{-\infty}^{\infty}dt\ e^{(i/\hbar)Et}K(0,0,t)$$
$$\times\int_{-\infty}^{\infty}dE'f(E'-E)e^{(i/\hbar)(E'-E)t}$$
$$=\int_{-\infty}^{\infty}dt\ e^{(i/\hbar)Et}K(0,0,t)\tilde{f}(t/\hbar) . \tag{13}$$

If $f(E'-E)$ is a broad, smooth function with a width Δ, then we expect $\tilde{f}(t/\hbar)$ to be a narrow function with a width $t\sim\hbar/\Delta$, and we will only need the short-time behavior of the propagators to calculate the energy averages. The free propagator $K_0(0,0,t)$ for a $q\bar{q}$ system with a reduced mass $m_q/2$ is

$$K_0(0,0,t)=\left(\frac{m_q}{4\pi\hbar(it+\epsilon)}\right)^{3/2} . \tag{14}$$

Our strategy is to relate K to K_0 for short times.

From Eq. (7) we express the propagator as

$$K(\vec{r}',\vec{r},t)=\frac{1}{(2\pi\hbar)^3}\int d^3p\; e^{-(i/\hbar)\vec{p}\cdot\vec{r}}\, e^{-(it/\hbar)H(\vec{r}')} \times e^{(i/\hbar)\vec{p}\cdot\vec{r}'} , \tag{15}$$

where

$$H(\vec{r}')=-\frac{\hbar^2}{m_q}\vec{\nabla}'^2+V(\vec{r}')=H_0+V(\vec{r}') . \tag{16}$$

We assume that $V(\vec{r}')$ is spherically symmetric and can be expanded in a Taylor series near $\vec{r}'=0$, a restriction we will remove later, and choose the energy scale so that $V(0)=0$. This assumption allows us to expand the factor $e^{-(it/\hbar)H(\vec{r}')}\, e^{(i/\hbar)\vec{p}\cdot\vec{r}'}$ in Eq. (15) as

$$e^{-(i/\hbar)Ht}e^{(i/\hbar)\vec{p}\cdot\vec{r}'}=e^{-(i/\hbar)(E+V)t}\left[e^{(i/\hbar)Vt}e^{-(i/\hbar)(H_0+V)t}e^{(i/\hbar)H_0t}\right]e^{(i/\hbar)\vec{p}\cdot\vec{r}'}$$

$$=e^{-(i/\hbar)(E+V)t}\left\{1-\tfrac{1}{2}(t/\hbar)^2[H_0,V]+\tfrac{1}{6}i(t/\hbar)^3([H_0,V],V]+[H_0,[H_0,V]])+\cdots\right\}e^{(i/\hbar)\vec{p}\cdot\vec{r}'} , \tag{17}$$

where $E=p^2/m_q$. The coefficient functions in the power series in $t/\hbar$ all involve derivatives of $V(r')$. When this series is substituted into Eq. (14), the first term gives the free propagator $K_0(\vec{r}',\vec{r},t)$ multiplied by $e^{-(i/\hbar)V(r')}$. We group the rest of the terms by powers of $t/\hbar\sim 1/\Delta$, set $\vec{r}'$ and $\vec{r}$ equal to zero, and find that to order $(t/\hbar)^3$, with $V(0)=0$,

$$K(0,0,t)\simeq K_0(0,0,t)\left\{1+\frac{1}{6}\left(\frac{t}{\hbar}\right)^2\frac{\hbar^2}{m_q}\nabla^2V(0)-\frac{i}{12}\left(\frac{t}{\hbar}\right)^3\frac{\hbar^2}{m_q}[\nabla V(0)]^2+\frac{i}{60}\left(\frac{t}{\hbar}\right)^3\frac{\hbar^4}{m_q^2}\nabla^2\nabla^2V(0)+O\left(\left(\frac{t}{\hbar}\right)^4\right)\right\}. \tag{18}$$

We have therefore shown that for suitable potentials $V(r)$ and for sufficiently small times t,

$$K(0,0,t)\simeq K_0(0,0,t) , \tag{19}$$

which implies from Eq. (13) that

$$\langle\tilde{K}(E)\rangle\simeq\langle\tilde{K}_0(E)\rangle . \tag{20}$$

This gives the simple duality relation

$$\langle\sigma_{\text{bound}}\rangle\simeq\langle\sigma_{\text{free}}\rangle \tag{21}$$

from Eqs. (8) and (9).

B. Discussion and interpretation

Assuming that the derivatives of $V(r)$ are finite at the origin, we can estimate the corrections to $\langle\tilde{K}(E)\rangle$,

$$\langle\tilde{K}(E)\rangle\simeq\int_{-\infty}^{\infty}dt\; e^{(i/\hbar)Et}K(0,0,t)\tilde{f}(t/\hbar)$$

$$=\langle\tilde{K}_0(E)\rangle+\int_{-\infty}^{\infty}dt\; e^{(i/\hbar)Et}K_0(0,0,t)\tilde{f}(t/\hbar)\left\{\frac{1}{6}\left(\frac{t}{\hbar}\right)^2\frac{\hbar^2}{m_q}\nabla^2V(0)-\frac{i}{12}\left(\frac{t}{\hbar}\right)^3\frac{\hbar^2}{m_q}[\nabla V(0)]^2\right.$$

$$\left.+\frac{i}{6}\left(\frac{t}{\hbar}\right)^3\frac{\hbar^4}{m_q^2}\nabla^2\nabla^2V(0)+\theta\left(\left(\frac{t}{\hbar}\right)^4\right)\right\} . \tag{22}$$

Upon integration each factor of $t/\hbar$ is replaced by $\sim 1/\Delta$, and the corrections can be made arbitrarily small by choosing $f(E'-E)$ broad enough [$\tilde{f}(t/\hbar)$ narrow enough], e.g., a Gaussian. For example, for a harmonic oscillator, $V(r)=\frac{1}{4}m_q\omega^2r^2$, only the $\nabla^2V(0)$ term is nonzero, and the correction is of order $(\hbar\omega/\Delta)^2$ which is small for Δ large compared with the oscillator energy spacing.

