Total absorption in quantum chromodynamics at high energies

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Using a calculational scheme developed earlier for unitarizing high-energy S-matrix amplitudes, we find that the high-energy elastic hadron-hadron scattering amplitude in quantum chromodynamics agrees with the expansion of a unitary eikonal formula up to the eighth perturbative order. This eikonal formula is a convolution of hadronic wave functions and the amplitude of scattering among constituent quarks $\langle S\rangle = \langle \exp(i\chi)\rangle$, where χ is a Hermitian operator in the Fock space of pionization particles (here vector gluons). Based on a classical-quark-charge representation of the eikonal operator χ, we derive a functional partial differential equation for $\langle S\rangle$, from which we prove that a color-singlet hadron becomes completely absorptive at high energies.

I. INTRODUCTION

Recently, using unitarity as a guiding principle, a high-energy approximation to (two-body) scattering amplitudes in gauge field theories was proposed[1] (high center-of-mass energy, fixed momentum transfers). A unitary amplitude was obtained in quantum electrodynamics (QED) and Yang-Mills (YM) theory,[1,2] expressible in an eikonal form with the eikonal as an operator in the Fock space of pionization products. This led to the development of an operator eikonal formalism,[3] which was used to prove in QED that a particle becomes completely absorptive at high energies.

Strong interactions, however, are most likely to be based on the theory of quantum chromodynamics (QCD), with quarks forming hadrons through a yet-unknown color-confinement dynamics. Hence it is important to know how to relate quark-quark scattering studied earlier with color-singlet hadron-hadron scattering, to study modifications of the operator eikonal formalism in the limit of zero gluon mass. More seriously, even in the non-Abelian gauge theories with massive vector mesons studied earlier,[2] that the eikonal operator is also a matrix in group representation space of the interacting particles has foiled analytical attempts to extract any experimental implications from the operator eikonal formalism. The situation in QCD can only be worse.

In this paper, we shall report on progress on the problems posed above. Our starting point is QCD, but we make no pretense on our ignorance of color-confinement dynamics. Rather, we assume confinement effects to be satisfactorily described by a hadron wave function or quark distribution function, and leave such a function practically unspecified. We shall first give a small mass to the color-gluons, via the Higgs mechanism for calculational purposes, then show how to obtain infrared-convergent expresssions for color-singlet–color-singlet elastic scattering, in the limit of zero gluon mass.

We use a scheme of diagrammatic calculations devised earlier for Yang-Mills theories[2] to show that in QCD the high-energy scattering amplitude up to the eighth perturbative order can again be expressed in an *operator eikonal form*, convoluted with hadron wave functions, in the form of an impact picture of scattering[4] (or impulse approximation) as given by Eqs. (2.2) through (2.13). This scheme of calculation consistently retains terms of the form

$$s\left(\frac{g^2t^2}{2\pi}\right)^n\left(\frac{g^2\ln s}{4\pi^2}\right)^m,$$

n, m being positive integers and t^2 the quadratic Casmir invariant of the quark representation $t_a\,(t_at_a=t^2\underline{1})$. In this paper, we shall call such terms the leading-unitary terms. Terms dropped, the nonleading terms, have additional factors of g^2. We review the justification and discuss the meaning of this approximation in Sec. II B.

For our present discussion, we want to emphasize that only the leading-unitary terms generated by the eikonal operator in the scattering amplitude are meaningful and determined uniquely by the diagrammatic calculations. Hence two representations of the eikonal operator which generate the same leading-unitary terms in the scattering amplitude, but different nonleading terms, are equally good representations of the eikonal operator at the present level of approximation.

We have found a classical-quark-charge representation of the eikonal operator which yields leading-unitary terms consistent with the diagrammatic calculation scheme in QCD, and which can be analyzed to the stage where we can prove that a color-singlet hadron becomes completely absorptive at high energies.

Our analysis proceeds by deriving a functional partial differential equation for a certain generating functional for the S matrix in impact-distance space. This is the functional-space analog of deriving Schrödinger's equation from the Feynman path integral, as the operator eikonal form is a kind of path integral. The partial differential equation describes how the S matrix behaves with increasing rapidity (rapidity is $T = \ln s/2\pi$, s being the center-of-mass energy of scattering). A lattice version of the partial differential equation is studied. In the continuum limit, the analysis enables us to arrive at the behavior of the S matrix at high energies stated earlier.

The paper is organized as follows. Section II is devoted to the discussion of the diagrammatic calculation in QCD incorporating hadron wave functions, and the summary of the eikonal-approximation results, some details of which are presented in Appendices A and B. We introduce the classical-source representation of the eikonal operator in Sec. III, with a proof that it yields leading-unitary terms consistent with the diagrammatic calculations of the previous section. A functional partial differential equation for the generating functional of the S matrix is derived in Sec. IV. Using this approach, we present a solvable model in Sec. V. In Sec. VI we analyze the partial differential equation in QCD, in the limit of zero gluon mass, to obtain the behavior of the color-singlet scattering amplitude at high energies. We conclude in Sec. VII with a brief discussion of the phenomenological aspects of our research.

II. EIKONAL APPROXIMATION IN QCD

A. Hadronic wave function

Before we present diagrammatic results in QCD, we shall describe how we propose to handle confinement dynamics in hadrons. In a high-energy collision between hadrons, there are two time scales for interactions. Interactions among quarks from the *same* hadron, the quark-confinement interactions, are associated with a long time scale, that of the hadron rest frame. On the other hand, interactions among quarks from *different* hadrons are instantaneous. Hence we shall assume that the confinement dynamics are adequately described by hadron wave functions, prescribing the distribution of quarks inside the incident and target hadrons, a long time before and after the high-energy instantaneous interactions. This is the physical picture that quarks from the same hadron do not interact among themselves during the short time scale when quarks from different hadrons interact. Hence in the Feynman diagrams for the scattering processes that we shall consider we do not put in explicit gluon exchanges between quarks from the same hadron.

With this in mind, it is possible to show that for high-energy and fixed-momentum-transfer processes, any given Feynman diagram incorporating hadron wave functions is factorizable as a convolution of hadronic impact factors[4] and an amplitude of scattering between constituent quarks of different hadrons. This is proved in Appendix A, where in the infinite-momentum, center-of-mass frame, any Feynman amplitude is obtained as in Eq. (A17):

$$\mathfrak{M}^F = \int \mathcal{G}_A(\{\vec{q}_{i\perp}\})\mathfrak{M}_0^F(\{\vec{q}_{i\perp}\},\{\vec{q}'_{k\perp}\})\mathcal{G}_B(\{\vec{q}'_{k\perp}\})\delta^{(2)}\left(\sum_i^M \vec{q}_{i\perp} - \vec{\Delta}\right)\prod_{i=1}^M d^2\vec{q}_{i\perp}\prod_{k=1}^N d^2\vec{q}_{k\perp}\,. \tag{2.1}$$

Here, $\mathcal{G}_A(\{\vec{q}_{i\perp}\})$ is the impact factor of hadron A carrying large plus momentum $\vec{P}_A$ ($\vec{P}_\pm = \vec{P}_t \pm \vec{P}_z$), and which is made up of M quarks each absorbing transverse momentum $\vec{q}_{i\perp}$ ($i = 1, \ldots, M$). Hadron B has N quarks, and a similar notation applies. $\mathfrak{M}_0^F$ is the amplitude of scattering between the M quarks of hadron A with the N quarks of hadron B, having the same topology as $\mathfrak{M}^F$, but with the hadronic wave functions truncated. The total momentum transferred to each hadron is fixed to be $\sum_{i=1}^M \vec{q}'_{i\perp} = \vec{\Delta} = -\sum_{k=1}^N \vec{q}'_{k\perp}$.

We shall proceed to compute $\mathfrak{M}_0$ perturbatively up to the 8th order in the coupling constant, using a scheme developed earlier[2] to compute leading-unitary terms in Yang-Mills theory. We shall find that $\mathfrak{M}_0$ is in an eikonal form in the impact-distance space,[5] with the eikonal being an operator, just as in QED and Yang-Mills theory studied earlier.[1,2]

B. Leading-unitary terms

In the discussion of high-energy scattering amplitudes with fixed momentum transfers, a high-energy approximation to the perturbation series is meaningful only if the result it gives dominates the sum of the terms being dropped. In contrast, in the leading-logarithm approximation, for example, where in each perturbation order we keep only the terms with the highest power of ($\ln s/2\pi$) (s being the center-of-mass energy), each of the terms we have dropped can be much larger than the final answer. Indeed, summing leading logarithms in both QED and non-Abelian gauge theories leads to results which violate the Froissart bound.[6] So the sum of nonleading terms must necessarily be as large as the leading terms to restore the unitarity bound. In other words, the leading logarithms by themselves cannot be meaningful at high energies.

On the other hand, consider summing leading terms defined by a high-energy approximation scheme which respects unitarity at every stage, and assume that the scattering amplitude so obtained is bounded by unitarity. Then in this sum there must be numerous cancellations among leading terms from the different orders of perturbation, which individually violates the Froissart bound. The dramatic cancellation in this case is a consequence of the constraints of unitarity, so it is perhaps reasonable to hope for a similar cancellation among the nonleading terms that we have dropped, due to unitarity constraints. If this happens, the leading terms which sum to a unitary answer, which we call *leading-unitary terms*, are probably meaningful and dominate the sum of nonleading terms.

This is borne out in QED, where we have recently proposed a procedure to sum leading terms, respecting s-channel and t-channel unitarity at every step.[1] The sum of the so-defined leading terms are explicitly unitary. Both elastic and inelastic amplitudes are summarized by an eikonal formula, in which the eikonal is an operator in the Fock space of pionization particles. A subsequent *nonperturbative* derivation[7] of this operator eikonal form (under general high-energy assumptions) shows that the sum of nonleading terms does not invalidate the leading-unitary terms. As to be expected, the nonleading terms modify the various functions in the functional form of the leading result in a perturbative fashion. The eikonal operator obtained from the leading-unitary terms emerges as a first approximation to the full nonperturbative eikonal operator.

Details of this unitarization procedure in QED are described in Ref. 1. Briefly, the leading-unitary terms of this scheme are the leading terms of a minimal set of Feynman diagrams which closes under unitarity, crossing, and gauge symmetry relations.

In the same spirit, a high-energy approximation scheme to calculate the leading-unitary terms diagrammatically in non-Abelian gauge theories was devised in Ref. 2 (where the vector mesons acquire masses through the Higgs mechanism). For fermion-fermion scattering, where the representation matrices of the fermions are t_a, the scattering amplitude for a particular channel in the t-channel is a function of the Casmir invariant t^2 ($t_a t_a = t^2 \underline{1}$). (This comes from the vector-meson–fermion vertex as the exchanged vector mesons are attached to the scattering fermions.) In fact, the scattering amplitude in each order of perturbation is a double series in t^2 and $(\ln s/2\pi)$ at high energies. The scheme in Ref. 2 is to approximate the coefficient of $(t^2)^n$ for fixed integer n in this double series by the term with the highest power of $(\ln s/2\pi)$. From calculation, these leading terms are of the form $s(g^2 t^2/2\pi)^n (g^2 \ln s/4\pi^2)^m$, having all powers of t^2. *A priori*, this mathematical prescription does not guarantee a unitary result, but for an amplitude $\mathfrak{M}$, which is a function of t^2 to satisfy the nonlinear unitarity condition $\mathrm{Im}\mathfrak{M} = \mathfrak{M}^\dagger \mathfrak{M}$, it must at least contain all powers of t^2. Indeed, these leading terms are summarized by a unitary eikonal formula, with an eikonal operator involving Reggeized vector mesons. In contrast, the leading-logarithm approximation retains terms with only the first or second power of t^2, so the unitarity-violating result it gives is not unexpected.

A nonperturbative derivation of the eikonal form to justify this perturbative unitarization scheme in non-Abelian gauge theories is still under investigation. Owing to the similarities of this scheme with the unitarization procedure in QED, it is likely that this can be done soon.

We shall define leading terms for $M+N$ quark scattering in QCD in a similar manner as in Ref. 2, all quarks being in the same color-group representation with representation matrices t_a. Appendix B contains the details of the calculation to extract these leading terms from Feynman diagrams up to the eighth order,[8] using the methods of Ref. 2. These leading terms of the amplitude $\mathfrak{M}_0$ of Sec. A, having the form $s(g^2 t^2/2\pi)^n (g^2 \ln s/2\pi)^m$, again sums to an eikonal form.

C. The eikonal form in QCD

The impact-distance representation of the sum of leading-unitary terms is [see Eq. (A19)]

$$\mathfrak{M} = 2is \int I_A(\Delta, \{\vec{x}_j\}) I_B(-\Delta, \{\vec{y}_k\}) e^{-\vec{\Delta}_\perp \cdot \vec{b}_\perp} \times [1 - S(\vec{b}_\perp, \{\vec{x}_j\}, \{\vec{y}_k\}, T)] \times \delta^{(2)}\left(\sum_{j=1}^{M} \vec{x}_j\right) \delta^{(2)}\left(\sum_{k=1}^{N} \vec{y}_k\right) \prod_{j,k} d^2\vec{x}_j d^2\vec{y}_k d^2\vec{b}_\perp . \quad (2.2)$$

The scattering matrix $S(\vec{b}_\perp, \{\vec{x}_j\}, \{\vec{y}_k\}, T)$ describes scattering of two hadrons separated by $\vec{b}_\perp$ in transverse impact-distance space, at center-of-mass energy s ($T = \ln s/2\pi$), where the jth quark is at position $\vec{x}_j$ measured from the center of mass of the hadron it belongs to. For elastic scattering, it is the expectation value of a unitary operator:

$$S(\vec{b}_\perp, \{\vec{x}_j\}, \{\vec{y}_k\}, T) = \left\langle 0 \left| \exp\left[i \sum_{k=1}^{N} \sum_{j=1}^{M} \chi(\vec{b}_\perp + \vec{x}_j + \vec{y}_k, T)\right] \right| 0 \right\rangle . \quad (2.3)$$

The eikonal is a sum of Hermitian operators $\chi(\vec{b}_\perp + \vec{x}_j + \vec{y}_k, T)$, each associated with one of the ways M quarks can scatter from N quarks.

The eikonal operator χ is the same as that obtained earlier in non-Abelian gauge theories[2] (with massive vector mesons and Higgs scalars). The modification for hadron-hadron scattering is merely in the additional impact factors I_A and I_B to be convoluted with the quark-quark scattering amplitudes. Owing to the operator nature of χ, however, S includes numerous mixed interactions between the quarks. We shall postpone taking the zero-gluon-mass limit until Sec. VI, but for now, we shall invoke the Higgs mechanism to give the gluons masses. Then the eikonal operator χ between two quarks is explicitly given by the iterative solution of the following integral equation (2.5). We define χ_{ab} from χ as (where $T = \ln s/2\pi$ is the rapidity)

$$\chi(\vec{b}_\perp + \vec{x}_j + \vec{y}_k, T) = \chi_{ac}(\vec{b}_\perp + \vec{x}_j + \vec{y}_k, T)(-t_a^{(j)} t_c^{(k)}) . \quad (2.4)$$

Then

$$\chi_{ac}(\vec{b}_\perp, T) = \chi^0_{ac}(\vec{b}_\perp, T) + g^2 \int_0^T dT' \int d^2\vec{b}'_\perp \chi_{ab}(\vec{b}'_\perp, T') \times \Sigma^{bc}(\vec{b}_\perp, \vec{b}'_\perp; T, T') , \quad (2.5)$$

where the inhomogeneous term is

$$\chi^0_{ac}(\vec{b}_\perp, T) = g^2 \delta_{ac} \int \frac{d^2\vec{q}_\perp}{(2\pi)^2} e^{i\vec{q}_\perp \cdot \vec{b}_\perp} \frac{\exp\{[\alpha(\vec{q}_\perp) - 1]T\}}{(\vec{q}_\perp{}^2 + \lambda^2)} \quad (2.6)$$

with $\alpha(\vec{q}_\perp)$ being the Regge trajectory on which the vector meson lies [where $if_{abc} t_b t_c = C_A t_a$, so that $C_A = N/2$ in SU(N)]:

$$\alpha(\vec{q}_\perp) = 1 - C_A g^2(\vec{q}_\perp{}^2 + \lambda^2) \times \int \frac{d^2\vec{q}'_\perp}{(2\pi)^2} \frac{1}{(\vec{q}'^2_\perp + \lambda^2)[(\vec{q}_\perp - \vec{q}'_\perp) + \lambda^2]} . \quad (2.7)$$

The kernel Σ^{bc} is

$$\Sigma^{bc}(\vec{b}_\perp, \vec{b}'_\perp, T, T') = \int \frac{d^2\vec{q}_{1\perp}}{(2\pi)^2} \frac{d^2\vec{q}_{2\perp}}{(2\pi)^2} e^{-i\vec{q}_{1\perp}\cdot\vec{b}' + i\vec{q}_{2\perp}\cdot\vec{b}} \times [V^{bc}(\vec{q}_{1\perp}, \vec{q}_{2\perp}, T') + V_H^{bc}(T')] \times \left[\frac{\exp\{[\alpha(\vec{q}_{2\perp}) - 1](T - T')\}}{\vec{q}_{2\perp}{}^2 + \lambda^2}\right] . \quad (2.8)$$

The quantity in the first square brackets in the Fourier representation (2.8) is a sum of a momentum-dependent vertex V^{bc} for vector-meson creation and annihilation, and a similar vertex V_H^{bc} for Higgs scalar particles responsible for the vector-meson masses. The quantity in the second square brackets represents further Reggeized vector-meson exchange after particle production, just as in (2.6).

The vertex V^{bc} is expressed in terms of creation and annihilation operators $a_d^{\alpha\dagger}(\vec{k}_\perp, T)$, $a_d^\alpha(\vec{k}'_\perp, T')$ (f_{bcd} being the structure constant of the gauge group):

$$V^{bc}(\vec{q}_{1\perp}, \vec{q}_{2\perp}, T) = if_{bcd}\Gamma_\mu(\vec{q}_{1\perp}, \vec{q}_{2\perp})\epsilon^\alpha_\mu(q_1 - q_2) \times \frac{g}{\sqrt{2}}[a_d^{\alpha\dagger}(\vec{q}_{1\perp} - \vec{q}_{2\perp}, T) - a_d^\alpha(\vec{q}_{2\perp} - \vec{q}_{1\perp}, T)] . \quad (2.9)$$

$a_c^\alpha, a_d^{\beta\dagger}$ are δ-function normalized:

$$[a_c^\alpha(\vec{k}'_\perp, T'), a_d^{\beta\dagger}(\vec{k}, T)] = \delta_{\alpha\beta}\delta_{cd}(2\pi)^3\delta^{(2)}(\vec{k}_\perp - \vec{k}'_\perp)\delta(T - T') . \quad (2.10)$$

They operate on the Fock space of pionization particles, which here are vector mesons.