For a potential which varies as r^k with k large, all low-order correction terms vanish. (The larger the power of r, the more closely the potential approximates a square well. This explains the success of the duality relations for square wells noted by Novikov *et al.*[10]) The

physical interpretation of this result is that a $q\bar{q}$ pair produced as a localized wave packet at $r=0$ does not "feel" the confining potential, and moves nearly freely over some range of r. This range, and therefore the time during which a $q\bar{q}$ bound wave packet acts like a free $q\bar{q}$ system, is determined by the potential. The flatter the potential near $r=0$, the better the uncorrected duality relation holds.

For $k<2$, notably for the linear potential, the method of Eq. (22) does not give a useful estimate of the correction terms, since $\nabla^2 V(r)$ diverges as $r\to 0$. Even for large k, the series in Eqs. (18) and (22) are only asymptotic, and we can only calculate a finite number of correction terms.

More physically, for k small, a localized low-energy wave packet is affected by the confining potential even near $r=0$. This changes the duality relations near the threshold for free $q\bar{q}$ production. (The importance of the potential relative to the kinetic energy for k small is shown for the bound states by the virial theorem, $\langle V\rangle=(2/k)\langle T\rangle$ for $V\propto r^k$.) At energies well above threshold, the $q\bar{q}$ wave function is insensitive to the potential for r small, and the usual relations hold.

In these cases, it is necessary either to evaluate $K(0,0,t)$ more carefully for small t, e.g., as a Feynman path integral, to use a different method to compare $\langle\tilde{K}(E)\rangle$ with $\langle\tilde{K}_0(E)\rangle$, e.g., the Euler-Maclaurin summation formula,[22] or to treat the short-range part of the potential separately.

We will not discuss the path-integral method here except to observe that the correction terms in Eq. (22) can be identified with terms in the path integral. For example, the $\nabla^2 V(0)$ term results from the change in normalization of $K(0,0,t)$ due to fluctuations about the classical path for a quadratic potential. The $[V'(0)]^2$ term is of order $1/\hbar$ and is exactly the classical action for a linear potential. This term results from expansion of the semiclassical approximation to the path integral, $K\sim e^{(i/\hbar)S_{cl}}$.

The Euler-Maclaurin summation formula[22] gives an alternate way of comparing the averages of $\tilde{K}_0(0,0,E)$ with $\tilde{K}(0,0,E)$ [given by Eqs. (10) and (11)],

$$\langle\tilde{K}_0(E)\rangle=\frac{m_q^{3/2}}{2\pi\hbar^2}\int_0^\infty f(E'-E)\sqrt{E'}\,dE'\ , \tag{23}$$

$$\langle\tilde{K}(E)\rangle=2\pi\hbar\sum_n f(E_n-E)\,|\psi_n(0)|^2\ . \tag{24}$$

We use the Euler-Maclaurin formula to convert the sum in Eq. (24) to an integral which can be compared with Eq. (23), plus a remainder. We will illustrate this technique in Sec. IV for the linear potential.

C. Short-range corrections

If $V(r)$ is not analytic or is singular at $\vec{r}=0$, the expansion used to obtain Eq. (22) fails. However, we can get a duality relation by splitting the potential into a short-range nonconfining part V_S and a long-range confining part V_L, analytic at $\vec{r}=0$. We replace $K_0(\vec{r}',\vec{r},t)$ by a propagator $K_S(\vec{r}',\vec{r},t)$ constructed from the exact free solutions ψ_E to the Schrödinger equation for the Hamiltonian

$$H_S(\vec{r}')=-\frac{\hbar^2}{m_q}\nabla'^2+V_S(r')\ , \tag{25}$$

$$K_S(\vec{r}',\vec{r},t)=(2\pi\hbar)^{-3}\int d^3p\ \psi_E(\vec{r}')e^{-(i/\hbar)Et}\psi_E^*(\vec{r}), \tag{26}$$

where $E=p^2/m_q$. If there are bound states in the potential V_S, their contribution must be added to Eq. (26); see, e.g., Eq. (7).

The propagator for the full Hamiltonian $H(\vec{r}')=H_S(\vec{r}')+V_L(r')$ can be expressed in terms of the functions ψ_E as

$$K(\vec{r}',\vec{r},t)=\frac{1}{(2\pi\hbar)^3}\int d^3p[e^{-(i/\hbar)H(\vec{r}')t}\psi_E(\vec{r}')]\psi_E(\vec{r})\,. \tag{27}$$

It is clear that $K(\vec{r}',\vec{r},t)$ can be approximated by $K_S(\vec{r}',\vec{r},t)$ for t small. We can obtain the corrections for finite t by expanding the exponential operator using the identity

$$e^{(i/\hbar)(H_S+V_L)t}\psi_E=e^{(i/\hbar)(E+V_L)t}[e^{(i/\hbar)V_Lt}e^{-(i/\hbar)(H_S+V_L)t}e^{(i/\hbar)H_St}]\psi_E$$
$$=e^{(i/\hbar)(E+V_L)t}\left\{1-\frac{1}{2}\left(\frac{t}{\hbar}\right)^2[H_S,V_L]+\cdots\right\}\psi_E\ . \tag{28}$$

When this expansion is substituted in Eq. (27), we find that

$$K(\vec{r}',\vec{r},t)=e^{-(i/\hbar)V_L(r')t}\left\{1-\frac{1}{2}\left(\frac{t}{\hbar}\right)^2[H_S(\vec{r}'),V_L(\vec{r}')]+\cdots\right\}K_S(\vec{r}',\vec{r},t)\ . \tag{29}$$

This equation expresses $K(\vec{r}',\vec{r},t)$ in terms of $K_S(\vec{r}',\vec{r},t)$ and a series of corrections which depend on derivatives of V_L. When this expression is substituted in Eq. (13), we obtain a modified duality relation, $\langle\tilde{K}(E)\rangle\simeq\langle K_S[E-V_L(0)]\rangle$ plus corrections, which reduces to Eq. (22) for $V_S=0$. [Note that $V_L(0)$ need not vanish.]

D. Duality for general transition rates

An interesting corollary to our proof of duality for e^+e^- cross sections is a more general duality relation for transition rates involving any short-range operator $A(\vec{r})e^{-iEt}(\hbar=1)$. The transition rate Γ from an initial state ψ_0 to the bound state ψ_n is

$$\Gamma(E)=2\pi\sum_n |\langle\psi_n|A|\psi_0\rangle|^2\delta(E-E_n)\ . \tag{30}$$

It is easy to show that Γ can be expressed in terms of the Fourier transform of the propagator $K(\vec{r}',\vec{r},E)$ given in Eq. (11),

$$\Gamma(E)=\int d^3r'\int d^3r\ \psi_0^*(\vec{r}')A^\dagger(\vec{r}')\tilde{K}(\vec{r}',\vec{r},E)A(\vec{r})\psi_0(\vec{r}). \tag{31}$$

We define an energy-averaged transition rate

$$\langle\Gamma(E)\rangle=\int dE'\ f(E'-E)\Gamma(E')\ . \tag{32}$$

The technique we used to derive Eq. (18) for $K(0,0,t)$ can be used to express $K(\vec{r}',\vec{r},t)$ in terms of $K_0(\vec{r}',\vec{r},t)$ for $\vec{r}'$, $\vec{r}$, and t all small. Then for narrow functions $\tilde{f}(t)$, $\langle\tilde{K}(\vec{r}',\vec{r},E)\rangle\simeq\langle\tilde{K}_0(\vec{r}',\vec{r},E)\rangle$, and it follows that

$$\langle\Gamma(E)\rangle\simeq\langle\Gamma_0(E)\rangle \tag{33}$$

for short-range operators $A(\vec{r})$, where $\Gamma_0(E)$ is the decay rate to free states. Again, the correction terms must be estimated as discussed above.

IV. APPLICATIONS

As noted earlier, duality is used to compare QCD cross sections with experimental data, to determine quark masses, and to predict leptonic widths for $q\bar{q}$ resonances. Two methods are used. In one method, $f(E'-E)$ is chosen to be a smooth, broad function (e.g., a Gaussian or Lorentzian curve) and the smeared data are compared to QCD cross sections.[2,7] In the other method, several moments of the experimental and QCD cross sections, usually restricted to some finite-energy range, are compared.[5,8,9]

In this section we discuss the accuracy of these methods using the results of Sec. III applied to the (exactly solvable) linear and quadratic potentials. We conclude that the uncorrected moments method is not reliable for high moments, either positive or negative. On the other hand, Gaussian smearing works well, except near the $q\bar{q}$ threshold, provided the width of the Gaussian is comparable to the spacing between resonances. Both methods work very well when corrections are included. We conclude the section with an example of short-range corrections, using the exactly solvable Hulthén potential. The results are applicable to the Coulomb-plus-linear potential.

A. Corrected Gaussian smearing

Our first illustration of the duality relations is for the harmonic-oscillator problem with $V(r)=\frac{1}{4}m_q\omega^2r^2$ (the reduced mass of the $q\bar{q}$ system is $m_q/2$). We use a Gaussian for the smearing function $f(E'-E)$, so $\tilde{f}(t/\hbar)$ is also Gaussian,

$$f(E'-E)=\frac{1}{(2\pi\Delta^2)^{1/2}}\,e^{-(E'-E)^2/2\Delta^2}\ ,$$

$$\tilde{f}(t/\hbar)=e^{-\Delta^2t^2/2\hbar^2}\ . \tag{34}$$

In this case, Eqs. (10), (11), and (22) give the results

$$\langle\tilde{K}(E)\rangle_{\text{exact}}=2\pi\hbar\sum_n |\psi_n(0)|^2e^{-(E-E_n)^2/2\Delta^2}\ , \tag{35}$$

$$\langle\tilde{K}(E)\rangle_{\text{approx}}=\int_{-\infty}^{\infty}dt\ e^{(i/\hbar)Et-\Delta^2t^2/2\hbar^2}\left[\frac{m_q}{4\pi\hbar(it+\epsilon)}\right]^{3/2}\left[1+\frac{1}{4}\left(\frac{t}{\hbar}\right)^2(\hbar\omega)^2+\ldots\right]$$

$$=\frac{m_q{}^{3/2}}{2\pi\hbar^2}\int_0^\infty dE'\sqrt{E'}\,e^{-(E-E')^2/2\Delta^2}\left\{1+\frac{1}{4}\left(\frac{\hbar\omega}{\Delta}\right)^2\left[1-\frac{(E'-E)^2}{\Delta^2}\right]+\ldots\right\}, \tag{36}$$

and

$$\langle\tilde{K}_0(E)\rangle=\frac{m_q{}^{3/2}}{2\pi\hbar^2}\int_0^\infty dE'\,\sqrt{E'}\,e^{-(E-E')^2/2\Delta^2}\ . \tag{37}$$

In evaluating the expression for $\tilde{K}(E)_{\text{approx}}$ we have used the observation that the corrections in powers of $(t/\hbar)$ can be written in terms of derivatives of the exponential $iEt/\hbar$ with respect to E.

We give numerical results for the harmonic-oscillator duality relation in Table I. We have used $m_q=m_c=1.45$ GeV and a potential matched to the spacing of the ψ and ψ' particles, $E_2-E_1=E_{\psi'}-E_\psi$ or $2\hbar\omega=0.6$ GeV. We used $\Delta=0.6$ GeV, which corresponds to a full width at half-maximum for the Gaussian of 1.4 GeV. For the oscillator

$$|\psi_n(0)|^2=\left(\frac{m_q\omega}{2\hbar}\right)^{3/2}\frac{2}{\pi^2}\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}\ , \tag{38}$$

and $E_n=(2n-\frac{1}{2})\hbar\omega$. The results are excellent, even uncorrected, above the free-quark threshold $E=0$. Below threshold, the correction term becomes important and the corrected $\langle\tilde{K}(E)\rangle_{\text{approx}}$ agrees very well with $\langle\tilde{K}(E)\rangle_{\text{exact}}$.