The vertex factor of V^{bc} is the same as Eq. (3.8) of Ref. 2, with Γ_μ [where $\Gamma_\mu(q_1 - q_2)_\mu = 0$] given by

$$\Gamma_\mu(\vec{q}_{1\perp}, \vec{q}_{2\perp}) = g_{\mu\perp}(\vec{q}_{1\perp} + \vec{q}_{2\perp}) - g_{\mu+}(q_1 - q_2)_- \left[\frac{1}{2} - \frac{\vec{q}_{2\perp}{}^2 + \lambda^2}{(\vec{q}_{1\perp} - \vec{q}_{2\perp})^2 + \lambda^2}\right] + g_{\mu-}(q_1 - q_2)_+ \left[\frac{1}{2} - \frac{\vec{q}_{1\perp}{}^2 + \lambda^2}{(\vec{q}_{1\perp} - \vec{q}_{2\perp})^2 + \lambda^2}\right] . \quad (2.11)$$

Also, ϵ^α_μ for $\alpha = 1, 2, 3$ are three physical polarization vectors for the created massive vector meson, satisfying $k_\mu \epsilon^\alpha_\mu(k) = 0$, $\epsilon^2(k) = -1$.

By choosing $\epsilon^\alpha_\mu(k)$ to be the following four vectors:

$$\begin{aligned} k_\mu &= (k_+, k_-, k_x, k_y) , \\ \epsilon^{(1)}_\mu(k) &= \left(0, 0, -\frac{k_y}{|\vec{k}_\perp|}, \frac{k_x}{|\vec{k}_\perp|}\right) , \\ \epsilon^{(2)}_\mu(k) &= \left(0, \frac{2|\vec{k}_\perp|}{k_+}, \frac{k_x}{|\vec{k}_\perp|}, \frac{k_y}{|\vec{k}_\perp|}\right) , \\ \epsilon^{(3)}_\mu(k) &= \frac{1}{\lambda}\left(k_+, \frac{\vec{k}_\perp{}^2 - \lambda^2}{|k_+|}, k_x, k_y\right) \end{aligned} \quad (2.12)$$

we can express the product $\Gamma_\mu \epsilon^\alpha_\mu$ as

$$\Gamma_\mu(\vec{q}_{1\perp},\vec{q}_{2\perp})\epsilon_\mu^\alpha(q_1-q_2)=2\left[\frac{q_{1x}q_{2y}-q_{1y}q_{2x}}{|\vec{q}_{1\perp}-\vec{q}_{2\perp}|}\right]\delta_{\alpha,1}+\left[\frac{\vec{q}_{1\perp}^{\,2}-\vec{q}_{2\perp}^{\,2}}{|\vec{q}_{1\perp}-\vec{q}_{2\perp}|}+|\vec{q}_{1\perp}-\vec{q}_{2\perp}|\left(1-\frac{2(\vec{q}_{1\perp}^{\,2}+\lambda^2)}{(\vec{q}_{1\perp}-\vec{q}_{2\perp})^2+\lambda^2}\right)\right]\delta_{\alpha,2}$$

$$+\lambda\left[1-\frac{2(\vec{q}_{1\perp}^{\,2}+\lambda^2)}{(\vec{q}_{1\perp}-\vec{q}_{2\perp})^2+\lambda^2}\right]\delta_{\alpha,3}\,. \qquad (2.13)$$

Only the physical polarizations $\alpha=1,2$ contribute to (2.13) as $\lambda\to 0$.

Finally, the vertex V_H^{bc} in (2.8) describes creation/annihilation of Higgs scalar particles. As we shall take the limit of zero gluon mass $\lambda\to 0$ later, we shall not bother to exhibit V_H^{bc} explicitly; suffice it to note that V_H^{bc} is also an operator, but $V_H^{bc}\propto\lambda\to 0$ as $\lambda\to 0$.

The matrix element $\langle\vec{k}_1\cdots\vec{k}_n|\chi|\vec{k}_1'\cdots\vec{k}_m'\rangle$ of the eikonal operator χ between incoming m-vector-meson state $|\vec{k}_1'\cdots\vec{k}_m'\rangle$ and outgoing n-vector-meson state $\langle\vec{k}_1\cdots\vec{k}_n|$ is related to the scattering amplitude of

$$2\text{ fermion}+m\text{ vector meson}\to 2\text{ fermion}+n\text{ vector meson}. \qquad (2.14)$$

If $\alpha(\vec{q}_\perp)$, the Regge trajectory of the vector-meson, were one instead of being given by Eq. (2.7), $\langle\vec{k}_1\cdots\vec{k}_n|\chi|\vec{k}_1'\cdots\vec{k}_m'\rangle$ is in fact the lowest-order amplitude of the processes of (2.14). The creation and annihilation of vector mesons are described by the vertex factor V^{bc} of Eqs. (2.8) and (2.9). Hence we can think of the operator χ generated from the iterative solution to Eq. (2.5) as having matrix elements which are the lowest-order amplitudes of the processes of (2.14), but with the exchanged vector mesons Reggeized as given by Eq. (2.7). Examples of matrix elements of χ are given in Fig. 1.

Knowing the matrix elements of χ, we can evaluate S of Eq. (2.3) by first expanding the exponential, then inserting a complete set of states where necessary. For example, the χ^2 term is evaluated as

$$\langle 0|\chi^2|0\rangle=\sum_n\langle 0|\chi|n\rangle\langle n|\chi|0\rangle\,.$$

The leading-unitary terms generated by such an expansion of the eikonal in Eq. (2.3), i.e., terms of the form $s(g^2t^2/2\pi)^n(g^2\ln s/4\pi^2)^m$, can be compared with the result of the diagrammatic calculation. That they agree in the case of quark-quark scattering is shown in great detail in Ref. 2. For the case of $M+N$ quark scattering in QCD, the eikonal form of Eq. (2.3) can be shown to agree with the diagrammatic calculation using similar techniques. We shall not present the details here, as it is rather tedious but straightforward.

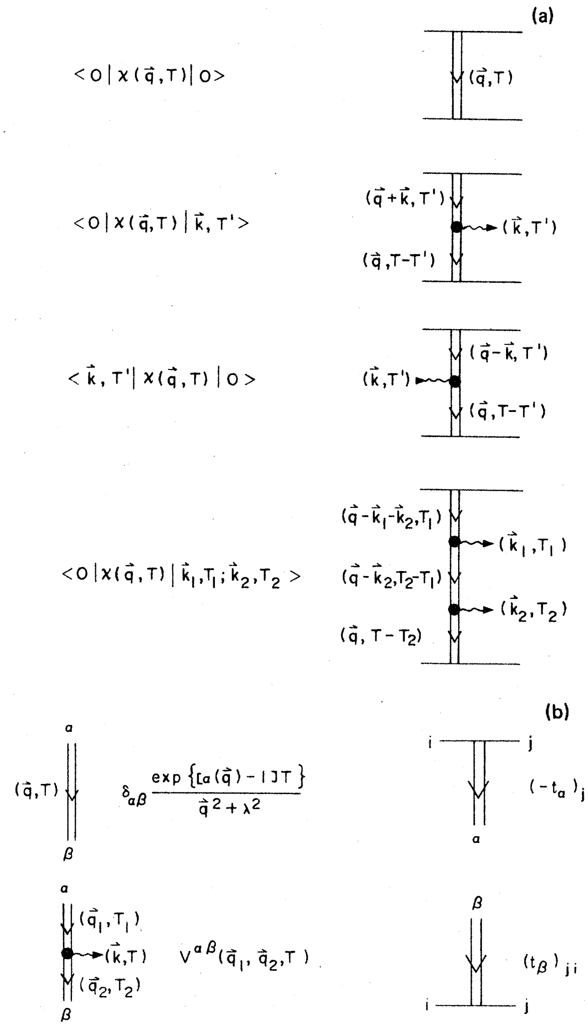

FIG. 1. (a) Examples of matrix elements of $\chi(\vec{q},T)$, the Fourier transform of $\chi(\vec{b},T)=\int[d^2\vec{q}/(2\pi)^2]e^{i\vec{q}\cdot\vec{b}}\chi(\vec{q},T)$. (b) Diagrammatic rules for Reggeized propagators, "Reggeon-Reggeon-meson" vertex function, and quark-quark-Reggeon vertex. [See Eqs. (2.7) and (2.9) for definitions of $\alpha(\vec{q})$ and $V^{\alpha\beta}$.]

III. CLASSICAL-SOURCE REPRESENTATION OF THE EIKONAL OPERATOR

The eikonal formula (2.2) of the previous section summarizes the leading-unitary terms of the diagrammatic calculation in the sense that if we compare terms of the form $s(g^2t^2/2\pi)^n(g^2\ln s/4\pi^2)^m$, both the eikonal formula and the diagrammatic calculation give the same result. However, this does not mean that the eikonal operator χ of (2.4) is determined uniquely by the diagrammatic calculation. For instance, consider an operator $\tilde{\chi}$ defined in exactly the same way as χ through Eqs. (2.4)–(2.10), but with an additional term of order g^4, independent of t or $\ln s$, added to the Regge trajectory $\alpha(\vec{q}_\perp)$ of (2.7). When put into the eikonal formula (2.2), both χ and $\tilde{\chi}$ generate results with the same functional dependence on $\alpha(\vec{q}_\perp)$, so the leading-unitary terms of the form $s(g^2t^2/2\pi)^n \times(g^2\ln s/4\pi^2)^m$ are the same. The presence of the term of order g^4 in the $\alpha(\vec{q}_\perp)$ of $\tilde{\chi}$ can only give rise to nonleading terms with extra factors of g^2, with no compensating factors of t^2 or $\ln s$. At the present level of approximation, where nonleading terms are not considered, χ and $\tilde{\chi}$ are equally good representations of the eikonal operator.

This nonuniqueness of the eikonal operator is far from saying that the eikonal formula (2.2) is not useful. That χ and $\tilde{\chi}$ discussed above are equally adequate representations of the eikonal operator hinges crucially on the fact that they have the same functional form. This particular functional form comes from the sum of leading-unitary terms and seems to be determined uniquely by them. Indeed, in QED, the functional form for the eikonal operator and eikonal formula, obtained from summing analogous leading-unitary terms, is preserved when nonleading terms are included.[7] So it is very reasonable to expect that (2.2) is a meaningful first approximation to the full high-energy scattering amplitude.

To extract experimental implications from the eikonal formula of (2.2), we would naturally want to consider the simplest representation of the eikonal operator which reproduces the leading-unitary terms of the diagrammatic calculation. Not only would the mathematical analysis be the simplest, but more seriously, if nonleading terms are included into the eikonal operator in an arbitrary manner, unwarranted infrared divergences are likely to appear (in the nonleading term sector) when we take the zero-gluon-mass limit.

Below we shall present a representation of the eikonal operator simpler than that given by Eqs. (2.4)–(2.10), but which still reproduces the required leading-unitary terms of the diagrammatic calculation. In essence, we shall treat the quarks of one of the hadrons as classical quarks in the eikonal operator, so the resulting representation is a *classical-source representation* of the eikonal operator.

Nevertheless, the representation presented in Sec. II remains the best representation of the eikonal operator for purposes of comparing with diagrammatic results. This is so because the corresponding eikonal formula (2.2) generates expressions in the form of a product of a momentum integral (represented by transverse-momenta Feynman diagrams) and a group-spin diagram, forms in which the diagrammatic results are expressed. For the classical-source representation, we need to project the amplitudes into the respective channels in the t channel before we can compare with diagrammatic results. To this we shall now turn.

First, we shall consider quark 1 scattered from quark 2. In Eq. (2.4), we set $j=1$, $k=2$, and $\vec{x}_i=\vec{y}_k=0$:

$$\chi(\vec{b}_\perp, T)=\chi_{ac}(\vec{b}_\perp, T)(-t_a^{(1)}t_c^{(2)})\,. \tag{3.1}$$

While χ_{ab} is an operator in the Fock space of pionization particles (vector mesons), χ is also a matrix in the product group space of the interacting quarks: $\chi\in t^{(1)}\otimes t^{(2)}$. This is the manifestation of the quantum nature of quark sources in QCD (and non-Abelian gauge field theories).

In classical non-Abelian gauge theories, a source has a classical charge which is specified by a vector Q_a in group space. This is, however, not a canonical description of the sources (see Sec. II of Ref. 9). A canonical description, necessary in the quantum theory, is obtained by promoting the vector Q_a to a set of matrices t_a, so that the quantum source transforms according to a particular representation of the gauge group:

$$[t_a, t_b]=if_{abc}t_c\,. \tag{3.2}$$

The classical-source representation of the eikonal operator is the representation in which we demote the quantum nature of the quarks of one hadron to the classical level in the eikonal operator.

For the example of quark-quark scattering, we replace $t_c^{(2)}$ in (3.1) by commuting vectors $Q_c^{(2)}$ in group space, with $Q_a^{(2)}Q_a^{(2)}=t^2$, but otherwise we keep the definition of χ_{ac} as in (2.5)–(2.10). We shall now show that the amplitude generated by the eikonal formula (2.2), when projected into the different exchange channels in the t channel, are the same for the eikonal of (3.1) and the eikonal of the classical-source representation, up to nonleading terms.

Consider the expressions generated by the eikonal formula (2.2) with (3.1) as the eikonal operator. [We drop the factors I_A, I_B in (2.2) for quark-quark scattering.] Each term is a subamplitude which we shall generically denote as $\mathfrak{M}_n$ if it comes from the matrix element $\langle 0|\chi^n|0\rangle$ in the expansion of $\langle 0|\exp(i\chi)|0\rangle$. $\mathfrak{M}_n$ is the product of a group-spin factor and a space-time factor. The group-spin factor is of the general form

$$I_n = (g^2)^n t^{(1)}_{i_1} t^{(1)}_{i_2} \cdots t^{(1)}_{i_n} \Omega^{i_1,\ldots,i_n}_{j_1,\ldots,j_n} t^{(2)}_{j_1} \cdots t^{(2)}_{j_n}, \quad (3.3)$$

where $\Omega^{\{i\}}_{\{j\}}$ is some invariant tensor of the gauge group, depending on the subamplitude. The space-time factor is in the form of momentum integrals, and has a general form $F_n(g^2 \ln s/4\pi^2)$, a function of the combination $g^2 \ln s/4\pi^2$. So each subamplitude is expressed as

$$\mathfrak{M}_n = I_n F_n\left(\frac{g^2 \ln s}{4\pi^2}\right). \quad (3.4)$$

When $\mathfrak{M}_n$ is projected into the various exchange channels (μ) in the t channel, with projection matrices $P_{(\mu)}$, the projected amplitude $\mathfrak{M}_n^{(\mu)}$ is given by

$$\mathfrak{M}_n = \sum_\mu \mathfrak{M}_n^{(\mu)} P_{(\mu)}, \qquad \mathfrak{M}_n^{(\mu)} = \frac{\mathrm{Tr}(P_{(\mu)}\mathfrak{M}_n)}{\mathrm{Tr}(P^2_{(\mu)})}. \quad (3.5)$$

Tr is the trace in the product space $t^{(1)}\otimes t^{(2)}$, as both $P_{(\mu)}$ and $\mathfrak{M}_n$ are matrices in this space. We make use of the fact that $P_{(\mu)}$ is a tensor product of projection operators $P^{(1)}_{(\mu);k_1,\ldots,k_m}$ and $P^{(2)}_{(\mu);k_1,\ldots,k_m}$ with m indices in the t channel $(k_1, k_2, \ldots, k_m)$, where $P^{(1)}_{(\mu)} \in t^{(1)}$, $P^{(2)}_{(\mu)} \in t^{(2)}$ ($t^{(1)}$, $t^{(2)}$ being the separate group-representation space of the quarks):

$$P_{(\mu)} = \sum_{\{k\}} P^{(1)}_{(\mu);k_1,\ldots,k_m} \otimes P^{(2)}_{(\mu);k_1,\ldots,k_m}. \quad (3.6)$$

For example, in SU(2), the isospin-1 channel (μ =triplet) in the t channel has a projection operator which is $P_3 \propto t^{(1)}_k t^{(2)}_k$, so that $P^{(1)}_{3;k} = t^{(1)}_k$, $P^{(2)}_{3;k} = t^{(2)}_k$. In the vacuum channel (μ =singlet), $P_1 = \underline{1}^{(1)}\underline{1}^{(2)}$, so $m=0$ and $P^{(1)}_1 = \underline{1}^{(1)}$, $P^{(2)}_1 = \underline{1}^{(2)}$. ($\underline{1}$ is the unit matrix.)

From (3.4), (3.5), and (3.6), we obtain

$$\mathrm{Tr}(P_{(\mu)}\mathfrak{M}_n) = (g^2)^n F_n\left(\frac{g^2 \ln s}{4\pi^2}\right)\Omega^{i_1,\ldots,i_n}_{j_1,\ldots,j_n} \times \mathrm{Tr}(P^{(1)}_{(\mu);k_1,\ldots,k_m} t^{(1)}_{i_1}\cdots t^{(1)}_{i_n}) \times \mathrm{Tr}(P^{(2)}_{(\mu);k_1,\ldots,k_m} t^{(2)}_{j_1}\cdots t^{(2)}_{j_n}). \quad (3.7)$$

Equation (3.7) together with (3.5) clearly show that $\mathfrak{M}_n^{(\mu)}$ is a polynomial in t^2 ($t^{(1)}_a t^{(1)}_a = t^{(2)}_a t^{(2)}_a = t^2\underline{1}$). The highest power of t^2 in this polynomial is at most n, as there are only $2n$ matrices t_a in the product of (3.7). It is this term with the power $(t^2)^n$ which we need to retain as a leading-unitary term, for only this term is of the form $(g^2t^2)^n F_n(g^2 \ln s/4\pi^2)$.

In fact, the two trace factors of (3.7) can each be expressed as a sum of invariant tensors of the gauge group. In this sum, the t^n term is multiplied by tensors of one specific kind, namely, the invariant tensors symmetric with respect to all the indices $\{k_1, \ldots, k_m, i_1, \ldots, i_n\}$ or $\{k_1, \ldots, k_m, j_1, \ldots, j_n\}$:

$$\mathrm{Tr}(P^{(1)}_{(\mu);k_1,\ldots,k_m} t^{(1)}_{i_1}\cdots t^{(1)}_{i_n}) = A_n t^n \times (\text{invariant symmetric tensor } S^{(1)}_{k_1,\ldots,k_m,i_1,\ldots,i_n}) + R^{(1)}_{k_1,\ldots,k_m;i_1,\ldots,i_n}. \quad (3.8)$$

$R^{(1)}_{\{k\};\{i\}}$ denotes remaining terms with powers of t^2 less than n. These are associated with nonleading terms that we have no interest in. (If A_n happens to be zero for the $\mathfrak{M}_n^{(\mu)}$ of a subamplitude, then this subamplitude $\mathfrak{M}_n$ does not contribute to the leading-unitary terms in the μ channel of the t channel.)