Our second illustration involves the linear potential, $V(r)=r/a^2$. In this case Eq. (22) fails and we have investigated the duality relations for the smeared cross sections by using the Euler-Maclaurin formula[22] to convert the sum in Eq. (35) to an integral. In this case

$$|\psi_n(0)|^2=\frac{m_q}{4\pi a^2\hbar^2} \tag{39}$$

for all n. We have used the JWKB formula for the energies E_n,

$$E_n=\left(\frac{9\pi^2\hbar^2}{4m_q a^4}\right)^{1/3}(n-1/4)^{2/3}[1+O(n^{-2})]\ , \tag{40}$$

which is accurate to 0.8% for $n=1$ and is better for higher n. The result is

$$\langle\tilde{K}(E)\rangle_{\text{approx}}\simeq\frac{m_q{}^{3/2}}{2\pi\hbar^2}\int_0^\infty dE'\sqrt{E'}\,e^{-(E-E')^2/\Delta^2}$$
$$-\frac{m_q{}^{3/2}}{2\pi\hbar^2}\int_0^{E_1}dE'\sqrt{E'}\,e^{-(E-E')^2/2\Delta^2}$$
$$+\frac{m_q}{4a^2\hbar}\,e^{-(E-E_1)^2/2\Delta^2}+\cdots\ . \tag{41}$$

The first term in this equation is just $\langle\tilde{K}_0(E)\rangle$. The presence of this term is guaranteed by the fact that $K(0,0,t)\simeq K_0(0,0,t)$ for t sufficiently small, independently of the potential. The second term corrects for the fact that the first resonance for the bound system is above the threshold for production of the free system. The third term is the first Euler-Maclaurin correction. All higher corrections involve derivatives of the summand at $n=1$, i.e., at $E'=E_1$.

We give numerical results for the linear potential in Table II. We have chosen the parameter $a=2.06$ GeV^{-1} to fit the ψ and ψ' spacing, $E_2=E_1=0.6$ GeV with $m_q=1.45$ GeV, and $\Delta=0.6$ GeV. The uncorrected duality relation is inaccurate by 7% at the first resonance and is much worse below that. At threshold the error is 55%. The Euler-Maclaurin corrections bring $\langle\tilde{K}(E)\rangle_{\text{approx}}$ into agreement with $\langle\tilde{K}(E)\rangle_{\text{exact}}$ to better than 1.5%. The corrections are obviously important in this case.

Since the commonly used confining potentials which best fit the ψ and Υ data are closer to linear than quadratic, correction terms should always be included in duality calculations. The threshold region is especially sensitive to corrections. One can always make these corrections using the Euler-Maclaurin formula. The corrections depend primarily on σ_{free} (assumed to be known), and on the energy and cross section for the first resonance. The higher corrections in the Euler-Maclaurin series involve derivatives which can be estimated by using data for more than one resonance.

B. Moments and finite-energy sum rules

The method of moments or finite-energy sum rules corresponds to the choice of smearing

TABLE I. Duality relation for the three-dimensional harmonic-oscillator potential using Gaussian smearing. The spacing $2\hbar\omega$ of the oscillator levels $E_n=(2n-\frac{1}{2})\hbar\omega$ was chosen to match E_2-E_1 to $E_{\psi'}-E_\psi$, $2\hbar\omega=0.6$ GeV. We used $\Delta=0.6$ GeV and $m_q=m_c=1.45$ GeV. $\langle\tilde{K}(E)\rangle_{\text{exact}}$ and $\langle\tilde{K}(E)\rangle_{\text{approx}}$ were calculated using Eqs. (35) and (38), and Eq. (36), respectively.

	E (GeV)	$\langle\tilde{K}_0(E)\rangle$ (GeV^3)	Correction [Eq. (36)]	$\langle\tilde{K}(E)\rangle_{\text{approx}}$ (GeV^3)	$\langle\tilde{K}(E)\rangle_{\text{exact}}$ (GeV^3)
	−0.4	0.0574	−0.0054	0.0520	0.0519
	0	0.1329	−0.0042	0.1287	0.1286
E_1 →	0.4	0.2365	−0.0005	0.2360	0.2363
E_2 →	0.8	0.3446	+0.0021	0.3467	0.3470
	1.2	0.4390	0.0024	0.4414	0.4416
E_3 →	1.6	0.5173	0.0017	0.5190	0.5191
E_4 →	2.0	0.5836	0.0011	0.5847	0.5847
	2.4	0.6418	0.0009	0.6427	0.6428

TABLE II. Duality relations for the three-dimensional linear potential, calculated using Gaussian smearing and the Euler-Maclaurin method. The parameter a in the potential $V = r/a^2$ was chosen to match $E_2 - E_1$ to $E_{\psi'} - E_{\psi}$, $a = 2.06$ GeV^{-1}, $E_2 - E_1 = 0.59$ GeV. We used $\Delta = 0.6$ GeV and $m_q = m_c = 1.45$ GeV. $\langle \tilde{K}(E) \rangle_{\text{approx}}$ was calculated using Eq. (41) and $\langle \tilde{K}(E) \rangle_{\text{exact}}$ using Eqs. (35), (39), and (40).

	E (GeV)	$\langle \tilde{K}_0(E) \rangle$ (GeV^3)	Correction [Eq. (41)]	$\langle \tilde{K}(E) \rangle_{\text{approx}}$ (GeV^3)	$\langle \tilde{K}(E) \rangle_{\text{exact}}$ (GeV^3)
	−0.4	0.0574	−0.0314	0.0260	0.0264
	0	0.1329	−0.0482	0.0847	0.0860
	0.4	0.2365	−0.0441	0.1924	0.1942
E_1 →	0.8	0.3446	−0.0220	0.3226	0.3235
	1.2	0.4390	−0.0042	0.4348	0.4346
E_2 →	1.6	0.5173	+0.0008	0.5165	0.5180
E_3 →	2.0	0.5836	0.0004	0.5832	0.5844
E_4 →	2.4	0.6418	0.0000	0.6418	0.6425
E_5 →					

function

$$f(E'-E) = (E'-E)^k \theta((E_{\max}-E) - (E'-E)) . \quad (42)$$

E is given a value well below both the free and bound spectra, and k ranges over both positive and negative integers. $E_{\max}$ is usually taken to be above the resonance region. With this choice of f (and $\hbar = 1$),

$$\langle \tilde{K}(E) \rangle_{\text{exact}} = 2\pi \sum_{n=1}^{N} |\psi_n(0)|^2 (E_n - E)^k \quad (43)$$

and

$$\langle \tilde{K}_0(E) \rangle = \frac{m_q^{3/2}}{2\pi} \int_0^{E_{\max}} \sqrt{E'} (E'-E)^k dE' . \quad (44)$$