To summarize, we have shown that the leading-unitary term of $\mathfrak{M}_n^{(\mu)}$ in (3.5) comes from the contraction of $\Omega^{\{i\}}_{\{j\}}$ in (3.7) with invariant symmetric tensors $S^{(1)}_{k_1,\ldots,k_m,i_1,\ldots,i_n}$ and $S^{(2)}_{k_1,\ldots,k_m,j_1,\ldots,j_n}$ of (3.8). All other invariant tensors in the trace factors of (3.7) can be dropped.

On the other hand, had we started with the classical-source representation of the eikonal operator instead of (3.1), the only change in the above discussion is to replace matrices $t^{(2)}_a$ everywhere by vectors $Q^{(2)}_a$ [see discussion after (3.2)]. For example, the tensor product in (3.6) is not needed:

$$P^Q_{(\mu)} = \sum_{\{k\}} P^{(1)}_{(\mu);k_1,\ldots,k_m} P^Q_{(\mu);k_1,\ldots,k_m}. \quad (3.9)$$

In SU(2), the t-channel projection operator for the isospin-1 channel becomes $P^Q_3 \propto t^{(1)}_k Q^{(2)}_k$, so that $P^Q_{3;k} = Q^{(2)}_k$. Similarly, the last trace in (3.7) is unnecessary, so

$$\mathrm{Tr}(P^Q_{(\mu)}\mathfrak{M}_n) = (g^2)^n F_n\left(\frac{g^2 \ln s}{4\pi^2}\right) \times \Omega^{i_1,\ldots,i_n}_{j_1,\ldots,j_n} \mathrm{Tr}(P^{(1)}_{(\mu);k_1,\ldots,k_m} t^{(1)}_{i_1}\cdots t^{(1)}_{i_n}) \times P^Q_{(\mu);k_1,\ldots,k_m} Q^{(2)}_{j_1}\cdots Q^{(2)}_{j_n}. \quad (3.10)$$

Since the $Q^{(2)}_a$ are commuting vectors in group space, the last factor $P^{Q}_{(\mu);k_1,\ldots,k_m}Q^{(2)}_{j_1}\cdots Q^{(2)}_{j_n}$, being a product of $Q^{(2)}_a$'s, is, in fact, an invariant tensor symmetric in the indices $\{k_1,\ldots,k_m,j_1,\ldots,j_n\}$, just like $S^{(1)}_{k_1,\ldots,k_m,i_1,\ldots,i_n}$ of (3.8), or $S^{(2)}_{k_1,\ldots,k_m,j_1,\ldots,j_n}$. So in (3.10), this tensor $P^{Q}_{(\mu);\{k\}}Q^{(2)}_{j_1}\cdots Q^{(2)}_{j_n}$ projects out the term with the power $(t^2)^n$ in the same manner as the tensor $S^{(2)}_{\{k\},j_1,\ldots,j_n}$ does in (3.7), because they are both completely symmetric and $Q^{(2)}_aQ^{(2)}_a=t^2$. Furthermore, since the normalization of these two symmetric tensors are conveniently incorporated in Eq. (3.5), the leading-unitary term of $\mathfrak{M}^{(\mu)}_n$ is the same whichever one of these two tensors we use to contract with $\Omega^{\{i\}}_{\{j\}}$ and $S^{(1)}_{\{k\},\{i\}}$.

In other words, whether we use the representation (3.1) or the classical-source representation of the eikonal operator makes no difference to the leading-unitary terms generated in the scattering amplitude. So the classical-source representation is also consistent with the diagrammatic calculation for quark-quark scattering. An explicit verification has also been made in SU(2) Yang-Mills theory, using the results of Ref. 2.

This conclusion is easily generalized to the case of M quarks of hadron A scattering from N quarks of hadron B. The M quarks of hadron A, say, belong to the same group representation with matrices $t^{(A)}_a$. So the projection of the group-spin states of the quarks into the μ channel in the t channel (with m indices as before) involves the trace $\mathrm{Tr}(t^{(A)}_{k_1}\cdots t^{(A)}_{k_m}t^{(A)}_{i_1}\cdots t^{(A)}_{i_n})$, where the matrices $\{t^A_k\}$ come from the projection operator $P^{(A)}_{(\mu)}$. This is the same trace factor as in (3.7) for quark-quark scattering. For example, to project into the color-singlet channel of three-quark states in color SU(3), with two of the quarks each emitting one color gluon, we evaluate

$$\epsilon_{ijk}(t_\alpha)_{ii'}(t_\beta)_{jj'}\epsilon_{i'j'k}=-\mathrm{Tr}(t_\alpha t_\beta) \tag{3.11}$$

as shown in Fig. 2. So what we have said about quark-quark scattering applies directly to hadron-hadron scattering, by replacing (A) for (1) and (B) for (2) in the above discussion.

The classical quark-source representation of the eikonal operator in QCD is obtained from (2.4) by replacing $t^{(k)}_c$ by color group-spin vectors $Q^{(k)}_c$ (indices k denotes quarks in hadron B):

$$\chi(\vec{b}_\perp+\vec{x}_j+\vec{y}_k,T)=\chi_{ac}(\vec{b}_\perp+\vec{x}_j+\vec{y}_k,T)\times(-t^{(j)}_aQ^{(k)}_c). \tag{3.12}$$

The scattering amplitude so obtained by substituting (3.12) in (2.2) agrees with diagrammatic calculations of Sec. II. We shall turn to the analysis of the scattering amplitude in the next few sections.

IV. PARTIAL DIFFERENTIAL EQUATION FOR A GENERATING FUNCTIONAL FOR THE S MATRIX

The analysis of the elastic scattering amplitude $S(\vec{b},T)$ given by (2.3) that we shall present is based on the observation that S is a sort of Feynman history path integral. The rapidity variable $T=\ln s/2\pi$ plays the role of time. Indeed from (2.5), we see that the eikonal $\chi(\vec{b},T)$ at a specific T value, defined through $\chi_{ab}(\vec{b},T)$ of (2.4), is a function of $\chi(\vec{b}',T')$ for all rapidities T' less than T. A sum over histories is involved in evaluating $\chi(\vec{b},T)$, if T were the time.

In quantum mechanics, the Feynman path integral representation of the wave function can be studied by first deriving the Schrödinger equation for the wave function from the path integral. In the same manner, we shall show how to derive a partial differential equation for the scattering amplitude S based on the classical-source representation of the eikonal operator of Eq. (3.12). In the field theory case, this is a functional partial differential equation.[3]

$S(\vec{b},T)$ is not a convenient function to study. Instead, we define the generating functional for the S matrix to be

$$S[Q_c(\vec{b}');T]=\left\langle 0\left|\exp\left\{i\int\chi_c(\vec{b}',T)Q_c(\vec{b}')d^2\vec{b}'\right\}\right|0\right\rangle, \tag{4.1}$$

where we have introduced matrices $\chi_c(\vec{b}',T)$ in quark representation space as

$$\chi_c(\vec{b}',T)=\sum_{j=1}^{M}(-t^{(j)}_a)\chi_{ac}(\vec{b}'+\vec{x}_j,T) \tag{4.2}$$

with χ_{ac} satisfying (2.5). $Q_c(\vec{b}')$ are functions of $\vec{b}'_\perp$, and which transforms as vectors in group space. $S(\vec{b},T)$ can be recovered from $S[Q_c(\vec{b}');T]$ by setting

$$Q_c(\vec{b}')=Q^B_c(\vec{b}')=\sum_{k=1}^{N}Q^{(k)}_c\delta(\vec{b}'-\vec{b}-\vec{y}_k) \tag{4.3}$$

with $Q^{(k)}_c$ as the classical quark sources in (3.12):

$$S[Q_c(\vec{b}')=Q^B_c(\vec{b}');T]=S(\vec{b},T). \tag{4.4}$$

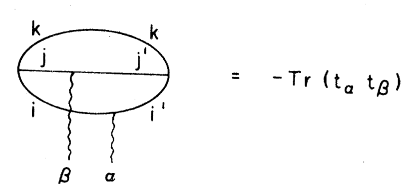

FIG. 2. Example of color-singlet projection of three-quark states [see Eq. (3.11)]. t_α is the representation matrix for the quarks.

We can interpret $Q_c(\vec{b}')$ as a vector source function, and $Q_c^B(\vec{b}')$ as the source function of hadron B, with classical, nonradiating quark sources $Q_c^{(k)}$ located at $\vec{b}+\vec{y}_k$, from the center of mass of hadron A.

Discussion in momentum space proved to be easier algebraically. To avoid having a proliferation of symbols, we shall use the same notation for a function in $\vec{b}$ space as its Fourier transform in $\vec{q}$ space, making a distinction only in the argument of the functions. For example,

$$\chi_c(\vec{b},T)=\int\frac{d^2\vec{q}}{(2\pi)^2}\,e^{i\vec{q}\cdot\vec{b}}\chi_c(\vec{q},T)\,, \tag{4.5a}$$

$$Q_c(\vec{b})=\int\frac{d^2\vec{q}}{(2\pi)^2}\,e^{i\vec{q}\cdot\vec{b}}Q_c(\vec{q}) \tag{4.5b}$$

so that

$$S[Q_c(\vec{b}');T]=S[Q_c(\vec{q});T]$$
$$=\left\langle 0\left|\exp\left\{i\int\chi_c(\vec{q},T)Q_c(\vec{q})\,\frac{d^2\vec{q}}{(2\pi)^2}\right\}\right|0\right\rangle. \tag{4.5c}$$

The integral equation for $\chi_c(\vec{b},T)$ is derived by contracting Eq. (2.5) with $(-t_a^{(j)})$, and then summing over j as in (4.2). This leads to the integral equation for the Fourier transform $\chi_c(\vec{q},T)$:

$$\chi_c(\vec{q},T)=g^2\chi_c^0(\vec{q},T)+\int_0^T dT'\int\frac{d^2\vec{q}'}{(2\pi)^2}\,\chi_a(\vec{q}',T')\times[V^{ab}(\vec{q}',\vec{q},T')+V_H^{ab}(T')]\chi_{bc}^0(\vec{q},T-T'), \tag{4.6}$$

where V^{ab} and V_H^{ab} are defined as in (2.8), (2.9), and

$$\chi_{bc}^0(\vec{q},T)=\delta_{bc}\,\frac{\exp\{[\alpha(\vec{q})-1]T\}}{\vec{q}^{\,2}+\lambda^2}\,, \tag{4.7}$$

$$\chi_c^0(\vec{q},T)=\sum_{j=1}^{M}(-t_a^{(j)})\chi_{ac}^0(\vec{q},T)e^{i\vec{q}\cdot\vec{z}_j}\,. \tag{4.8}$$

It must be remembered that $\chi_c(\vec{b},T)$, $\chi_c(\vec{q},T)$, and $\chi_c^0(\vec{q},T)$ are all matrices.

To derive a Schrödinger-type equation for the generating functional S of (4.1), we divide the rapidity T space into small regions and approximate integrals over T by sums, as in the quantum-mechanical case. This is to replace T by a one-dimensional lattice with lattice spacing ϵ, so that $T=N\epsilon$, with N an integer. The lattice version of $S[Q_c(\vec{q});T]$ is $S[Q_c(\vec{q});N]$, and similarly for other functions. By evaluating the difference

$$S[Q_c(\vec{q});N+1]-S[Q_c(\vec{q});N] \tag{4.9}$$

we obtain a difference equation, which would be a partial differential equation for S in the continuum limit $\epsilon\to 0$ with T fixed.

To evaluate the difference of (4.9), we first express $\chi_c(\vec{q},N+1)$ in terms of $\chi_c(\vec{q},N)$. This is obtained from the lattice version of (4.6):

$$\chi_c(\vec{q},N)=g^2\chi_c^0(\vec{q},N)+\sqrt{\epsilon}\sum_{l=0}^{N-1}\frac{d^2\vec{q}'}{(2\pi)^2}\,\chi_a(\vec{q}',l)\times[V^{ab}(\vec{q}',\vec{q},l+1)+V_H^{ab}(l+1)]\chi_{bc}^0(\vec{q},N-l-1)\,. \tag{4.10}$$

In (4.10), $V^{ab}(\vec{q}',\vec{q},l')$ and $V_H^{ab}(l')$ are defined as in (2.9), but with the creation and annihilation operators satisfying unit normalization in the lattice T-space sector:

$$[a_c^\alpha(\vec{k}',l'),a_d^{\beta\dagger}(\vec{k},l)]=\delta_{\alpha\beta}\delta_{cd}(2\pi)^2\delta(\vec{k}-\vec{k}')\delta_{ll'}\,, \tag{4.11}$$

where

$$a_c^\alpha(\vec{k},l)=\frac{1}{\sqrt{\epsilon}}\int_{l-1}^{l}\frac{dT'}{2\pi}\,a_c^\alpha(\vec{k},T')\,,$$
$$a_d^{\beta\dagger}(\vec{k},l)=\frac{1}{\sqrt{\epsilon}}\int_{l-1}^{l}\frac{dT'}{2\pi}\,a_d^{\beta\dagger}(\vec{k},T')\,. \tag{4.12}$$

Writing an equation for $\chi_c(\vec{q},N+1)$ similar to (4.10), then subtracting (4.10) from it, we obtain

$$\chi_c(\vec{q},N+1)-\chi_c(\vec{q},N)=g^2[\chi_c^0(\vec{q},N+1)-\chi_c^0(\vec{q},N)]+\sqrt{\epsilon}\sum_{l=0}^{N-1}\int\frac{d^2\vec{q}'}{(2\pi)^2}\,\chi_a(\vec{q}',l)[V^{ab}(\vec{q}',\vec{q},l+1)+V_H^{ab}(l+1)]\,[\chi_{bc}^0(\vec{q},N-l)-\chi_{bc}^0(\vec{q},N-l-1)]+\sqrt{\epsilon}\int\frac{d^2\vec{q}'}{(2\pi)^2}\,\chi_a(\vec{q}',N)[V^{ab}(\vec{q}',\vec{q},N+1)+V_H^{ab}(N+1)]\chi_{bc}^0(\vec{q},N-N)\,. \tag{4.13}$$

From the lattice version of (4.7) and (4.8), the differences $[\chi_c^0(\vec{q},N+1)-\chi_c^0(\vec{q},N)]$ and $[\chi_{bc}^0(\vec{q},N-l)-\chi_{bc}^0(\vec{q},N-l-1)]$ are linear functions of $\chi_c^0(\vec{q},N)$ and $\chi_{bc}^0(\vec{q},N-l-1)$, respectively. Using (4.10) again, and keeping terms up to order ϵ only, we arrive at

$$\chi_c(\vec{q},N+1)=\chi_c(\vec{q},N)+\sqrt{\epsilon}\int\frac{d^2\vec{q}'}{(2\pi)^2}\Sigma^{ac}(\vec{q},\vec{q}';N+1)\chi_a(\vec{q}',N)+\epsilon[\alpha(\vec{q})-1]\chi_c(\vec{q},N), \tag{4.14}$$

where

$$\Sigma^{ac}(\vec{q},\vec{q}';N+1)=\Sigma^{ac}(\vec{q},\vec{q}';N+1,N+1) \tag{4.15a}$$

is the Fourier transform of $\Sigma^{ac}(\vec{b},\vec{b}';T,T)$ of (2.8):

$$\Sigma^{ac}(\vec{q},\vec{q}';T,T')=[V^{bc}(\vec{q}',\vec{q},T')+V_H^{bc}(T')]\frac{\exp\{[\alpha(\vec{q})-1](T-T')\}}{\vec{q}^{\,2}+\lambda^2}. \tag{4.15b}$$

Putting $\chi_c(\vec{q},N+1)$ of (4.14) into the lattice version of (4.5c), we obtain

$$S[Q_c(\vec{q});N+1]=\left\langle 0\left|\exp\left[i\int\frac{d^2\vec{q}}{(2\pi)^2}Q_c(\vec{q})\left\{\chi_c(\vec{q},N)+\sqrt{\epsilon}\int\frac{d^2\vec{q}'}{(2\pi)^2}\Sigma^{ac}(\vec{q},\vec{q}';N+1)\chi_a(\vec{q}',N)+\epsilon[\alpha(\vec{q})-1]\chi_c(\vec{q},N)\right\}\right]\right|0\right\rangle. \tag{4.16}$$

The exponent in (4.16) is a sum of operator valued, noncommuting matrices in group space, so the explicit dependence of $S[Q_c(\vec{q});N+1]$ on $S[Q_c(\vec{q});N]$ is obscured.[10]

Fortunately, by using functional derivatives, (4.16) can be expressed explicitly in terms of $S[Q_c(\vec{q});N]$ as follows:

$$S[Q_c(\vec{q}'');N+1]=\left\langle 0\left|\left\{:\exp\left[\int\frac{d^2\vec{q}}{(2\pi)^2}Q_c(\vec{q})\left\{\sqrt{\epsilon}\int\frac{d^2\vec{q}'}{(2\pi)^2}\Sigma^{ac}(\vec{q},\vec{q}';N+1)\frac{\delta}{\delta Q_a(\vec{q}')}+\epsilon[\alpha(\vec{q})-1]\frac{\delta}{\delta Q_c(\vec{q})}\right\}\right]:\right\}\times\exp\left[i\int\frac{d^2\vec{q}''}{(2\pi)^2}Q_c(\vec{q}'')\chi_c(\vec{q}'',N)\right]\right|0\right\rangle. \tag{4.17}$$

We have introduced a normal-ordering prescription :[ ]:, which ensures that inside the square brackets, all functions $Q_c(\vec{q})$ stand to the left of all functional derivatives. Equation (4.17) can be shown to be equivalent to Eq. (4.16) by comparing the direct expansions of both expressions.[11]

On expanding the first exponential operator in (4.17), in powers of ϵ, we obtain a difference equation. Unlike the exponent in (4.16), this first exponent of (4.17) is no longer a matrix in group space, so the associated exponential can be easily expanded in powers of ϵ. It is only necessary to keep terms up to order ϵ, for our purpose of deriving a partial differential equation in the continuum limit. A further simplification is possible by making use of the fact that the vacuum state $|0\rangle$ is a tensor product of all the vacuum states $|0\rangle_l$ at each point l on the T-space lattice. So in (4.17), $\Sigma^{ac}(\vec{q},\vec{q}';N+1)$ operates only on the state $|0\rangle_{N+1}$, whereas $\chi_c(\vec{q}'',N)$ operates on all states $|0\rangle_l$ with $l\le N$, but not on $|0\rangle_{N+1}$. The difference equation takes the final form $({}_l\langle 0|0\rangle_l=1)$

$$S[Q_c(\vec{q}'');N+1]-S[Q_c(\vec{q}'');N]=\epsilon HS[Q_c(\vec{q}'');N], \tag{4.18}$$

where

$$H=\int\frac{d^2\vec{q}}{(2\pi)^2}[\alpha(\vec{q})-1]Q_c(\vec{q})\frac{\delta}{\delta Q_c(\vec{q})}+\tfrac{1}{2}\int\prod_{j=1}^{4}\frac{d^2\vec{q}_j}{(2\pi)^2}\langle 0|\Sigma^{ac}(\vec{q}_3,\vec{q}_1;N+1)\Sigma^{bd}(\vec{q}_4,\vec{q}_2;N+1)|0\rangle Q_c(\vec{q}_3)Q_d(\vec{q}_4)\frac{\delta^2}{\delta Q_a(\vec{q}_1)\delta Q_b(\vec{q}_2)}. \tag{4.19}$$

We have dropped the term multiplying $\sqrt{\epsilon}$, as $\Sigma^{ac}(\vec{q},\vec{q}';N+1)$ has zero vacuum-expectation value $\langle 0|\Sigma^{ac}(\vec{q},\vec{q}';N+1)|0\rangle=0$, being linear in the creation and annihilation operators $a_d^{\beta\dagger}(\vec{q}'-\vec{q},N+1)$ and $a_d^{\beta}(\vec{q}-\vec{q}';N+1)$ of (4.11) [see eqs. (2.8) and (2.9)].

From (4.15), (2.9), and (2.13) we obtain

$$\langle 0|\Sigma^{ac}(\vec{q}_3,\vec{q}_1;N+1)\Sigma^{bd}(\vec{q}_4,\vec{q}_2;N+1)|0\rangle = f_{ace}f_{bde}\langle\vec{q}_3,\vec{q}_4|\vec{q}_1,\vec{q}_2\rangle + \langle 0|V_H^{ac}(N+1)V_H^{bd}(N+1)|0\rangle\,, \tag{4.20}$$

where $\langle\vec{q}_3,\vec{q}_4|\vec{q}_1,\vec{q}_2\rangle$ is defined as

$$\langle\vec{q}_3,\vec{q}_4|\vec{q}_1,\vec{q}_2\rangle = g^2(2\pi)^2\delta^{(2)}(\vec{q}_3+\vec{q}_4-\vec{q}_1-\vec{q}_2)\frac{1}{(\vec{q}_3^{\,2}+\lambda^2)(\vec{q}_4^{\,2}+\lambda^2)}$$
$$\times\left\{\frac{(\vec{q}_1^{\,2}+\lambda^2)(\vec{q}_4^{\,2}+\lambda^2)+(\vec{q}_2^{\,2}+\lambda^2)(\vec{q}_3^{\,2}+\lambda^2)}{(\vec{q}_3-\vec{q}_1)^2+\lambda^2} - [(\vec{q}_1+\vec{q}_2)^2+\lambda^2] - \frac{\lambda^2}{2}\right\} \tag{4.21}$$

and

$$\langle 0|V_H^{ac}(N+1)\,V_H^{bd}(N+1)|0\rangle = O(\lambda^2)\,. \tag{4.22}$$

Hence

$$H = \int\frac{d^2\vec{q}}{(2\pi)^2}[\alpha(\vec{q})-1]Q_c(\vec{q})\frac{\delta}{\delta Q_c(\vec{q})}$$
$$+\tfrac{1}{2}\int\left[\prod_{j=1}^{4}\frac{d^2\vec{q}_j}{(2\pi)^2}\right]\langle\vec{q}_3,\vec{q}_4|\vec{q}_1,\vec{q}_2\rangle f_{ace}f_{bde}Q_c(\vec{q}_3)Q_d(\vec{q}_4)\frac{\delta^2}{\delta Q_a(\vec{q}_1)\delta Q_b(\vec{q}_2)} + O(\lambda^2)\,. \tag{4.23}$$

In the continuum limit, we obtain the desired functional partial differential equation for $S[Q_c(\vec{q}\,'');T]$:

$$\frac{\partial}{\partial T}S[Q_c(\vec{q}\,'');T] = HS[Q_c(\vec{q}\,'');T]\,. \tag{4.24}$$

The generating functional $S[Q_c(\vec{q}\,'');T]$ in this equation is still a matrix in the group-representation space of the quarks of hadron A. It is useful to consider the projection $S^{(\mu)}$ of the generating functional into the (μ) channel of the t channel, as in (3.5):

$$S[Q_c(\vec{q}\,'');T] = \sum_\mu S^{(\mu)}[Q_c(\vec{q}\,'');T]P^Q_{(\mu)}\,. \tag{4.25}$$

$P^c_{(\mu)}$ are t-channel projection operators defined in Eq. (3.9). Since $P^Q_{(\mu)}$ are not functions of $Q_c(\vec{q})$, but only of $Q_c^{(k)}$ of (4.3), $S^{(\mu)}$ satisfy similar partial differential equations as (4.24):

$$\frac{\partial}{\partial T}S^{(\mu)}[Q_c(\vec{q}\,'');T] = HS^{(\mu)}[Q_c(\vec{q}\,'');T]\,. \tag{4.26}$$

As in (3.5), $S^{(\mu)}$ are no longer matrices in group-representation space:

$$S^{(\mu)}[Q_c(\vec{q}\,'');T] = \frac{\mathrm{Tr}\{P^Q_{(\mu)}S[Q_c(\vec{q}\,'');T]\}}{\mathrm{Tr}\{P^Q_{(\mu)}P^Q_{(\mu)}\}}\,. \tag{4.27}$$

V. A SOLVABLE MODEL

We shall illustrate the discussion of the last section by a solvable model[12] based on the SU(2) group. It is the case in which $\vec{q}$ space (or similarly $\vec{b}$ space) consists of only one point, instead of a two-dimensional continuum.

Consider the scattering of two particles,[13] each being in the spin-j representation of SU(2) with matrix t_a ($t_a t_a = t^2\underline{1}$, so that Tr $\underline{1} = 2j+1$). The leading-unitary terms of this model are summarized by an eikonal form,

$$\mathfrak{M} \sim 2is[1-S(T)]\,, \tag{5.1}$$

where

$$S(T) = \langle 0|\exp[i\chi(T)]|0\rangle\,. \tag{5.2}$$

We consider the classical-source representation of the eikonal operator $\chi(T) = \chi_a(T)Q_a$, where Q_a is the classical-source vector in SU(2), and $\chi_a(T)$ satisfies the integral equation

$$\chi_c(T) = g^2 t_c \exp[(\alpha-1)T] + \int_0^T dT'\chi_a(T')V^{ac}(T')\times\exp[(\alpha-1)(T-T')]\,. \tag{5.3}$$

This is Eq. (4.6) in the case of a one-point $\vec{q}$ space. Here

$$V^{ac}(T') = ig\epsilon_{acd}[a_d^\dagger(T') - a_d(T')] \tag{5.4}$$

describes the creation and annihilation of vector mesons which are the pionization particles, to be compared with (2.9) in the case of continuum $\vec{q}$ space. The operators $a_d^\dagger(T')$ and $a_d(T')$ satisfy δ-function normalization $[a_d(T), a_d^\dagger(T')] = \delta(T-T')$. The vector mesons in this model lie on the Regge trajectory α, so in view of (2.7), we define

$$\alpha = 1 - g^2\beta\,. \tag{5.5}$$

Replacing T space by a one-dimensional lattice with lattice spacing ϵ, so that $T=N\epsilon$, we can derive[14] a partial differential equation for the generating functional for the S matrix, following the procedures of Sec. IV:

$$S[Q_c;T]=\langle 0|\exp[i\chi_c(T)Q_c]|0\rangle \tag{5.6}$$

so that

$$\frac{\partial}{\partial T}S[Q_c;T]=HS[Q_c;T] \tag{5.7}$$

with

$$H=-g^2\beta Q_c\frac{\partial}{\partial Q_c}+\tfrac{1}{2}g^2\epsilon_{ace}\epsilon_{bde}Q_cQ_d\frac{\partial^2}{\partial Q_a\partial Q_b}. \tag{5.8}$$

The first term in H reflects the Reggeization of the vector meson, while the second term in (5.8) is associated with the creation and annihilation of vector mesons as pionization particles. [Also see Eq. (4.23).]

We shall study the projection of the generating functional S into the various (μ) channels of the t channel. An appropriate choice for the t-channel projection operators in SU(2) is the set of irreducible spherical tensor operators[15] of rank l for the channel $\mu=l$. For the operator $P^{(1)}_{(l);k_1,\ldots,k_m}$ defined in (3.9), the irreducible tensor operators are formed from the group matrices t_k, whereas for $P^{Q}_{(l);k_1,\ldots,k_m}$ in (3.9), they are formed from vectors Q_k. We write these operators, respectively, as $P^{(1)}_{l,m}$ and $P^{Q}_{l,m}$, where $m=-l,-l+1,\ldots,0,\ldots,l$ labels each of the $(2l+1)$ irreducible tensors of rank l. An example is the triplet channel of the t channel, where

$$P^{(1)}_{1,0}=t_z,\quad P^{(1)}_{1,1}=-\frac{1}{\sqrt{2}}(t_x+it_y),\qquad P^{(1)}_{1,-1}=\frac{1}{\sqrt{2}}(t_x-it_y), \tag{5.9a}$$

$$P^{Q}_{1,0}=Q_z,\quad P^{Q}_{1,1}=-\frac{1}{\sqrt{2}}(Q_x+iQ_y),\qquad P^{Q}_{1,-1}=\frac{1}{\sqrt{2}}(Q_x-iQ_y) \tag{5.9b}$$

so that $t_kQ_k=\sum_m P^{(1)*}_{1,m}P^{Q}_{1,m}$ is the triplet projection operator.

We define

$$S_{l,m}[Q_c;T]=\frac{\mathrm{Tr}\{P^{(1)}_{l,m}S[Q_c;T]\}}{\mathrm{Tr}\{P^{(1)*}_{l,m}P^{(1)}_{l,m}\}} \tag{5.10}$$

so that

$$S[Q_c;T]=\sum_{l,m}P^{(1)*}_{l,m}P^{Q}_{l,m}\left\{\frac{\sum_{m'}P^{Q*}_{l,m'}S_{l,m'}[Q_c;T]}{\sum_{m''}P^{Q*}_{l,m''}P^{Q}_{l,m''}}\right\}. \tag{5.11}$$

The projected generating functional satisfies

$$\frac{\partial}{\partial T}S_{l,m}[Q_c;T]=HS_{l,m}[Q_c;T] \tag{5.12}$$

from (5.7).

The partial differential equation (5.12) can be solved exactly. It is convenient to change to vector notation $Q_c=(\vec{Q})_c$, $c=1,2,3$, so that

$$\begin{aligned}H&=g^2(1-\beta)Q_c\frac{\partial}{\partial Q_c}+\tfrac{1}{2}g^2\left(Q_c\epsilon_{ace}\frac{\partial}{\partial Q_a}\right)\left(Q_d\epsilon_{bde}\frac{\partial}{\partial Q_b}\right)\\&=g^2(1-\beta)\vec{Q}\cdot\vec{\nabla}_Q-\tfrac{1}{2}g^2\vec{L}_Q^{\,2},\end{aligned} \tag{5.13}$$

where $\vec{L}_Q$ is just the angular momentum operator in three-dimensional $\vec{Q}$ space [for a SU(2) classical Yang-Mills source]:

$$\vec{L}_Q=\frac{1}{i}\vec{Q}\times\vec{\nabla}_Q. \tag{5.14}$$

Since $\vec{Q}\cdot\vec{\nabla}_Q$ commutes with $\vec{L}_Q^{\,2}$, the solution to (5.12) and (5.13) is a product of a radial function R in $\vec{Q}$ space, and an angular function Θ, $(Q=|\vec{Q}|)$:

$$S_{l,m}[\vec{Q};T]=R[Qe^{g^2(1-\beta)T}]\,\Theta_{l,m}(\theta,\phi,T). \tag{5.15}$$

With the specific dependence of the radial function shown in (5.15), Eq. (5.12) becomes simply

$$\frac{\partial}{\partial T}\Theta_{l,m}=-\tfrac{1}{2}g^2\vec{L}_Q^{\,2}\Theta_{l,m}. \tag{5.16}$$

This equation can be solved easily when we take into account the initial condition for $S_{l,m}[\vec{Q};T]$. We show in Appendix C that

$$S_{l,m}[\vec{Q};T=0]=R_l[\vec{Q}]Y_{lm}(\theta,\phi), \tag{5.17}$$

$Y_{lm}(\theta,\phi)$ being the spherical harmonics in $\vec{Q}$ space [see Eq. (C9)]. Hence we can identify $\Theta_{l,m}(\theta,\phi,T=0)=Y_{lm}(\theta,\phi)$.

Therefore, the complete solution is

$$S_{l,m}[\vec{Q};T]=R_l[Qe^{g^2(1-\beta)T}]e^{-1/2g^2l(l+1)T}Y_{lm}(\theta,\phi), \tag{5.18}$$

with R_l defined in (5.17), which is evaluated from Eqs. (C8), (C9) of Appendix C and (5.10).

From the result (5.18), we can study the high-energy behavior of the various SU(2) t-channel exchange amplitudes. For instance, the vacuum-channel $(l=0)$ amplitude $\mathfrak{M}_0$, and the triplet $(l=1)$ channel amplitude $\mathfrak{M}_1$ is obtained from (5.1), (5.11), (5.17), and (5.18) as

$$\mathfrak{M}_0\sim 2is\left\{1-\frac{1}{\sqrt{4\pi}}R_0[Qs^{g^2(1-\beta)/2\pi}]\right\}, \tag{5.19}$$

$$\mathfrak{M}_1\sim -2is(3/4\pi)^{1/2}\frac{1}{Q}R_1[Qs^{g^2(1-\beta)/2\pi}]s^{-g^2/2\pi}t_k^{(1)}Q_k. \tag{5.20}$$

[We have used the relation $Y_{lm}=(3/4\pi)^{1/2}(1/Q)P^{Q}_{l,m}$, obtained from equation (5.9b), as well as $T=\ln s/2\pi$.]

In Appendix C, we have computed R_0 and R_1 to be [see equations (C11) and (C14), respectively]

$$\frac{1}{\sqrt{4\pi}}R_0[Q]=\frac{1}{2j+1}\times\begin{cases}1+2\sum_{n=1}^{j}\cos(ng^2Q), & j=\text{integer}\\ 2\sum_{n=1}^{j}\cos(ng^2Q), & j=\frac{1}{2}\text{ integer}\end{cases}\tag{5.21}$$

$$(3/4\pi)^{1/2}R_1[Q]=\frac{1}{2j+1}\sum_{\substack{n=1\\(n=1/2)}}^{j} in\sin(ng^2Q),\quad j=\text{integer}\ (\tfrac{1}{2}\text{ integer}).\tag{5.22}$$

For $\beta=1$, $\mathfrak{M}_0$ of (5.19) and $\mathfrak{M}_1$ of (5.20) exhibit Regge-pole behavior, with trajectories 1 and $1-g^2/2\pi$, respectively (the latter being the trajectory on which the exchanged vector-mesons lie). In fact, since R_l in (5.18) is no longer dependent on s (where $T=\ln s/2\pi$), the l-channel amplitude shows a Regge-pole behavior with trajectory $1-l(l+1)g^2/4\pi$. This result is also obtained in Ref. 12, in a different context and from different analytical considerations.

The case $\beta>1$ is similar to the previous case $\beta=1$. $\mathfrak{M}_0$ and $\mathfrak{M}_1$ have the same type of Regge-pole behavior as before (as $s\to\infty$, $Qs^{-g^{2}(\beta-1)/2\pi}\to 0$ and is only weakly s dependent). For $\beta<1$ however, $\mathfrak{M}_0$ and $\mathfrak{M}_1$ have in addition a rapidly oscillatory component of $\sum_n \cos(ng^2Qs^{g^2(1-\beta)/2\pi})$ and $\sum_n n\sin(ng^2Qs^{g^2(1-\beta)/2\pi})$, respectively.

VI. COLOR-SINGLET SCATTERING IN QCD

What we have presented so far in this paper is applicable to high-energy scattering regardless of what group spin is exchanged in the t channel. In this section, we shall focus our attention to the physical situation that hadrons are color singlets and that the exchanged color gluons are massless. The functional partial differential equation (4.24) or (4.26) summarizes the behavior of the S matrix as the center-of-mass energy s ($T=\ln s/2\pi$) increases. Hence we shall analyze this differential equation, concentrating on the vacuum exchange channel in the t channel (the singlet channel $\mu=1$). In addition, we shall display the infared convergence of our expressions in the zero-gluon-mass limit ($\lambda\to 0$).

A. Initial condition for the differential equation

From equations (4.5), (4.7), (4.8), and (4.27), the generating functional for the singlet t channel $S^{(\mu=1)}$ at $T=0$ is

$$S^{(1)}[Q_c(\vec{q}\,''); T=0]=\frac{1}{D_F}\operatorname{Tr}\left\{\exp\left[ig^2\sum_{j=1}^{M}\int\frac{d^2\vec{q}}{(2\pi)^2}\frac{e^{i\vec{q}\cdot\vec{x}_j}}{\vec{q}^{\,2}+\lambda^2}(-t_a^{(j)})Q_a(\vec{q})\right]\right\}\tag{6.1}$$

as $P_{(1)}^{Q}=\underline{1}$, the unit matrix, and D_F is the normalization of $P_{(1)}^{Q}$, $D_F=\operatorname{Tr}\underline{1}$, which is the dimension of the representation t_a. It is clear from (6.1) that in the limit of zero gluon mass $\lambda\to 0$, $S^{(1)}[Q_c(\vec{q}\,''); T=0]$ is infrared finite. This is so because the infrared divergence of the exponent inside the trace comes from the region of small $\vec{q}$, where $e^{i\vec{q}\cdot\vec{x}_j}]\sim 1$, so that we can first sum over $t_a^{(j)}$ in the exponent. For a color-singlet hadron, $\sum_{j=1}^{M}t_a^{(j)}$ has zero eigenvalue, hence killing the infrared divergence in the exponent.

We do not have to worry about the region $|\vec{x}|$ being very large, so that $e^{i\vec{q}\cdot\vec{x}_j}\neq 1$ even though $\vec{q}$ is small, for the impact factor $I_A(\Delta,\{\vec{x}_j\})$ of (2.2) which we eventually must convolute with, provides a cutoff for large $|\vec{x}|$. This is the statement that the quarks inside a hadron cannot get too far away from each other, owing to confinement dynamics.