For $E = -2m_q$ and $k = -l$, the approximate relation $\langle \tilde{K}(E) \rangle \simeq \langle \tilde{K}_0(E) \rangle$ and Eq. (4) give the sum rule for leptonic widths suggested by several authors[1,3] and derived in the JWKB approximation by Quigg and Rosner,[11]

$$\int_0^{E_{\max}} \frac{\sqrt{E'}\, dE'}{(E' + 2m_q)^l} = \sum_{n=1}^{N} \frac{\pi m_q^{1/2}}{\alpha^2 e_q^2} \frac{\Gamma_n(e^+e^-)}{M_n^l} . \quad (45)$$

This sum rule has been used to determine the quark mass in terms of resonance masses M_n and leptonic widths Γ_n.[8,9]

The accuracy of the moments method has apparently not been checked previously for potential models. We have compared the results in Eqs. (43) and (44) (the uncorrected duality relation) for the linear potential $V(r) = r/a^2$ with $a = 2.06$ GeV^{-1}, $m_q = 1.45$ GeV, $E_{\max} = 2.77$ GeV, and $E = 2.31$ GeV (this choice of E gives the correct ψ and ψ' masses). The value of $E_{\max}$ was chosen somewhat above the fifth bound state, and adjusted so that the $k = 0$ sum rule is essentially exact. The results of the calculation are shown in Table III.

TABLE III. Duality relations for the three-dimensional linear potential using the moments defined in Eqs. (43) and (44) with $N = 5$, $E_{\max} = 2.77$ GeV, and $E = -2.31$ GeV. The remaining parameters are given in Table II. The end-point corrections change the interval of integration $0 \leq E' \leq E_{\max}$ which appears in the definition of $\langle \tilde{K}_0(E) \rangle$, Eq. (44), to the range $E_1 \leq E' \leq E_N$ which occurs in $\langle \tilde{K}(E) \rangle_{\text{approx}}$, Eq. (46). We have included only the first two Euler-Maclaurin corrections. These are given in Eq. (46).

k	$M_1^{-k} \langle \tilde{K}_0(E) \rangle$ (GeV^3)	End-point corrections	Euler-Maclaurin corrections	$M_1^{-k} \langle \tilde{K}(E) \rangle_{\text{approx}}$ (GeV^3)	$M_1^{-k} \langle \tilde{K}(E) \rangle_{\text{exact}}$ (GeV^3)
−10	0.626	−0.526	0.118	0.218	0.214
− 8	0.508	−0.379	0.113	0.242	0.240
− 6	0.458	−0.281	0.109	0.286	0.285
− 4	0.474	−0.217	0.110	0.367	0.368
− 2	0.583	−0.179	0.124	0.528	0.528
0	0.852	−0.171	0.171	0.852	0.853
2	1.447	−0.216	0.305	1.536	1.522
4	2.748	−0.381	0.692	3.059	3.045
6	5.639	−0.849	1.662	6.452	6.457
8	12.203	−2.111	4.293	14.385	14.392
10	27.403	−5.462	11.193	33.133	33.222

The uncorrected sum rule for leptonic widths, Eq. (45), requires that $\langle K_0\rangle = \langle \tilde{K}\rangle_{\text{exact}}$. This equality fails spectacularly for the high moments: $\langle K_0\rangle$ and $\langle K\rangle_{\text{exact}}$ differ by a factor of 3 for $k=-10$ and by more than 20% for $k=+10$. The corrections are clearly important.

We can derive an approximate expression for $\langle \tilde{K}(E)\rangle$ in terms of $\langle \tilde{K}_0(E)\rangle$ by using the Euler-Maclaurin formula and the results for $|\psi_n(0)|^2$ and E_n in Eqs. (39) and (40) to convert the sum in Eq. (43) into an integral,

$$\langle \tilde{K}(E)\rangle_{\text{approx}} = \frac{m_q^{3/2}}{2\pi}\int_{E_1}^{E_N}\sqrt{E'}(E'-E)^k dE' + \frac{1}{4}\frac{m_q}{a^2}\left[(E_1-E)^k + (E_N-E)^k\right] - \frac{1}{27}\frac{m_q}{a^2}kE_1\left[(E_1-E)^{k-1} - \left(\frac{E_1}{E_N}\right)^{1/2}(E_N-E)^{k-1}\right] + \cdots . \tag{46}$$

Except for the range of integration, the integral above is just $\langle \tilde{K}_0(E)\rangle$, Eq. (44). Thus, $\langle \tilde{K}(E)\rangle_{\text{approx}}$ is equal to $\langle K_0(E)\rangle$ plus the end-point corrections necessary to change the range of integration from $0 \le E' \le E_{\max}$ to $E_1 \le E' \le E_N$, plus the first two terms in the Euler-Maclaurin series.

We have calculated the moments $\langle \tilde{K}(E)\rangle_{\text{approx}}$ using Eq. (46) for $|k| \le 10$. The approximate results given in Table III agree with the exact moments to better than 1% for all values of k except -10.

We can understand the breakdown of the simple duality relation $\langle \tilde{K}(E)\rangle \simeq \langle \tilde{K}_0(E)\rangle$ for moments if we note that the function $f(E'-E) \sim (E'-E)^k$ varies rapidly for $|k|$ large. Our proof of the duality relation required a smooth, broad function $f(E'-E)$, with a sharply peaked Fourier transform $\tilde{f}(t/\hbar)$. This condition is clearly violated.

We can get further insight into the growth of the error with $|k|$ by considering the $k=0$ calculation. $\langle \tilde{K}_0(E)\rangle$ and $\langle \tilde{K}(E)\rangle_{\text{exact}}$ are asymptotically equal for N and $E_{\max}$ very large [see Eqs. (43) and (44)]:

$$2\pi\sum_{n=1}^{N}|\psi_n(0)|^2 \sim \frac{m_q^{3/2}}{2\pi}\int_0^{E_{\max}}\sqrt{E}\,dE . \tag{47}$$

This corresponds to $f(E'-E)$ becoming very broad, that is, to the very short-time behavior of the propagators. We can make Eq. (47) be an exact equality by choosing a set of intervals $E'_{n-1} \le E \le E'_n$ such that[23]

$$2\pi|\psi_n(0)|^2 = \frac{m_q^{3/2}}{2\pi}\int_{E'_{n-1}}^{E'_n}\sqrt{E}\,dE = \frac{m_q^{3/2}}{3\pi}\left(E_n'^{3/2} - E_{n-1}'^{3/2}\right) . \tag{48}$$

It is usually assumed that each resonance can be associated with such an interval ("local duality") and furthermore that the resonance energy E_n lies in the middle of the interval. The second assumption is generally incorrect. The relation of E_n to E'_n is shown in Figs. 1(a) and 1(b) for the linear and harmonic-oscillator potentials.

For the linear potential the E_n's are very close to the E'_n's. For large n, it is easy to show that

$$E_n = E'_n - \tfrac{1}{4}(E'_n - E'_{n-1}) . \tag{49}$$

The displacement of E_n from the center of the interval leads to systematic discrepancies when sums over the resonances are compared with integrals with weighting functions which vary

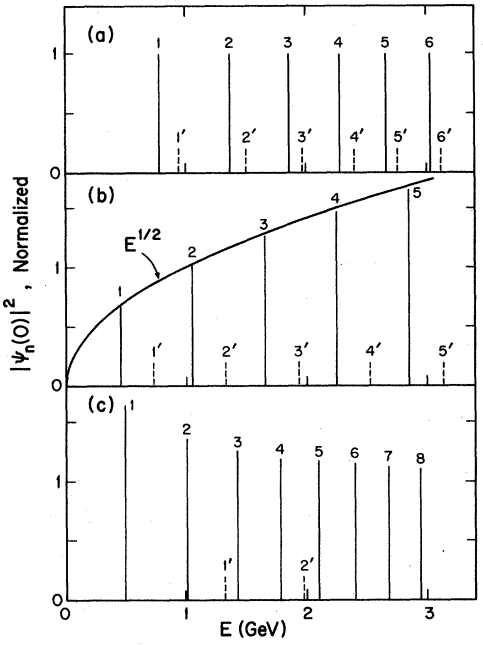

FIG. 1. Comparison of energy eigenvalues, $|\psi_n(0)|^2$, and "duality intervals" for (a) the linear potential $V=r/a^2$ with $a=2.06$ GeV^{-1}, $m_q=1.45$ GeV; (b) the oscillator potential with $\omega=0.3$ GeV, $m_q=1.45$ GeV; (c) the linear-plus-Coulomb potential of Bell and Pasupathy, Ref. 16. $|\psi_n(0)|^2$ is normalized in all cases to the value $m_q/4\pi a^2$ for the linear potential. The curve labeled $E^{1/2}$ in (b) gives the corresponding result for the free $q\bar{q}$ system. The duality intervals E'_n, E'_{n-1} are defined so that the integral of $E^{1/2}$ over the interval gives $|\psi_n(0)|^2$ [see Eq. (47)]. Note in case (c) that $E'_1 > E_2$, $E'_2 > E_4$ [failure of local duality for the free (noninteracting) $q\bar{q}$ system].

rapidly over the interval, e.g., high-order moments. We illustrate the problem with the $k=-10$ moment in Fig. 2, where the weighting function and the integrand of Eq. (44) are shown. In contrast, the harmonic-oscillator resonances are essentially at the centers of the intervals for $n>2$, so we would expect the simple moments method to work much better for that potential.

C. Short-range effects

The confining potential commonly used in models describing the ψ and Υ systems is[10-12]

$$V(r)=-\frac{\alpha}{r}+\frac{r}{a^2}\,, \tag{50}$$

which includes the color Coulomb singularity of QCD ($\alpha=4\alpha_s/3$). As discussed in Sec. IIIC, the duality relation holds only if $V(r)$ is separated into a (nonconfining) short-range part V_S which includes the Coulomb singularity, and a long-range confining potential V_L which is well behaved at $r=0$. The duality relation is given in this case by Eq. (22) with $K_0(0,0,t)$ replaced by $K_S(0,0,t)$ from Eq. (26),

$$\langle \tilde{K}(E)\rangle \simeq \langle \tilde{K}_S[E-V_L(0)]\rangle + \cdots\,, \tag{51}$$

where

$$\langle \tilde{K}_S(E)\rangle = \frac{m_q^{3/2}}{2\pi}\int_0^\infty f(E'-E)\,|\psi_{E'}(0)|^2\sqrt{E'}\,dE'\,. \tag{52}$$

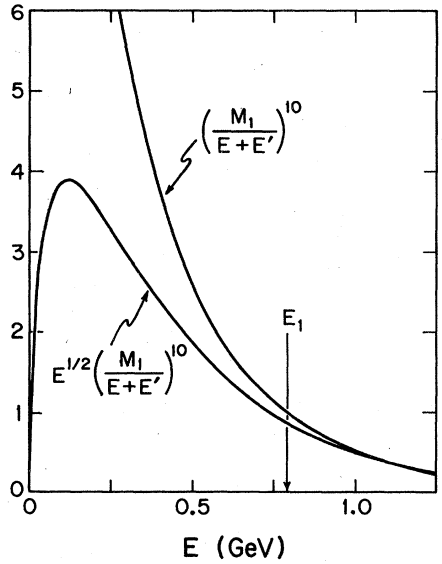

FIG. 2. Plot of the weighting function and integrand used to calculate the moments defined in Eqs. (43) and (44) for $k=-10$. The location of the first bound state in the linear potential is shown.

Here $\psi_E(r)$ is the free wave function with energy E in the potential V_S. (If there are bound states in V_S, their discrete contributions must be added to the integral.) The correction terms in Eq. (51) can be calculated or estimated using the methods discussed earlier.