B. Infrared finiteness of the generating functional

The coefficients of each of the two differential operators of H in (4.23) are separately infrared divergent in the zero-gluon-mass limit $\lambda\to 0$. We shall show that H admits a simple infrared regularization so that all expressions in H are infrared convergent, when H operates on the singlet projection of the generating functional $S^{(1)}[Q_c(\vec{q}\,''); T]$. In essence, the separate infrared-divergent expressions in H cancel when operating on $S^{(1)}[Q_c(\vec{q}\,''); T]$.

From the form of the initial condition ($T=0$) of $S^{(1)}[Q_c(\vec{q}\,''); T]$, and the form of the differential equation (4.23), we can deduce that $S^{(1)}[Q_c(\vec{q}\,''); T]$ can be expressed as [see (3.10) with all the quarks $j=1,\ldots,M$ in hadron A belonging to the same color-group representation]

$$S^{(1)}[Q_c(\vec{q}\,''); T]=\sum_{n,\Omega}\int\left[\prod_{l}^{n}\frac{d^2\vec{q}_l}{(2\pi)^2}\right]F_\Omega(\vec{q}_1,\ldots,\vec{q}_n; T)\Psi_\Omega[Q_{j_1}(\vec{q}_1),\ldots,Q_{j_n}(\vec{q}_n)],\tag{6.2a}$$

where

$$\Psi_\Omega[Q_{j_1}(\vec{q}_1),\ldots,Q_{j_n}(\vec{q}_n)]=Q_{j_1}(\vec{q}_1)\cdots Q_{j_n}(\vec{q}_n)\Omega^{i_1,\ldots,i_n}_{j_1,\ldots,j_n}\,\mathrm{Tr}\{t_{i_1}\cdots t_{i_n}\} \tag{6.2b}$$

and $\Omega^{\{i\}}_{\{j\}}$ is again some appropriate invariant tensor of the gauge group. Ψ_Ω satisfies the following eigenvalue relation, which we prove in Appendix D:

$$h\Psi_\Omega[Q_{j_1}(\vec{q}_1),\ldots,Q_{j_n}(\vec{q}_n)]=0, \tag{6.3a}$$

where

$$h=\tfrac{1}{2}\int\frac{d^2\vec{q}_1'}{(2\pi)^2}\frac{d^2\vec{q}_2'}{(2\pi)^2}\left[f_{ace}Q_c(\vec{q}_1')\frac{\delta}{\delta Q_a(\vec{q}_1')}\right]\left[f_{bde}Q_d(\vec{q}_2')\frac{\delta}{\delta Q_b(\vec{q}_2')}\right] \tag{6.3b}$$

$$=\left\{\int\frac{d^2\vec{q}_1'}{(2\pi)^2}\frac{d^2\vec{q}_2'}{(2\pi)^2}\left[\tfrac{1}{2}f_{ace}f_{bde}Q_c(\vec{q}_1')Q_d(\vec{q}_2')\frac{\delta^2}{\delta Q_a(\vec{q}_1')Q_b(\vec{q}_2')}\right]-C_A\int\frac{d^2\vec{q}'}{(2\pi)^2}Q_c(\vec{q}')\frac{\delta}{\delta Q_c(\vec{q}')}\right\} \tag{6.3c}$$

and C_A is the Casmir invariant $if_{abc}t_bt_c=C_At_a$ [$C_A=N/2$ in SU(N)]. Physically, h as defined in (6.3b) is proportional to the invariant quadratic Casmir operator in functional space (see the discussion at the end of Appendix D). Ψ of (6.2b) is a singlet state, just as $S^{(1)}$ of (6.2a) is, so naturally $h\Psi=0$.

Therefore, if we add to H an operator h_R of the following form:

$$h_R=R(\lambda)h, \tag{6.4}$$

then

$$\{H-h_R\}S^{(1)}[Q_c(\vec{q}'');T]=HS^{(1)}[Q_c(\vec{q}'');T] \tag{6.5}$$

from (6.2) and (6.3). The function $R(\lambda)$ we shall choose to be a function of the gluon mass λ, such that

$$\lim_{\lambda\to 0}\{H-h_R\}=\text{infrared-finite operator } H_R\,. \tag{6.6}$$

Using the operator H_R, defined in (6.6) in the zero gluon-mass limit, in place of H in the differential equation of (4.26) guarantees that $S^{(1)}[Q_c(\vec{q}'');T]$ is infrared finite, since it is infrared finite at $T=0$ from Sec. VI A.

We only need to exhibit the choice of $R(\lambda)$ which has the required property

$$R(\lambda)=2g^2\int\frac{d^2\vec{k}}{(2\pi)^2}\frac{\overline{R}(\vec{k})}{\vec{k}^2+\lambda^2}, \tag{6.7}$$

where $\overline{R}(\vec{k})$ is any function ensuring the convergence of the integral for finite λ, and such that $\overline{R}(0)=1$. For example, $\overline{R}(\vec{k})=\exp(-\vec{k}^2/\Lambda^2)$ with a regulator mass Λ. This leads to an infrared-finite operator [from (4.21) and (4.23)]:

$$H_R=\int\frac{d^2\vec{q}}{(2\pi)^2}[\alpha_R(\vec{q})-1]Q_c(\vec{q})\frac{\delta}{\delta Q_c(\vec{q})}$$
$$+\tfrac{1}{2}\int\left[\prod_{j=1}^{4}\frac{d^2\vec{q}_j}{(2\pi)^2}\right]\langle\vec{q}_3,\vec{q}_4|\vec{q}_1,\vec{q}_2\rangle_R\,Q_c(\vec{q}_3)Q_d(\vec{q}_4)f_{ace}f_{bde}\frac{\delta^2}{\delta Q_a(\vec{q}_1)\delta Q_b(\vec{q}_2)}, \tag{6.8}$$

where

$$\alpha_R(\vec{q})=1-g^2C_A\int\frac{d^2\vec{q}'}{(2\pi)^2}\frac{\vec{q}^2-(\vec{q}-\vec{q}')^2\overline{R}(\vec{q}')-\vec{q}'^2\overline{R}(\vec{q}-\vec{q}')}{\vec{q}'^2[(\vec{q}-\vec{q}')^2]} \tag{6.9}$$

and

$$\int\frac{d^2\vec{q}_3}{(2\pi)^2}\frac{d^2\vec{q}_4}{(2\pi)^2}\langle\vec{q}_3,\vec{q}_4|\vec{q}_1,\vec{q}_2\rangle_R Q_c(\vec{q}_3)Q_d(\vec{q}_4)$$
$$=\int\frac{d^2\vec{q}_3}{(2\pi)^2}\frac{d^2\vec{q}_4}{(2\pi)^2}\left\{\langle\vec{q}_3,\vec{q}_4|\vec{q}_1,\vec{q}_2\rangle Q_c(\vec{q}_3)Q_d(\vec{q}_4)-2g^2\frac{\overline{R}(\vec{q}_3-\vec{q}_1)}{(\vec{q}_3-\vec{q}_1)^2}Q_c(\vec{q}_1)Q_d(\vec{q}_2)\right\} \tag{6.10a}$$

$$= g^2 \int \frac{d^2\vec{q}_3}{(2\pi)^2} \left\{ \frac{1}{(\vec{q}_3 - \vec{q}_1)\vec{q}_3{}^2} [\vec{q}_1{}^2 Q_c(\vec{q}_3) Q_d(\vec{q}_1 + \vec{q}_2 - \vec{q}_3) - \vec{q}_3{}^2 \bar{R}(\vec{q}_3 - \vec{q}_1) Q_c(\vec{q}_1) Q_d(\vec{q}_2)] \right.$$

$$+ \frac{1}{(\vec{q}_3 - \vec{q}_1)^2 (\vec{q}_1 + \vec{q}_2 - \vec{q}_3)^2} [\vec{q}_2{}^2 Q_c(\vec{q}_3) Q_d(\vec{q}_1 + \vec{q}_2 - \vec{q}_3)$$

$$- (\vec{q}_1 + \vec{q}_2 - \vec{q}_3)^2 \bar{R}(\vec{q}_3 - \vec{q}_1) Q_c(\vec{q}_1) Q_d(\vec{q}_2)]$$

$$\left. - \frac{(\vec{q}_1 + \vec{q}_2)^2}{\vec{q}_3{}^2 (\vec{q}_1 + \vec{q}_2 - \vec{q}_3)^2} Q_c(\vec{q}_3) Q_d(\vec{q}_1 + \vec{q}_2 - \vec{q}_3) \right\} . \qquad (6.10b)$$

Possible divergences from the regions of small $\vec{q}'^2$, or $(\vec{q} - \vec{q}')^2$ in (6.9) and small $\vec{q}_3^2$, or $(\vec{q}_1 + \vec{q}_2 - \vec{q}_3)^2$, or $(\vec{q}_3 - \vec{q}_1)^2$ in (6.10) are explicitly canceled among the various terms in either expression.

C. Behavior of the singlet scattering amplitude with increasing energy

The scattering amplitude for color-singlet scattering is obtained from (2.2) by replacing $S(\vec{b}, \{\vec{x}_j\}, \{\vec{y}_k\}, T)$ with its projection $S^{(1)}(\vec{b}, \{\vec{x}_j\}, \{\vec{y}_k\}, T)$ into the singlet t channel. In momentum space, using (4.3) and (4.4),

$$S^{(1)}(\vec{b}, \{\vec{x}_j\}, \{\vec{y}_k\}, T) = S^{(1)}[Q_c(\vec{q}'') = Q_c^B(\vec{q}''); T], \qquad (6.11)$$

where

$$Q_c^B(\vec{q}) = \sum_{k=1}^{N} Q_c^{(k)} \exp[-i\vec{q}\cdot(\vec{b} + \vec{y}_k)] . \qquad (6.12)$$

At large impact distance, where there is little or no scattering, we expect from (2.2) that $S^{(1)}(\vec{b}, \{\vec{x}_j\}, \{\vec{y}_k\}, T)$ is close to unity. On the other hand, at small impact distance in the region of appreciable scattering, $S^{(1)}$ may be anything less than unity. The behavior of the total scattering cross section as energy increases [related to the imaginary part of (2.2) at $\Delta = 0$], depends directly on the change in extent of this region where $S^{(1)}$ differs appreciably from unity. We shall show that at fixed impact distance $\vec{b}$, $S^{(1)}$ decreases to zero as T increases, which means that the region of scattering grows with increasing energy as the target hadron becomes completely absorptive at high energies.

The behavior of $S^{(1)}(\vec{b}, \{\vec{x}_j\}, \{\vec{y}_k\}, T)$ is obtained through the infrared-finite generating functional $S^{(1)}[Q_c(\vec{q}''); T]$ which satisfies

$$\frac{\partial}{\partial T} S^{(1)}[Q_c(\vec{q}''); T] = H_R S^{(1)}[Q_c(\vec{q}''); T] \qquad (6.13)$$

with H_R given by (6.8), (6.9), and (6.10). We shall show that H_R is both negative definite and Hermitian. That H_R is negative definite means that the generating functional $S^{(1)}[Q_c(\vec{q}''); T]$ decreases with increasing T, from (6.13). From the Hermitian property of H_R, we are able to study features of its eigenfunctions to conclude that $S^{(1)}[Q_c(\vec{q}''); T]$ indeed decreases to zero as T becomes infinite. Such considerations in a model based on ϕ^3 theory, as well as in QED, have already been presented in Ref. (3).

First, we observe that H_R may be expressed as

$$H_R = \tfrac{1}{2} \int \left[\prod_{j=1}^{4} \frac{d^2\vec{q}_j}{(2\pi)^2} \right] \langle \vec{q}_3, \vec{q}_4 | \vec{q}_1, \vec{q}_2 \rangle_R \times \left[f_{ace} Q_c(\vec{q}_3) \frac{\delta}{\delta Q_a(\vec{q}_1)} \right] \times \left[f_{bde} Q_d(\vec{q}_4) \frac{\delta}{\delta Q_b(\vec{q}_2)} \right] . \qquad (6.14)$$

This is an explicitly Hermitian form due to the totally antisymmetric property of the structure constants f_{ace}, and the symmetry of $\langle \vec{q}_3, \vec{q}_4 | \vec{q}_1, \vec{q}_2 \rangle_R$:

$$\langle \vec{q}_3, \vec{q}_4 | \vec{q}_1, \vec{q}_2 \rangle_R = \langle \vec{q}_4, \vec{q}_3 | \vec{q}_2, \vec{q}_1 \rangle_R \qquad (6.15)$$

which is obtained directly from (6.10), with $\vec{q}_3 + \vec{q}_4 - \vec{q}_1 - \vec{q}_2 = 0$. To see that (6.14) follows from (6.8), we need only to show

$$\tfrac{1}{2} \int \left[\prod_{j=1}^{4} \frac{d^2\vec{q}_j}{(2\pi)^2} \right] \langle \vec{q}_3, \vec{q}_4 | \vec{q}_1, \vec{q}_2 \rangle_R f_{ace} \left[Q_c(\vec{q}_3) \frac{\delta}{\delta Q_a(\vec{q}_1)} Q_d(\vec{q}_4) \right] f_{bde} \frac{\delta}{\delta Q_b(\vec{q}_2)} = \int \frac{d^2\vec{q}}{(2\pi)^2} [\alpha_R(\vec{q}) - 1] Q_c(\vec{q}) \frac{\delta}{\delta Q_c(\vec{q})} . \qquad (6.16)$$

The equality results if we put into (6.16) the following definition of $\langle \vec{q}_3, \vec{q}_4 | \vec{q}_1, \vec{q}_2 \rangle_R$, obtained from (6.10b) by changing integration variables $\vec{q}_3 \to \vec{q}_3 + \vec{q}_1$ in the middle of the three terms in the curly brackets (permissible for convergent integrals):

$$\int \frac{d^2\vec{q}_3}{(2\pi)^2}\frac{d^2\vec{q}_4}{(2\pi)^2}\langle\vec{q}_3,\vec{q}_4|\vec{q}_1,\vec{q}_2\rangle_R Q_c(\vec{q}_3)Q_d(\vec{q}_4)$$

$$=g^2\int\frac{d^2\vec{q}_3}{(2\pi)^2}\left\{\frac{1}{(\vec{q}_3-\vec{q}_1)^2\vec{q}_3{}^2}[\vec{q}_1{}^2Q_c(\vec{q}_3)Q_d(\vec{q}_1+\vec{q}_2-\vec{q}_3)-\vec{q}_3{}^2\bar{R}(\vec{q}_3-\vec{q}_1)Q_c(\vec{q}_1)Q_d(\vec{q}_2)]\right.$$

$$+\frac{1}{\vec{q}_3{}^2(\vec{q}_2-\vec{q}_3)^2}[\vec{q}_2{}^2Q_c(\vec{q}_3+\vec{q}_1)Q_d(\vec{q}_2-\vec{q}_3)-(\vec{q}_2-\vec{q}_3)^2\bar{R}(\vec{q}_3)Q_c(\vec{q}_1)Q_d(\vec{q}_2)]$$

$$\left.-\frac{(\vec{q}_1+\vec{q}_2)^2}{\vec{q}_3{}^2(\vec{q}_1+\vec{q}_2-\vec{q}_3)^2}Q_c(\vec{q}_3)Q_d(\vec{q}_1+\vec{q}_2-\vec{q}_3)\right\}. \tag{6.17}$$

[(6.17) is used instead of (6.10b) because the latter introduces ambiguous differences of divergent integrals when evaluated in (6.16).] We also use the relation $f_{ace}f_{bae}=-2C_A\delta_{bc}$.

Next, we shall prove that all eigenvalues of H_R which contribute in the eigenfunction expansion of $S^{(1)}[Q_c(\vec{q}\,'');T]$ is negative definite. Let such a contributing normalized eigenfunction be $\Phi_W[Q_c(\vec{q}\,'')]$ with eigenvalue W. Then since $h_R S^{(1)}[Q_c(\vec{q}\,'');T]=0$ from (6.3), (6.4), we must also have

$$h_R\Phi_W[Q_c(\vec{q}\,'')]=0 \tag{6.18}$$

so that

$$H_R\Phi_W[Q_c(\vec{q}\,'')]=\lim_{\lambda\to 0}H\Phi_W[Q_c(\vec{q}\,'')] \tag{6.19}$$

from (6.6). (6.19) means that even though H does not have a limit as $\lambda\to 0$, the separately infrared-divergent expressions cancel when operating on $\Phi_W[Q_c(\vec{q}\,'')]$, and (6.19) has a limit as $\lambda\to 0$. Let us define

$$\lim_{\lambda\to 0}H\Phi_W[Q_c(\vec{q}\,'')]=H_0\Phi_W[Q_c(\vec{q}\,'')]. \tag{6.20}$$

In terms of H_0, the eigenvalue W is

$$W=\int[dQ]\Phi_W^*[Q_c(\vec{q}\,')]H_R\Phi_W[Q_c(\vec{q}\,'')] \tag{6.21a}$$

$$=\int[dQ]\Phi_W^*[Q_c(\vec{q}\,')]H_0\Phi_W[Q_c(\vec{q}\,'')]. \tag{6.21b}$$

We observe that a formal expression for H_0 can be obtained directly from H_R by setting $\bar{R}(\vec{k})$, defined in (6.7), to be zero. From (6.14) and (4.21), we have the formal expression

$$H_0=\tfrac{1}{2}\int\left[\prod_{j=1}^{4}\frac{d^2\vec{q}_j}{(2\pi)^2}\right]\langle\vec{q}_3,\vec{q}_4|\vec{q}_1,\vec{q}_2\rangle_0 \times\left[f_{ace}Q_c(\vec{q}_3)\frac{\delta}{\delta Q_a(\vec{q}_1)}\right]\times\left[f_{bde}Q_d(\vec{q}_4)\frac{\delta}{\delta Q_b(\vec{q}_2)}\right], \tag{6.22}$$

where

$$f_{ace}f_{bde}\langle\vec{q}_3,\vec{q}_4|\vec{q}_1,\vec{q}_2\rangle_0=f_{ace}f_{bde}\lim_{\lambda\to 0}\langle\vec{q}_3,\vec{q}_4|\vec{q}_1,\vec{q}_2\rangle \tag{6.23a}$$

$$=\lim_{\lambda\to 0}\langle 0|\Sigma^{ac}(\vec{q}_3,\vec{q}_1;N+1)\Sigma^{bd}(\vec{q}_4,\vec{q}_2;N+1)|0\rangle. \tag{6.23b}$$

(6.23b) follows from the definition (4.20), as $\lim_{\lambda\to 0}V_H\to 0$.

Using the explicit expressions (6.22) and (6.23) in (6.21b), we may finally express the eigenvalue W in a well-defined negative semi-definite form, after an integration by parts:

$$W=-\int[dQ]\lim_{\lambda\to 0}\langle 0|\kappa^\dagger\kappa|0\rangle\leq 0, \tag{6.24}$$

where

$$\kappa^\dagger=\int\frac{d^2\vec{q}_3}{(2\pi)^2}\frac{d^2\vec{q}_1}{(2\pi)^2}Q_c(\vec{q}_3)\Sigma^{ac}(\vec{q}_3,\vec{q}_1;N+1)\times\left\{\frac{\delta}{\delta Q_a(\vec{q}_1)}\Phi_W^*[Q_c(\vec{q}\,')]\right\}$$

and

$$\kappa=\int\frac{d^2\vec{q}_4}{(2\pi)^2}\frac{d^2\vec{q}_2}{(2\pi)^2}Q_d(\vec{q}_4)\Sigma^{bd}(\vec{q}_4,\vec{q}_2;N+1)\times\left\{\frac{\delta}{\delta Q_b(\vec{q}_2)}\Phi_W[Q_c(\vec{q}\,'')]\right\} \tag{6.25}$$

with Σ being Hermitian ($\Sigma^\dagger=\Sigma$). In (6.24), the equality holds if and only if Φ_W is a constant. A similar situation occurs in the ϕ^3-theory model and QED studied earlier.[3] Again it is possible to show that this eigenfunction does not contribute to the asymptotic behavior of $S^{(1)}[Q_c(\vec{q}\,'');T]$ as $T\to\infty$, so that $S^{(1)}[Q_c(\vec{q}\,'');T]\to 0$ in this limit.

To proceed as in Ref. 3, we replace the continuum $\vec{q}$ space by a two-dimensional lattice with

M lattice points. It is useful to define a radial variable[16] r and angular variables ω_j^c;

$$r^2=\sum_{c,j} Q_c(\vec{q}_j)Q_c(\vec{q}_j), \quad \omega_j^c=\frac{1}{r}Q_c(\vec{q}_j). \qquad (6.26)$$

In terms of these variables, $S^{(1)}$ at $T=0$ given by (6.1) can be written in Mellin-transform representation

$$S^{(1)}[Q_c(\vec{q}_l);T=0] = \int_{L-i\infty}^{L+i\infty}\frac{d\eta}{2\pi i}e^{i\pi\eta/2}\Gamma(\eta)\frac{1}{D_F} \times \mathrm{Tr}\{[rU\omega_l^c\chi_c^0(\vec{q}_l,T=0)U^{-1}]^{-\eta}\} \qquad (6.27)$$

using (4.7) and (4.8). In (6.27), we have defined a matrix U in group-representation space which diagonalizes the matrix $\omega_l^c\chi_c^0(\vec{q}_l,T=0)$.

It is easy to check that the lattice version of H_R commutes with

$$r\frac{\partial}{\partial r}=\sum_{c,j}Q_c(\vec{q}_j)\frac{\partial}{\partial Q_c(\vec{q}_j)},$$

the generator for scale transformations, so that the eigenfunctions of H_R are of the form $r^{-\eta}F(\omega_l^c)$. Hence we define $\mathcal{H}_R(\eta)$ operating on angular functions $F(\omega_l^c)$ as

$$H_R r^{-\eta}F(\omega_l^c)=r^{-\eta}\mathcal{H}_R(\eta)F(\omega_l^c). \qquad (6.28)$$

It is only necessary to discuss a complete set of these eigenfunctions useful in the discussion of the differential equation (6.13) with initial condition (6.27). For this purpose, we choose $\eta=D_AM/2+ip$ (p real, $-\infty<p<\infty$), where D_A is the number of generators in the gauge group, owing to the following consideration. ω_l^c are the angular variables of a D_AM-dimensional space, so from (6.8) and $H_R^\dagger=H_R$, it can be shown that $\mathcal{H}_R(\eta)=\mathcal{H}_R(D_AM-\eta^*)$ (the differential-volume element in this space being $d^{(D_AM)}r\Pi d\omega_l^c$). Hence $\mathcal{H}_R(D_AM/2+ip)$ is Hermitian, having real eigenvalues and a complete set of eigenfunctions in the angular sector. In the radial sector, the eigenfunctions $r^{-1/2D_AM-ip}$ also form a complete set, with δ-function normalization. Furthermore, that such eigenfunctions have radial derivatives proportional to M implies that the real eigenvalues W of (6.24) approach $-\infty$ as $M\to\infty$.

To express (6.27) in terms of these eigenfunctions, we move the contour L to the line $\mathrm{Re}\eta=\frac{1}{2}D_AM$. Then the solution to the lattice version of the differential equation (6.13) is given in the Laplace-transform representation as

$$S^{(1)}[Q_c(\vec{q}_l);T]=\int_{\delta-i\infty}^{\delta+i\infty}\frac{dw}{2\pi i}e^{wT}\tilde{S}^{(1)}[Q_c(\vec{q}_l);w] \qquad (6.29)$$

with

$$\tilde{S}^{(1)}[Q_c(\vec{q}_l);w]=-\int_{-\infty}^{\infty}\frac{dp}{2\pi}(re^{-i\pi/2})^{-1/2D_AM-ip}\Gamma(D_AM/2+ip)\frac{1}{D_F} \times[\mathcal{H}_R(D_AM/2+ip)-w]^{-1}\mathrm{Tr}\{[U\omega_l^c\chi_c^0(\vec{q}_l,T=0)U^{-1}]^{-1/2D_AM-ip}\}. \qquad (6.30)$$

As in the ϕ^3 model and QED studied earlier,[3] the Laplace transform of the generating functional $\tilde{S}^{(1)}[Q_c(\mathrm{q}_l);w]$ is an entire function of w in the continuum limit $M\to\infty$, since all singularities at the locations of the eigenvalues of $\mathcal{H}_R$ are at infinity in the complex w plane. Therefore, we can integrate along the imaginary axis in (6.29), where w is pure imaginary, and conclude that

$$S^{(1)}[Q_c(\vec{q}\,'');T]\to 0 \quad \text{as } T\to\infty \qquad (6.31)$$

due to the rapid oscillations in the integral (6.29).

VII. CONCLUSIONS AND PHENOMENOLOGICAL ASPECTS

We have concluded in the last section that for color-singlet scattering in QCD,[17] the region of scattering in impact-distance space grows with increasing energy as the target hadron becomes completely absorptive at very high energies. This picture of high-energy scattering is derived from the operator eikonal form of equations (2.2) and (2.3), using a classical-source representation of the eikonal operator. Unlike other operator eikonal forms proposed by assumption,[18] Eqs. (2.2) and (2.3) result from high-energy perturbative calculations in QCD reported earlier in this paper.

The perturbative results presented, however, do not yet establish Eqs. (2.2) and (2.3) beyond a doubt. First, the calculations must be extended to all perturbative orders. Second and more significantly, in these calculations we must go beyond the leading-unitary terms discussed in this paper. It is important to see if indeed the nonleading terms do not destroy the leading behavior that we have derived, but merely modify the functions in the functional form of Eqs. (2.2) and (2.3) in a perturbative fashion, as we have argued in Secs. II and III. Owing to inherent difficulties, pursuing higher-order calculations and extending calculational schemes to address nonleading terms are not the best ways to proceed. Rather, deriving the eikonal form of (2.2) and (2.3) nonperturbatively will settle our doubts, and promises to shed light on the important physics responsible for such a result. The physical understanding gained may even provide a simple physical explanation for the emergence of a Reggeon in the eikonal operator in non-Abelian gauge theories, and the totally absorptive behavior of the singlet scattering in QCD.[19] Investigation in QCD along the lines of the nonperturbative derivation of the operator eikonal form in QED[7] is under way.

From an experimental standpoint, much research is needed to extend our discussion to inelastic scattering. For elastic scattering and the high-energy behavior of the total scattering cross section, we have at least arrived at a qualitative picture. More specifically, although we have been able to show that the singlet scattering S matrix $S^{(1)}[Q_c(\vec{q}\,''); T]$ vanishes as T becomes infinite, we do not yet know the rate at which it vanishes. This rate determines how fast the total cross section rises with increasing s, as well as other features of the elastic scattering amplitude near the forward direction, and hence can be tested against present and future experimental data. A possibly fruitful direction to determine this rate[3] for QCD is the computer evaluation of $S^{(1)}$, treating both $\vec{q}$ space and T space as discrete lattice spaces.[3,20] Preliminary results are encouraging.[21]

Finally, even though both QED and QCD exhibit a totally absorptive behavior at high energies, the rates at which total absorption are approached in the two theories would be different. That the rates are different is at least true in the simple though unrealistic case in which $\vec{q}$ space consists of only one point instead of a two-dimensional continuum. In this case, the solvable non-Abelian SU(2) model of Sec. V yields a constant total cross section for singlet scattering, whereas a similar solvable Abelian QED-type model[22] is associated with a total cross section which decreases as a negative power of s.[20] Phenomenologically, it is important to exhibit the quantitative differences between the high-energy behavior of QED and QCD. These predictions could serve as other tests of the non-Abelian nature of hadron dynamics.

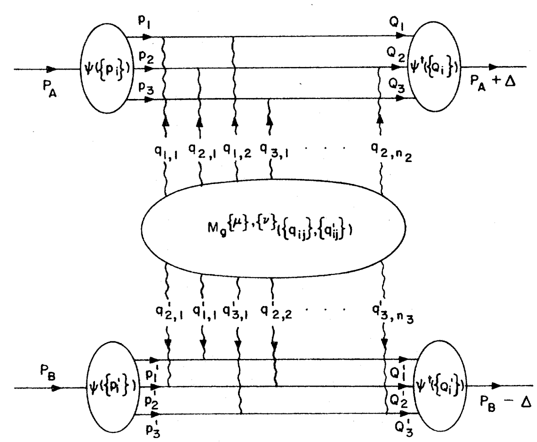

FIG. 3. A general Feynman diagram for hadron-hadron scattering, incorporating hadronic wave functions. $\vec{q}_{ij}$ refers to the momentum of the jth gluon attached to the ith quark. [See Appendix A where ψ and M_g are defined, as in Eq. (A7).]

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APPENDIX A: INCORPORATION OF HADRON WAVE FUNCTIONS

In the infinite-momentum center-of-mass frame for two colliding hadrons, the plus and minus momentum components are $p_\pm = p_t \pm p_z$. Let us focus on the hadron with large plus momentum P_+. It is made up of M color quarks, each of which carries fractional plus momentum $p_{i+} = \beta_i P_+$, and exchanges n_i gluons with quarks from the other hadron. In the Feynman scattering amplitude, each quark line contributes a factor $N_i D_i(n_i)$, to be convoluted with the hadron wave function. Here

$$N_i = (i)\not{p}_i \gamma_{\mu_1}(\not{p}_i + \not{q}_{i1} + m)\gamma_{\mu_2}(\not{p}_i + \not{q}_{i1} + \not{q}_{i2} + m)\gamma_{\mu_3}\cdots\gamma_{\mu_{n_i}}\left(\not{p}_i + \sum_{j=1}^{n_i}\not{q}_{ij} + m\right)(t_{\alpha_1}\cdots t_{\alpha_{n_i}}) \tag{A1}$$

and

$$D_i(n_i) = [p_i^2 - m^2]^{-1}[(p_i + q_{i1})^2 - m^2]^{-1}\cdots\left[\left(p_i + \sum_{j=1}^{n_i} q_{ij}\right)^2 - m^2\right]^{-1} \tag{A2}$$

as in Fig. 3.

For large $p_{i+}, D_i(n_i)$ can be approximated as

$$D_i(n_i) = (p_{i+})^{-n_i+1}(p_{i+}p_{i-} - \vec{p}_{i\perp}^{\,2} - m^2 + i\epsilon)^{-1}(p_{i+}Q_{i-} - \vec{Q}_{i\perp}^{\,2} - m^2 + i\epsilon)^{-1}\left[(p_{i-} + q_{i1-} + i\epsilon)^{-1}\cdots\left(p_{i-} + \sum_{j=1}^{n_i-1} q_{ij-} + i\epsilon\right)^{-1}\right], \tag{A3}$$

where $Q_i = (p_i + \sum_{j=1}^{n_i} q_{ij})$ and $Q_{i+} \sim p_{i+}$, assuming $q_{ij+} \sim 0$ small. The product in square brackets $[\cdots]$ above has the following Fourier representation[2,23]:

$$2\pi\delta\left(p_{i-} + \sum_{j=1}^{n_i} q_{ij-} - Q_i\right)G(p_{i-}, q_{i-}, \ldots, q_{i(n-1)-}) = 2\pi\delta\left(p_{i-} + \sum_{j=1}^{n_i} q_{ij-} - Q_i\right)\left[(p_{i-} + q_{i1-} + i\epsilon)^{-1}\cdots\left(p_{i-} + \sum_{j=1}^{n_i-1} q_{ij-} + i\epsilon\right)^{-1}\right]$$

$$= \int \exp\left(-i\sum_{j=1}^{n_i} q_{ij-}\xi_j - ip_{i-}\xi_1 + iQ_i\xi_{n_i}\right)(\xi_1, \ldots, \xi_n)\prod_{j=1}^{n_i} d\xi_j, \tag{A4}$$

where

$$(\xi_1, \ldots, \xi_n) = \begin{cases} (-i)^{n-1} & \text{if } \xi_n > \xi_{n-1} > \cdots > \xi_1 \\ 0 & \text{otherwise.} \end{cases} \tag{A5}$$

This representation can be proved by a series of contour integrations (temporarily suppressing all i index):

$$\int \exp\left(i\sum_{j=1}^{n} q_{j-}\xi_j\right)(2\pi)\delta\left(p_- + \sum_{j=1}^{n} q_{j-} - Q\right)G(p_-, q_{1-}, \ldots, q_{(n-1)-})\prod_{j=1}^{n}\frac{dq_{j-}}{(2\pi)}$$

$$= \exp(iQ\xi_n)\int \exp\left[i\sum_{j=1}^{n-1} q_{j-}(\xi_j - \xi_n) - ip_-\xi_n\right]G(p_-, q_{1-}, \ldots, q_{(n-1)-})\prod_{j=1}^{n-1}\frac{dq_{j-}}{(2\pi)}$$

$$= \exp(iQ\xi_n)\int \exp\left[i\sum_{j=1}^{n-2} q_{j-}(\xi_n - \xi_{n-1}) - ip_-\xi_{n-1}\right]G(p_-, q_{1-}, \ldots, q_{(n-2)-})(\xi_{n-1}, \xi_n)\prod_{j=1}^{n-2}\frac{dq_{j-}}{(2\pi)}$$

$$= \cdots$$

$$= \exp(-ip_-\xi_1 + iQ\xi_n) \times (\xi_1, \ldots, \xi_n). \tag{A6}$$

We are in a position to convolute the factors from the quark lines with the hadron wave function $\psi(\{p_i\})$, defined with truncated external propagators. We shall only consider fixed momentum transfer between the colliding hadrons, such that $\sum_{ij} q_{ij} = \Delta$, with $\Delta_+ = \Delta_- = 0$. The Feynman scattering amplitude is a convolution of three factors $\mathfrak{M}_A(\{q_{ij}\})$, $\mathfrak{M}_B(\{q'_{kl}\})$, coming from hadron wave functions and quark-line propagators of hadron A and B, and $M_g(\{q_{ij}\}, \{q'_{kl}\})$ describing gluon interactions after being emitted from the various quarks, as in Fig. (3).

$$\mathfrak{M} = \int (2\pi)^4\delta^{(4)}\left(\sum_{i,j} q_{ij} - \Delta\right)\sum_{i,j}\frac{d^4q_{ij}}{(2\pi)^4}\frac{d^4q'_{kl}}{(2\pi)^4}\mathfrak{M}_A(\{q_{ij}\})M_g(\{q_{ij}\}, \{q'_{kl}\})\mathfrak{M}_B(\{q'_{kl}\}). \tag{A7a}$$

For hadron A, we define the factor $\mathfrak{M}_A(\{q_{ij}\})$ as

$$2\pi\delta\left(\sum_{i,j} q_{ij-}\right)\mathfrak{M}_A(\{q_{ij}\}) = (2\pi)\delta\left(\sum_{i,j} q_{ij-}\right)\int (2\pi)^8\delta^{(4)}\left(\sum_{i=1}^{M} p_i - p\right)\delta^{(4)}\left(p_i - \sum_{j=1}^{n_i} q_{ij} - Q_i\right)$$

$$\times \psi(\{p_i\})N_iD_i(n_i)\psi^\dagger(\{Q_i\})\prod_{i=1}^{M}\frac{d^4p_i}{(2\pi)^4}\frac{d^4Q_i}{(2\pi)^4}, \tag{A7b}$$

with $N_iD_i(n_i)$ coming from the quark-line propagators.