We will illustrate this technique by treating the Coulomb singularity in the potential of Eq. (50), using the exactly solvable Hulthén potential[24,25] $V_H(r)$ for V_S,

$$V_H(r)=-\frac{\alpha\lambda e^{-\lambda r}}{1-e^{-\lambda r}}\,. \tag{53}$$

V_H approaches the Coulomb potential $-\alpha/r$ for $r\to 0$, and is cut off exponentially for $r>\lambda^{-1}$. We note that in Eq. (50), $V(r)=0$ at $r\sim 1$ GeV^{-1} for parameters in the usual range,[10-12] $\alpha\sim 0.25$, $a\sim 2$ GeV^{-1}. For $r<1$ GeV^{-1}, the Coulomb term is dominant, and the full potential varies rapidly, while for $r>1$ GeV^{-1}, the linear confining potential is dominant, and V varies slowly and smoothy. We therefore choose $\lambda=1$ GeV in V_H. The resulting Hulthén potential mocks the Coulomb potential for $r<1$ GeV^{-1}, and dies out rapidly for $r>1$ GeV^{-1}. The full potential $V(r)$, $V_H(r)$, and the long-range potential $V_L(r)$ defined by their difference, are shown in Figs. 3(a) and 3(b). Note that V_L is nearly linear over the range shown, with $V_L(0)<0$, and approaches V smoothly

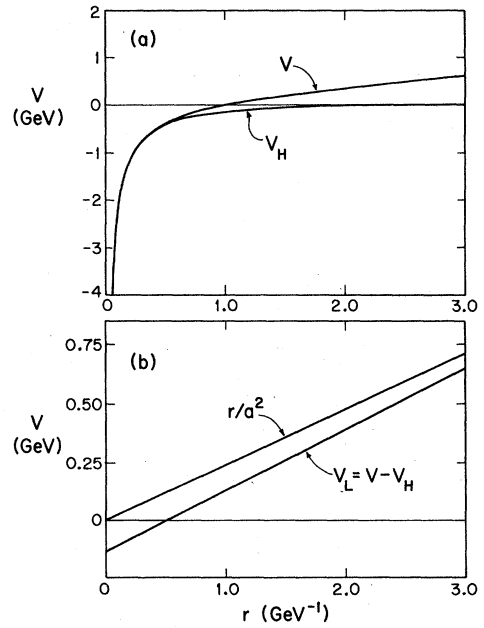

FIG. 3. (a) Comparison of the linear-plus-Coulomb potential $V(r)$ of Eq. (49) ($\alpha=0.25$, $a=2.06$ GeV^{-1}) with the Hulthén potential of Eq. (51) ($\alpha=0.25$, $\lambda=1.0$ GeV). (b) Plot of the long-range potential $V_L=V-V_H$. The linear term in V is shown for comparison.

for large r.

The exact S-state wave functions for the Hulthén potential are given in Ref. 25. For $r=0$,

$$|\psi_E(0)|^2 = \left| \frac{\Gamma(1+i\nu_-)\Gamma(1-i\nu_+)}{\Gamma(1-2i\kappa)} \right|^2 , \tag{54}$$

where

$$\nu_+ = \lambda^{-1}[m_q(E-\alpha\lambda)]^{1/2} \pm (m_q E/\lambda^2)^{1/2} ,$$
$$\kappa = (m_q E/\lambda^2)^{1/2} . \tag{55}$$

The number n of bound S states is the largest integer $n \geq 0$ such that $n < (m_q\alpha/\lambda)^{1/2}$. For $\alpha = 0.25$ and $\lambda = 1$ GeV, there are no bound states for $m_q = m_c = 1.45$ GeV, and one bound state for $m_q = m_b = 4.58$ GeV.

In Fig. 4(a) we compare the exact $|\psi_E(0)|^2$ for V_H to the bound state $|\psi_n(0)|^2$ for the Coulomb-plus-linear potential as given by Bell and Pasupathy.[16] We note that the values of $|\psi_E(0)|^2$ track the Coulomb-plus-linear values very nicely, and that the local duality intervals defined by the equation

$$2\pi |\psi_n(0)|^2 = \frac{m_q^{3/2}}{2\pi} \int_{E'_{n-1}}^{E'_n} |\psi_E(0)|^2 E^{1/2} dE \tag{56}$$

bracket the actual bound-state energies. This is in sharp contrast to the intervals calculated using Eq. (48) for free (noninteracting) quarks, which are shown in Fig. 1(c). In Fig. 4(b) we show the finite enhancement of $|\psi_E(0)|^2$ for the Hulthén potential relative to the case of no potential. The enhancement is large and causes $\langle \tilde{K}_S(E) \rangle$ in the duality relation in Eq. (51) to be significantly larger than $\langle \tilde{K}_0(E) \rangle$.

A different approach to the Coulomb singularity was suggested by Novikov *et al.*[10] and tested by Bell and Bertlmann.[19] Those authors used the wave functions for a pure, infinite-range Coulomb potential to calculate their comparison cross section. We show $|\psi_E(0)|^2$ for a pure Coulomb potential in Fig. 4(b). As shown, $|\psi_E(0)|^2$ diverges as $1/v$ for $v \to 0$. It is also necessary in this approach to include the infinite number of bound states in the Coulomb potential in the duality relation. Although this method appears to work well,[19] we regard it as somewhat cumbersome. It is also unphysical since the confined quarks do not feel the long-range part of the Coulomb potential. More generally, we would argue that *all* long-range effects in perturbative calculations of the $q\bar{q}$ cross section should properly be cut off since these are presumably included in the empirical confining interaction.