Consider first the integration over all minus momenta p_{i-}, Q_{i-}. We define the appropriate piece $\mathfrak{N}_1$ in the integral of (A7b):

$$\mathfrak{M}_1=\int(2\pi)^3\delta\left(\sum_{i=1}^{M}p_{i-}\right)\delta\left(p_{i-}+\sum_{j=1}^{n_i}q_{ij-}-Q_{i-}\right)\delta\left(\sum_i Q_{i-}\right)\psi(\{p_i\})N_i\psi^\dagger(\{Q_i\})$$
$$\times\prod_{i=1}^{M}\left[\frac{dp_i}{2\pi}\frac{dQ_{i-}}{2\pi}(i)(p_{i+})^{-n_i+1}G(p_{i-},q_{i1-},\ldots,q_{i(n-1)-})\right.$$
$$\left.\times(p_{i+}p_{i-}-\vec{p}_{i\perp}^{\,2}-m^2+i\epsilon)^{-1}(p_{i+}Q_{i-}-\vec{Q}_{i\perp}^{\,2}-m^2+i\epsilon)^{-1}\right]. \tag{A8}$$

Using the Fourier representation of (A4), this is equal to

$$\mathfrak{M}_1=\int(2\pi)\delta\left(\sum_{i=1}^{M}p_{i-}\right)\psi(\{p_i\})\prod_{i=1}^{M}\left[\frac{dp_{i-}}{2\pi}\exp(-ip_{i-}\xi_1^i)(p_{i+}p_{i-}-\vec{p}_{i\perp}^{\,2}-m^2+i\epsilon)^{-1}\right]$$
$$\times(i)^M\sum_{i=1}^{M}\left[N_i(p_i)^{-n_i+1}\exp\left(-i\sum_{j=1}^{n_i}q_{ij-}\xi_j^i\right)(\xi_1^i,\ldots,\xi_{n_i}^i)\prod_{j=1}^{n_i}d\xi_j^i\right]$$
$$\times(2\pi)\delta\left(\sum_{i=1}^{M}Q_{i-}\right)\psi^\dagger(\{Q_i\})$$
$$\times\prod_{i=1}^{M}\left[\frac{dQ_{i-}}{2\pi}\exp(iQ_{i-}\xi_{n_i}^i)(p_{i+}Q_{i-}-\vec{Q}_{i\perp}^{\,2}-m^2+i\epsilon)^{-1}\right]. \tag{A9}$$

We can now integrate over all p_{i-} easily if we assume the singularities in p_{i-} of the wave function $\psi(\{p_i\})$ do not contribute to the integral. Again we define the relevant piece $\mathfrak{M}_2$ of the integral of (A9):

$$\mathfrak{M}_2=\int2\pi\delta\left(\sum_{i=1}^{M}p_{i-}\right)\psi(\{p_i\})\prod_{i=1}^{M}\left[\frac{dp_{i-}}{2\pi}\exp(-ip_{i-}\xi_1^i)(p_{i+}p_{i-}-\vec{p}_{i\perp}^{\,2}-m^2+i\epsilon)^{-1}\right]$$
$$=\int\psi(\{p_i\})\prod_{i=1}^{M-1}\left\{\frac{dp_{i-}}{2\pi}\exp[-ip_{i-}(\xi_1^i-\xi_1^M)]\right.$$
$$\left.\times(p_{i+}p_{i-}-\vec{p}_{i\perp}^{\,2}-m^2+i\epsilon)^{-1}\left(-p_{M+}\sum_{i=1}^{M-1}p_{i-}-\vec{p}_{M\perp}^{\,2}-m^2+i\epsilon\right)^{-1}\right\}. \tag{A10}$$

Since all p_{i+} is positive and large, in each p_{i-} integration, we can close the contour in the upper half plane if $\xi_1^M\geq\xi_1^i$, and in the lower half plane if $\xi_1^M<\xi_1^i$. In either case we obtain the same result. Hence we have

$$\mathfrak{M}_2=\psi(\{p_i\})(-i)^{M+1}(-1)\left[\left(\prod_{i=1}^{M}p_{i+}\right)\left(\sum_{j=1}^{M}\frac{\vec{p}_{j\perp}^{\,2}+m^2}{p_{j+}}\right)\right]^{-1}. \tag{A11}$$

The Q_{i-} integration is done similarly and we have

$$\int2\pi\delta\left(\sum_{i=1}^{M}Q_{i-}\right)\psi^\dagger(\{Q_i\})\prod_{i=1}^{M}\left[\frac{dQ_{i-}}{2\pi}\exp(iQ_{i-}\xi_{n_i}^i)(p_{i+}Q_{i-}-\vec{Q}_{i\perp}^{\,2}-m^2+i\epsilon)^{-1}\right]$$
$$=\psi^\dagger(\{Q_i\})(i)^{M-1}(-1)\left[\left(\prod_{i=1}^{M}p_{i+}\right)\left(\sum_{j=1}^{M}\frac{\vec{Q}_{j\perp}^{\,2}+m^2}{p_{j+}}\right)\right]^{-1}. \tag{A12}$$

Since the main contribution comes from the region p_{i-} of order $1/P_+$, we can neglect p_i^2, $p_i\cdot q_{ij}$ compared with P_+. Hence we can approximate N_i by $(p_i)^{n_i}p_i\delta_{\mu_1+}\delta_{\mu_2+}\cdots\delta_{\mu_i+}\times$ (group-spin factor).

Finally, we obtain (in terms of longitudinal momentum fractions β_i)

$$\delta\left(\sum_{i,j}q_{ij-}\right)\mathfrak{M}_A(\{q_{ij}\})=P_+\left\{\int\delta\left(1-\sum_{i=1}^{M}\beta_i\right)\delta^{(2)}\left(\sum_{i=1}^{M}\vec{p}_{i\perp}-\vec{P}_\perp\right)\prod_{i=1}^{M}\left[d\beta_i\frac{d^2\vec{p}_{i\perp}d^2\vec{Q}_{i\perp}}{(2\pi)^2(2\pi)^2}\delta^{(2)}\left(\vec{p}_{i\perp}+\sum_{j=1}^{n_i}\vec{q}_{ij\perp}-\vec{Q}_{i\perp}\right)\right]\psi(\{\vec{p}_{i\perp}\},\{\beta_i\})\right.$$
$$\left.\times\prod_{i=1}^{M}(i\not{p}_i)\psi^\dagger(\{\vec{Q}_{i\perp}\},\{\beta_i\})\left[\left(\prod_{i=1}^{M}\beta_i\right)\sum_{j=1}^{M}\frac{\vec{p}_{j\perp}^{\,2}+m^2}{\beta_j}\right]^{-1}\left[\left(\prod_{i=1}^{M}\beta_i\right)\sum_{j=1}^{M}\frac{\vec{Q}_{j\perp}^{\,2}+m^2}{\beta_j}\right]^{-1}\right\}$$
$$\times\prod_{i=1}^{n_i}G(q_{i1-},\ldots,q_{in_i-})\delta_{\mu_1+}\delta_{\mu_2+}\cdots\delta_{\mu_{n_i}+}\times\text{(group-spin factor)}, \tag{A13}$$

where

$$G(q_1,\ldots,q_n)=\int\left(\prod_{j=1}^{n}d\xi_j\right)(\xi_1,\ldots,\xi_n)\exp\left(-i\sum_{j=1}^{n}q_j\xi_j\right)$$
$$=\delta\left(\sum_{j=1}^{n}q_j\right)(q_1+i\epsilon)^{-1}(q_1+q_2+i\epsilon)^{-1}\cdots\left(\sum_{j=1}^{n-1}q_j+i\epsilon\right)^{-1} \tag{A14}$$

is the Fourier transform of $(\xi_1,\ldots,\xi_n)$.

The integral inside the large curly brackets { } in (A13) in fact is the impact factor $\mathcal{I}_A(\{q_{i\perp}\})$ of hadron A, where we have defined $\vec{q}_{i\perp}=\sum_{j=1}^{n_i}\vec{q}_{ij\perp}$:

$$\delta\left(\sum_{i,j}q_{ij-}\right)\mathfrak{M}_A(\{q_{ij}\})=P_{A+}\mathcal{I}_A(\{\vec{q}_{i\perp}\})\prod_{i=1}^{M}[G(q_{i1-},\ldots,q_{in_i-})\delta_{\mu_1+}\cdots\delta_{\mu n_i+}]\times(\text{group-spin factor}). \tag{A15}$$

The factor $\mathfrak{M}_B(\{q'_{kl}\})$ associated with hadron B can be similarly evaluated (noting that $\sum_{ij}q_{ij}=\Delta=-\sum_k q'_{kl}$, and defining $\vec{q}'_{k\perp}=\sum_{l=1}^{n_l}\vec{q}'_{kl\perp}$):

$$\delta\left(\sum_{k,l}q'_{kl+}\right)\mathfrak{M}_B(\{q'_{kl}\})=P_{B-}\mathcal{I}_B(\{\vec{q}'_{k\perp}\})\prod_{k=1}^{N}[G(q'_{kl+},\ldots,q'_{kn_k+})\times\delta_{\nu_1-}\cdots\delta_{\nu_{n_k}-}]\times(\text{group-spin factor}). \tag{A16}$$

So from Eq. (A7),

$$\mathfrak{M}=\int\delta^{(2)}\left(\sum_i\vec{q}_{i\perp}-\vec{\Delta}\right)\prod_{i,k}\frac{d^2\vec{q}_{i\perp}}{(2\pi)^2}\frac{d^2\vec{q}'_{k\perp}}{(2\pi)^2}\mathcal{I}_A(\{\vec{q}_{i\perp}\})\mathfrak{M}_0(\{\vec{q}_{i\perp}\},\{\vec{q}'_{k\perp}\})\mathcal{I}_B(\{\vec{q}'_{k\perp}\}), \tag{A17}$$

where $\mathfrak{M}_0(\{\vec{q}_{i\perp}\},\{\vec{q}'_{k\perp}\})$ is just the high-energy scattering matrix of the $M+N$ quarks with truncated external legs (writing $s=P_{1+}P_{2-}$):

$$\mathfrak{M}_0(\{\vec{q}_{i\perp}\},\{\vec{q}'_{k\perp}\})=s\int\left[\prod_{i=1}^{M}\delta^{(2)}\left(\vec{q}_{i\perp}-\sum_{j=1}^{n_i}\vec{q}_{ij}\right)\prod_{j=1}^{n_i}\frac{d^4q_{ij}}{(2\pi)^4}\right]$$
$$\times\left[\prod_{k=1}^{N}\delta^{(2)}\left(\vec{q}'_{k\perp}-\sum_{l=1}^{n_l}\vec{q}'_{kl\perp}\right)\prod_{l=1}^{n_l}\frac{d^4q'_{kl}}{(2\pi)^4}\right]\left[\prod_{i=1}^{M}G(q_{i1-},\ldots,q_{in_i-})(\delta_{\mu_1+}\cdots\delta_{\mu_{n_i}+})\right]$$
$$\times(\text{group-spin factors }M_g^{\{\mu\}\{\nu\}}(\{q_{ij}\},\{q'_{kl}\})\left[\prod_{k=1}^{N}G(q'_{k1+},\ldots,q'_{kn_k+})(\delta_{\nu_1-}\cdots\delta_{\nu_{n_k}-})\right]. \tag{A18}$$

The impact-distance representation of Eq. (A17) is

$$\mathfrak{M}=2is\int I_A(\Delta,\{\vec{x}_j\})I_B(-\Delta,\{\vec{y}_k\})e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp}[1-S(\vec{b}_\perp,\{\vec{x}_j\},\{\vec{y}_k\})]\,\delta^{(2)}\left(\sum_{j=1}^{M}\vec{x}_j\right)\delta^{(2)}\left(\sum_{k=1}^{N}\vec{y}_k\right)\prod_{j,k}d^2\vec{x}_jd^2\vec{y}_kd^2\vec{b}_\perp\ , \tag{A19}$$

where

$$[1-S(\vec{b}_\perp,\{\vec{x}_j\},\{\vec{y}_k\})]=\frac{1}{2i}\int\exp\left(i\sum_j\vec{q}_{j\perp}\cdot(\vec{x}_j+\vec{b}_\perp)+i\sum_k\vec{q}'_{k\perp}\cdot\vec{y}_k\right)\mathfrak{M}_0(\{\vec{q}_{j\perp}\},\{\vec{q}'_{k\perp}\})\prod_{j,k}\frac{d^2\vec{q}_{j\perp}}{(2\pi)^2}\frac{d^2\vec{q}'_{k\perp}}{(2\pi)^2}. \tag{A20}$$

Here, $\vec{x}_i,\vec{y}_j$ are two-dimensional vectors transverse to the incident direction, in the impact-distance space. The position-space impact factor I_A of hadron A is the Fourier transform of $\mathcal{G}_A(\{\vec{q}_{j\perp}\})$ with respect to the $(M-1)$ independent relative momenta

$$\vec{\bar{q}}_{j\perp}:\ \left\{\vec{\bar{q}}_{j\perp}=\vec{q}_{j\perp}-\frac{1}{M}\vec{Q}_{\perp}\right\},$$

where $\vec{Q}_{\perp}$ is the total momenta $\vec{Q}_{\perp}=\sum_{j=1}^{M}\vec{q}_{j\perp}$. Hence $\mathcal{G}_A(\{\vec{q}_{j\perp}\})=\mathcal{G}_A(\vec{Q}_{\perp},\{\vec{\bar{q}}_{j\perp}\})$, $j=1,\ldots,M$, and

$$I_A(\vec{Q}_{\perp},\{\vec{x}_j\})=\int \exp\left(i\sum_{j=1}^{M}\vec{\bar{q}}_{j\perp}\cdot\vec{x}_j\right)\mathcal{G}_A(\vec{Q}_{\perp},\{\vec{\bar{q}}_{j\perp}\})\delta^{(2)}\left(\frac{1}{M}\sum_{j=1}^{M}\vec{\bar{q}}_{j\perp}\right)\prod_{j=1}^{M}\frac{d^2\vec{\bar{q}}_{j\perp}}{(2\pi)^2} \tag{A21}$$

so that

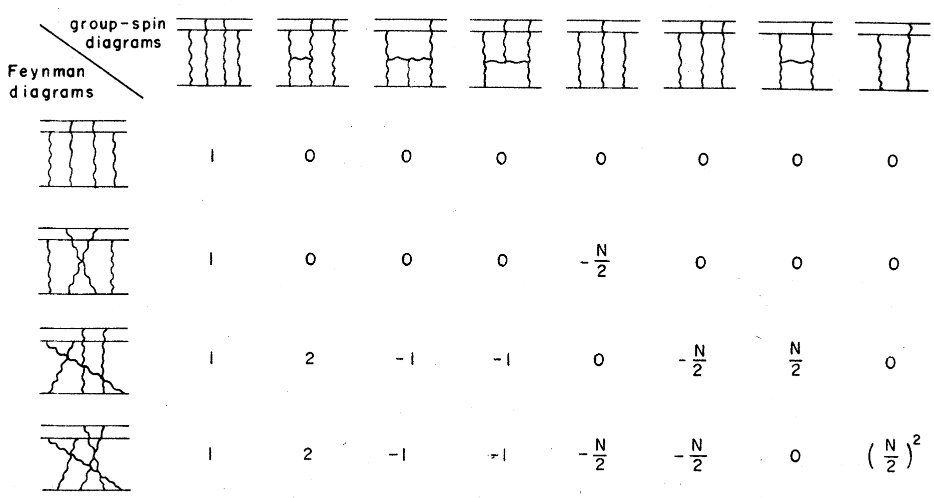
$$\mathcal{G}_A(\{\vec{q}_{j\perp}\})=\int \exp\left(-i\sum_{j=1}^{M}\vec{\bar{q}}_{j\perp}\cdot\vec{x}_j\right)I_A(\vec{Q}_{\perp},\{\vec{x}_j\})\delta^{(2)}\left(\sum_{j=1}^{M}\vec{x}_j\right)\prod_{j=1}^{M}d^2\vec{x}_j\,. \tag{A22}$$

I_B of hadron B is similarly defined.

APPENDIX B: CALCULATION OF LEADING-UNITARY TERMS

First, we shall review the salient features of the diagrammatic calculational scheme of Ref. 2. The amplitude of a Feynman diagram in non-Abelian gauge theories is a product of a momentum amplitude and a group-spin factor (e.g., isospin factor in Yang-Mills theory). The group-spin factor is best represented by a group-spin diagram having the same topology as the Feynman diagram. However, the amplitudes of Feynman diagrams cannot be easily compared since their group-spin diagrams have different topology. Especially in view of expected cancellations among amplitudes, it is desirable first to express the group-spin diagrams as linear sums of diagrams with a common topology. The set of *planar* diagrams were chosen as such a basis (they are the "box factors" of Ref. 2, having only vertical and horizontal lines for fermion-fermion scattering). Such a decomposition of group-spin diagrams into planar diagrams can be achieved by application of the Jacobi identity (in diagrammatic form). An example is given in Fig. 4.

The coefficients of a particular planar group-



	1	0	0	0	0	0	0	0
	1	0	0	0	$-\frac{N}{2}$	0	0	0
	1	2	-1	-1	0	$-\frac{N}{2}$	$\frac{N}{2}$	0
	1	2	-1	-1	$-\frac{N}{2}$	$-\frac{N}{2}$	0	$\left(\frac{N}{2}\right)^2$
$M = sM_7\times$	$-\frac{i}{12}g^8$	$-2g^6(g^2T)$	n-ℓ	n-ℓ	n-ℓ	$\frac{N}{2}g^6(g^2T)$	$-i\frac{N}{2}g^4(g^2T)^2$	$i\left(\frac{N}{2}\right)^2 g^4(g^2T)^2$

FIG. 4. The SU(N) group-spin diagrams of the four Feynman diagrams shown in the left-most column may be decomposed as linear sums of planar group-spin diagrams of the top row, with coefficients given in the table. The last row M shows the coefficients which multiply the group-spin diagrams of the respective columns when the four Feynman diagrams are added together. Here $T=\ln s/2\pi$, and M_7 is given in Fig. 7. (See Appendix B.) Also, n-l denotes nonleading terms.

spin diagram from all Feynman diagrams of the same perturbative order are then added together. The leading ln$s$ term of the resultant sum of momentum integrals can be evaluated by the infinite-momentum technique.[2,24] Such leading terms are the leading-unitary terms of the scheme of Ref. 2. The technique to evaluate sums of momentum integrals using a Fourier representation of products of momentum propagators (as in Appendix A) is detailed in Sec. IV (iii) of Ref. 2 and shall not be repeated here.

Part of our results in the 6th and 8th perturbative order is presented in Figs. 5 and 6. For clarity, we have omitted depicting the hadron wave function as well as quarks of hadron A and B not taking part in the instantaneous interaction (i.e., with no exchanged gluons attached). Figure 5 shows the case in which two quarks of hadron A exchange gluons with one quark of hadron B (represented, respectively, by two horizontal lines on top of each diagram, and one horizontal line at the bottom of each diagram). Figure 6 shows results of diagrams with gluons exchanged among two quarks from hadron A and two quarks from hadron B. (The case of one quark of hadron A scattered from one quark from hadron B is the same as quark-quark scattering of Ref. 2, so we shall not repeat it here.) Scattering involving more quarks has similarly been calculated. Owing to space limitations, we choose not to present our entire result.

APPENDIX C: INITIAL CONDITION FOR $S_{lm}[\vec{Q};T]$

We start with the initial condition for $S[\vec{Q};T]$ of Eq. (5.6). From (5.3),

$$S[\vec{Q};T=0]=\exp\{ig^2 t_c Q_c\}\,. \tag{C1}$$

(C1) is in fact a rotational matrix in SU(2). For convenience, we shall set $g=1$ and denote the representation of t_c as the spin-j representation, so that

$$S[\vec{Q};T=0]=D^{(j)}(R_1)=\exp\{it_cQ_c\}\,. \tag{C2}$$

Define a rotation from R_1 to R_2 by a unitary matrix U:

$$D^{(j)}(R_2)=\exp\{it_zQ_z\}=UD^{(j)}(R_1)U^{-1}\,, \tag{C3}$$

where t_z is the diagonal z-component matrix in the spin-j representation.

The initial condition for projection of the generating functional $S_{lm}[\vec{Q};T=0]$ is related to the following trace by (5.10):

$$\mathrm{Tr}\{P^{(1)}_{l,m}S[\vec{Q};T=0]\}=\mathrm{Tr}\{P^{(1)}_{l,m}D^{(j)}(R_1)\}=\mathrm{Tr}\{UP^{(1)}_{l,m}U^{-1}D^{(j)}(R_2)\}\,, \tag{C4}$$

using (C3).