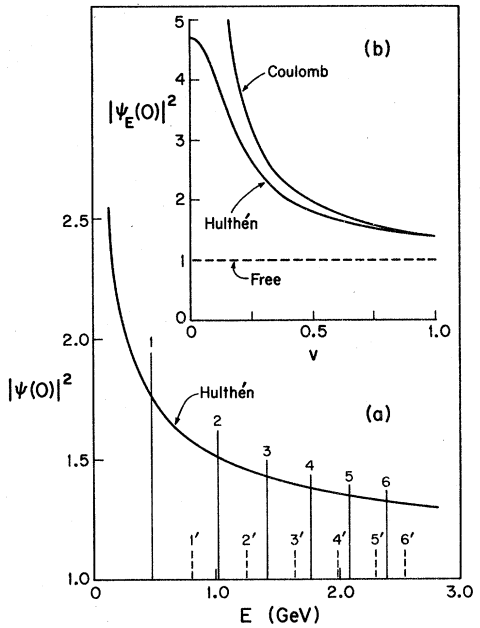

FIG. 4. (a) Comparison of $|\psi_E(0)|^2$ for the Hulthén potential of Eq. (51) ($\alpha = 0.25$, $\lambda = 1.0$ GeV) with $|\psi_n(0)|^2$ for the linear-plus-Coulomb potential of Bell and Pasupathy, Ref. 16. $|\psi_n(0)|^2$ is normalized to the value for the pure linear part of this potential to show the enhancement caused by the Coulomb interaction. (b) Comparison of $|\psi_E(0)|^2$ as a function of the nonrelativistic velocity of the quarks v for no interaction, a pure Coulomb interaction, and the short-range Hulthén interaction.

V. CONCLUSIONS

We have derived the duality relation in Sec. III for a class of potentials regular at the origin, then modified the derivation to allow for short-range singular potentials. The proof includes correction terms to the duality relation, which can be calculated from a commutator series or from the Euler-Maclaurin summation formula. In Sec. IV we have illustrated with exactly solvable potentials how to use the corrections and how to include a short-range potential. The corrections for potentials similar to those thought to work well for heavy-quark systems are large. This indicates that the correction terms should always be calculated in applying duality to data.

ACKNOWLEDGMENT

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[1]A. Bramon, E. Etim, and M. Greco, Phys. Lett. 41B, 609 (1972); M. Greco, Nucl. Phys. B63, 398 (1973); J. J. Sakurai, Phys. Lett. 46B, 207 (1973).
[2]E. C. Poggio, H. R. Quinn, and S. Weinberg, Phys. Rev. D 13, 1958 (1976).
[3]G. Farrar, V. A. Novikov, L. B. Okun', M. A. Shifman, M. B. Voloshin, and V. I. Zakarov, Phys. Lett. 71B, 115 (1977); V. A. Novikov, L. B. Okun', M. A. Shifman, A. I. Vainshtein, M. B. Voloshin, and V. I. Zakharov, Phys. Rev. Lett. 38, 626 (1977); 38, 791 (E) (1977).
[4]G. J. Gounaris, E. K. Manesis, and A. Verganelakis, Phys. Lett. 56B, 457 (1975); G. J. Gounaris, *ibid.* 72B, 91 (1977).
[5]V. Barger, W. F. Long, and M. G. Olsson, Phys. Lett. 57B, 452 (1975).
[6]F. E. Close, D. M. Scott, and D. Sivers, Nucl. Phys. B117, 134 (1976).
[7]R. M. Barnett, M. Dine, and L. McLerran, Phys. Rev. D 22, 594 (1980).
[8]M. Greco, Y. Srivastava, and G. Penso, Phys. Rev. D 21, 2520 (1980).
[9]K. J. Miller and M. G. Olsson, Wisconsin Report Nos. COO-881-136 and COO-881-137, 1980 (unpublished); Phys. Rev. D 22, 2137 (1980).
[10]V. A. Novikov, L. B. Okun', M. A. Shifman, A. I. Vainshtein, M. B. Voloshin, and V. I. Zakharov, Phys. Rep. 41, 1 (1978).
[11]C. Quigg and J. Rosner, Phys. Rep. 56, 167 (1979).
[12]See, for example, E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T.-M. Yan, Phys. Rev. D 17, 3090 (1978); 21, 203 (1980); A. Martin, CERN Report No. TH-2843, 1980, and the many references in those papers.
[13]See K. Berkelman, in *High Energy Physics—1980*, proceedings of the XXth International Conference on High Energy Physics, Madison, Wisconsin, 1980, edited by L. Durand and L. G. Pondrom (AIP, New York, 1981).
[14]M. Krammer and P. Leal Ferriera, Rev. Bras. Fis. 6, 7 (1976).
[15]C. Quigg and J. Rosner, Phys. Rev. D 17, 2364 (1978).
[16]J. S. Bell and J. Pasupathy, Z. Phys. C 2, 183 (1979).
[17]J. S. Bell and J. Pasupathy, Phys. Lett. 83B, 389 (1979).
[18]K. Ishikawa and J. J. Sakurai, Z. Phys. C 1, 117 (1979).
[19]J. S. Bell and R. A. Bertlmann, Z. Phys. C 4, 11 (1980).
[20]R. van Royen and V. F. Weisskopf, Nuovo Cimento 50A, 617 (1967); 51A, 583(E) (1967).
[21]This relation is usually stated with m_q replaced by $M_n/2$. This corresponds to an inconsistent treatment of spinors on the two sides of Eq. (4). We will discuss the effect of this inconsistency on the analysis of heavy-quark systems in a subsequent brief paper. In Sec. III of this paper, it will be clear that m_q is the proper factor.
[22]The Euler-Maclaurin summation formula relates a sum to an integral,
$$\sum_{n=1}^{N} f(n) = \int_1^N dn f(n) + \tfrac{1}{2} f(1) + \tfrac{1}{2} f(N) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} [f^{(2k-1)}(N) - f^{(2k-1)}(1)] + R_{2m},$$
where the B's are the Bernoulli numbers, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $\cdots$, and R_{2m} is a remainder term. The convergence of the series is generally only asymptotic.
[23]For E_n large compared to $\Delta E_n = E_n' - E_{n-1}'$, and $E_{n-1} \leq E_n \leq E_{n+1}'$, Eq. (47) gives the relation derived in Refs. 14, 15, and 17,
$$|\psi_n(0)|^2 = \frac{m_q^{3/2}}{4\pi^2} (E_n)^{1/2} \frac{dE_n}{dn}.$$
[24]L. Hulthén, Ark. Mat. Astron. Fys. A28, No. 5 (1942).
[25]L. Durand, Phys. Rev. 135, B310 (1964). See also U. Myhrman, J. Math. Phys. 21, 1732 (1980).