We can evaluate the trace in (C4) by explicitly putting in states corresponding to the eigenstates of the z component of the angular momentum in $\vec{Q}$ space:

$$\mathrm{Tr}\{UP^{(1)}_{l,m}U^{-1}D^{(j)}(R_2)\}=\sum_{m',m''}\langle jm'|UP^{(1)}_{l,m}U^{-1}|jm''\rangle\mathfrak{D}^{(j)}_{m''m'}(R_2)\,. \tag{C5}$$

Here, the conventional notation applies: j is the eigenvalue of the total angular momentum, and $\mathfrak{D}^{(j)}_{m''m'}(R_2)$ the standard matrix elements of the rotational operator $D^{(j)}(R_2)$.

$\mathfrak{D}^{(j)}_{m''m'}(R_2)$ is diagonal, as $D^{(j)}(R_2)$ of (C3) is, so we only need to evaluate the first matrix element of (C5) with $m''=m'$. The matrix element can be simplified further from the transformation equation of the irreducible spherical tensor operators $P^{(1)}_{l,m}$:

$$\langle jm'|UP^{(1)}_{l,m}U^{-1}|jm'\rangle=\sum_{m_1}\langle jm'|P^{(1)}_{l,m}|jm'\rangle\mathfrak{D}^{(l)}_{m_1,m}(U) \tag{C6a}$$

$$=\langle j\|P^{(1)}\|j\rangle\sum_{m_1}\langle jlm'm_1|jljm'\rangle\,\mathfrak{D}^{(l)}_{m_1,m}(U)\,. \tag{C6b}$$

We obtain (C6b) from (C6a) by use of the Wigner-Eckart theorem, where $\langle j\|P^{(1)}\|j\rangle$ is the reduced matrix element, and $\langle j_1j_2m_1m_2|j_1j_2jm\rangle$ is the Clebsch-Gordan coefficients for the product of representations j_1, j_2 to form representation $j=j_1+j_2$, with eigenvalues m_1, m_2, and m, respectively. However, the coefficient $\langle jlm'm_1|jljm'\rangle$ is nonzero only if $m'+m_1=m'$, or $m_1=0$.

Combining (C4), (C5), (C6), and the fact that

$$\mathfrak{D}^{(l)}_{0,m}(U)=\left(\frac{4\pi}{2l+1}\right)^{1/2}Y_{lm}(\theta,\phi)\,, \tag{C7}$$

$Y_{lm}(\theta,\phi)$ being the spherical harmonics, we finally have

$$\mathrm{Tr}\{P^{(1)}_{l,m}S[\vec{Q};T=0]\}=\left(\frac{4\pi}{2l+1}\right)^{1/2}Y_{lm}(\theta,\phi)\langle j\|P^{(1)}\|j\rangle\times\sum_{m'}\langle jlm',m_1=0|jljm'\rangle\times\mathfrak{D}^{(j)}_{m'm'}(R_2)\,. \tag{C8}$$

(C8) together with (5.10) clearly shows that

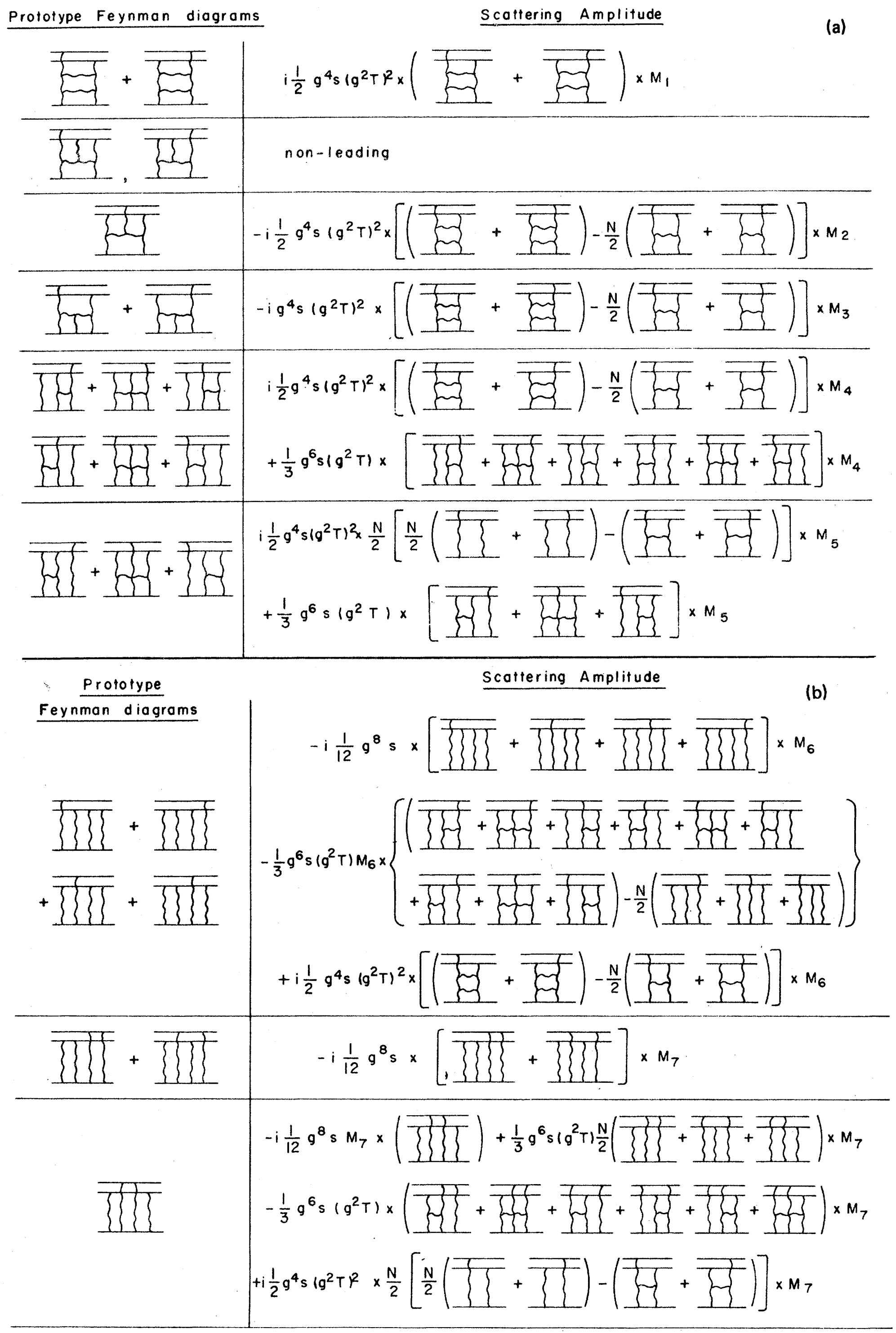


FIG. 5. A summary of the Feynman diagram calculation in eighth perturbative order for the case with two quarks of hadron A (upper horizontal lines) exchanging gluons with one quark of hadron B (lower horizontal line). For clarity, we have omitted the hadronic wave functions and other quarks with no exchanged gluons attached. The amplitudes on the right-hand side are expressed in terms of M_i of Fig. 7 and SU(N) group-spin diagrams depicted. These amplitudes come from the corresponding prototype Feynman diagram on the left, and all distinct Feynman diagrams obtained from it by any permutation of the order of attachment of gluons on each quark line. [For 5(b), we only consider permutations on the two upper quark lines.] Here $T = \ln s/2\pi$.

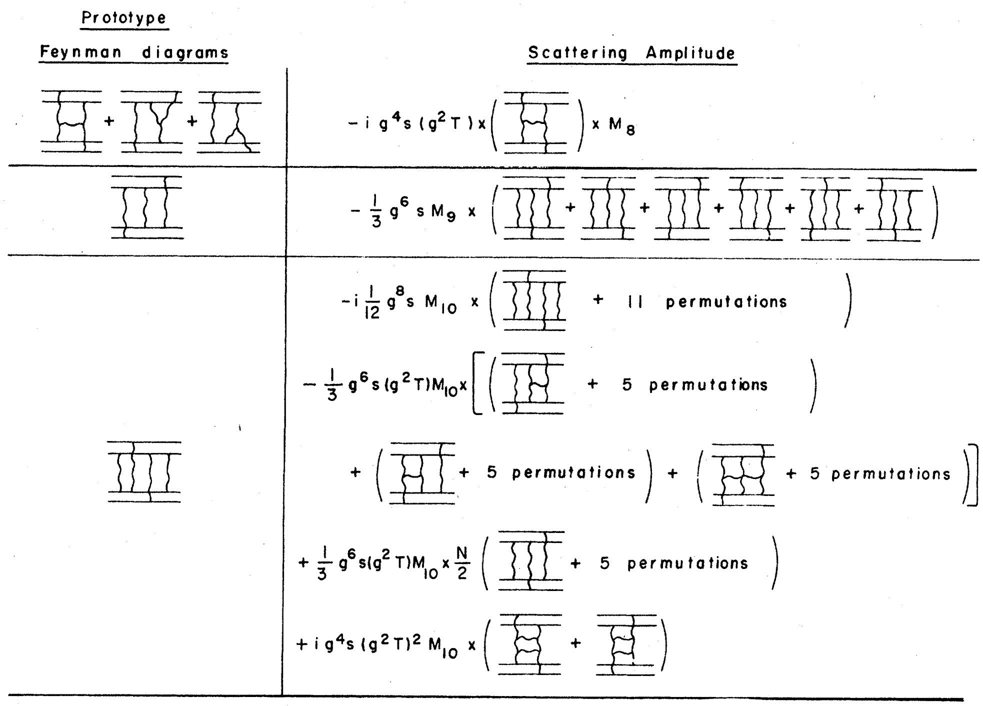


FIG. 6. A summary of part of the Feynman diagram calculation in the sixth and eighth order for the case with two quarks of hadron A (upper horizontal lines) exchanging gluons with two quarks of hadron B (lower horizontal lines). As in Fig. 5, we omit hadronic wave functions and extraneous quarks. The amplitudes on the right-hand side are expressed in terms of SU(N) group-spin diagrams, and "permutations" refer to planar group-spin diagrams obtained by permuting the order of the vertical gluons as in the second row. These amplitudes come from the prototype Feynman diagrams on the left, and all distinct diagrams obtained from it by permuting the order of attachment of gluons on each quark line.

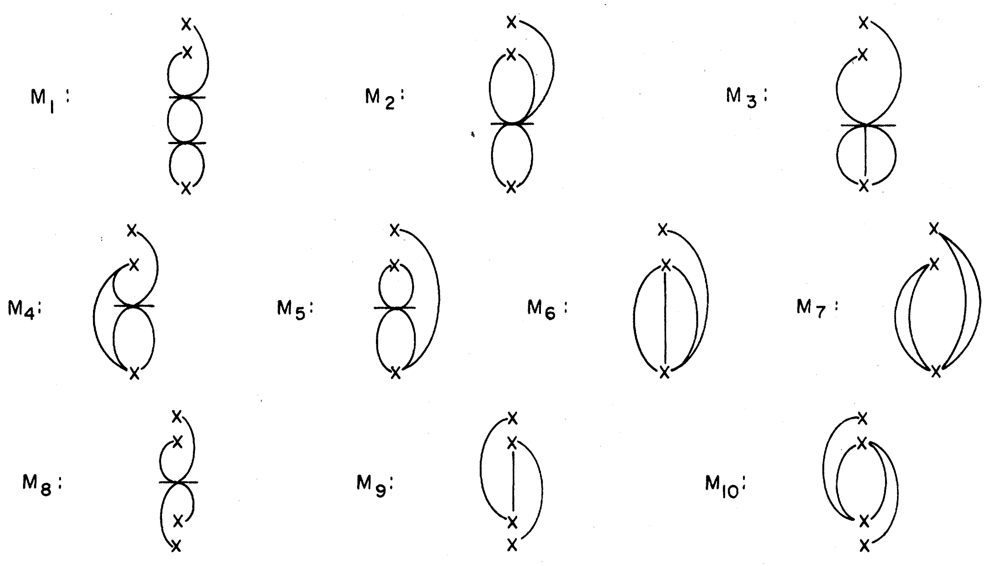


FIG. 7. Transverse-momentum diagrams used in Figs. 4, 5, and 6. All lines are gluon lines, associated with a factor of $(\vec{q}_\perp^{\,2}+\lambda^2)^{-1}$ if it carries momentum $\vec{q}_\perp$. They attach to the quark lines at the crosses (×) depicted. The total momentum entering each cross appears in the argument of the hadronic wave function $I_{A,B}(\{\vec{q}_{i\perp}\})$ of Appendix A, which we omit in the diagrams for clarity. For each horizontal bar through a vertex, there is a factor of $(\vec{q}_\perp+\lambda^2)$, where $\vec{q}_\perp$ is the sum of the momentum going vertically up (or down) the vertex. Finally, for each closed loop, we integrate over the loop momentum $\vec{k}_\perp$: $\int d^2\vec{k}_\perp/(2\pi)^2$.

$S_{l,m}[\vec{Q};T=0] \propto Y_{lm}(\theta,\phi)$. The coefficients of Y_{lm} in (C8) is a function of the radial variable $Q=|\vec{Q}|$ in $\vec{Q}$ space, so that if $P_l^{(1)}$ is given in terms of the matrices t_a, we can evaluate the radial function $R_l[Q]$ in

$$S_{l,m}[\vec{Q};T=0]=R_l[Q]Y_{lm}(\theta,\phi)\,. \quad \text{(C9)}$$

As an example, for $l=0$, it follows from (5.3) and (5.6) that (since $\mathrm{Tr}\,\underline{1}=2j+1$)

$$S_{0,0}[\vec{Q};T=0]=\frac{1}{2j+1}\mathrm{Tr}\{\exp[ig^2 t_c Q_c]\}$$

$$=\frac{1}{2j+1}\times\begin{cases}1+2\sum_{n=1}^{j}\cos(ng^2Q), & j=\text{integer} \quad \text{(C10)}\\ 2\sum_{n=1}^{j}\cos(ng^2Q), & j=\tfrac{1}{2}\text{ integer}\,. \quad \text{(C11)}\end{cases}$$

Comparing (C11) with (5.17) (with $Y_{00}=1/\sqrt{4\pi}$), we obtain (5.21).

For $l=1$, $m=0$,

$$S_{1,0}[\vec{Q};T=0]=\frac{1}{2j+1}\mathrm{Tr}\{t_z\exp[ig^2t_cQ_c]\} \quad \text{(C12)}$$

$$=\frac{1}{ig^2}\frac{\partial}{\partial Q_z}S_{0,0}[\vec{Q};T=0] \quad \text{(C13)}$$

$$=\frac{1}{2j+1}\sum_{\substack{n=1\\(n=1/2)}}^{j} in\sin(ng^2Q)\left(\frac{4\pi}{3}\right)^{1/2}Y_{1,0}$$

$$\text{for } j=\text{integer } (\tfrac{1}{2}\text{ integer})\,. \quad \text{(C14)}$$

Comparing (C14) and similar expressions for $S_{1,\pm1}[\vec{Q};T=0]$ with (5.17), we obtain (5.22).

APPENDIX D: $\Psi[Q_{j_1}(\vec{q}_1),\ldots,Q_{j_n}(\vec{q}_n)]$ AS AN EIGENFUNCTION

In this Appendix, we prove the eigenvalue relation

$$h\Psi_\Omega[Q_{j_1}(\vec{q}_1),\ldots,Q_{j_n}(\vec{q}_n)]=0\,, \quad \text{(D1)}$$

where

$$h=\tfrac{1}{2}\int\frac{d^2\vec{q}_1'}{(2\pi)^2}\frac{d^2\vec{q}_2'}{(2\pi)^2}\left[f_{ace}Q_c(\vec{q}_1')\frac{\delta}{\delta Q_a(\vec{q}_1')}\right]\left[f_{bde}Q_d(\vec{q}_2')\frac{\delta}{\delta Q_b(\vec{q}_2')}\right] \quad \text{(D2a)}$$

$$=\int\frac{d^2\vec{q}_1'}{(2\pi)^2}\frac{d^2\vec{q}_2'}{(2\pi)^2}\left[\tfrac{1}{2}f_{ace}f_{bde}Q_c(\vec{q}_1')Q_d(\vec{q}_2')\frac{\delta^2}{\delta Q_a(\vec{q}_1')\delta Q_b(\vec{q}_2')}\right]-C_A\int\frac{d^2\vec{q}'}{(2\pi)^2}Q_c(\vec{q}')\frac{\delta}{\delta Q_c(\vec{q}')} \quad \text{(D2b)}$$

and

$$\Psi_\Omega[Q_{j_1}(\vec{q}_1),\ldots,Q_{j_n}(\vec{q}_n)]=Q_{j_1}(\vec{q}_1)\cdots Q_{j_n}(\vec{q}_n)\Omega^{i_1,\ldots,i_n}_{j_1,\ldots,j_n}\mathrm{Tr}(t_{i_1}\cdots t_{i_n})\,. \quad \text{(D3)}$$

(D2b) is obtained from (D2a) by straightforward expansion, where $f_{ace}f_{bae}=-2C_A\delta_{ab}$ and $[\delta/\delta Q_c(\vec{q}')]Q_k(\vec{q}_{k'})=(2\pi)^2\delta_{kc}\delta^{(2)}(\vec{q}'-\vec{q}_{k'})$.

First consider Ψ of (D3) where $\Omega=\underline{1}$:

$$\Psi_0[Q_{j_1}(\vec{q}_1),\ldots,Q_{j_n}(\vec{q}_n)]=Q_{j_1}(\vec{q}_1)\cdots Q_{j_n}(\vec{q}_n)\,\mathrm{Tr}(t_{j_1}\cdots t_{j_n})\,. \quad \text{(D4)}$$

Then

$$\int\frac{d^2\vec{q}_2'}{(2\pi)^2}f_{bde}Q_d(\vec{q}_2')\frac{\delta}{\delta Q_b(\vec{q}_2')}\Psi_0[Q_{j_1}(\vec{q}_1),\ldots,Q_{j_n}(\vec{q}_n)]$$

$$=Q_{j_1}(\vec{q}_1)\cdots Q_{j_n}(\vec{q}_n)[f_{j_1'j_1e}\mathrm{Tr}(t_{j_1'}t_{j_2}\cdots t_{j_n})+f_{j_2'j_2e}\mathrm{Tr}(t_{j_1}t_{j_2'}\cdots t_{j_n})$$

$$+\cdots+f_{j_n'j_ne}\mathrm{Tr}(t_{j_1}\cdots t_{j_{n-1}}t_{j_n'})] \quad \text{(D5a)}$$

$$=Q_{j_1}(\vec{q}_1)\cdots Q_{j_n}(\vec{q}_n)(-i)[\mathrm{Tr}\{[t_{j_1},t_e]t_{j_2}\cdots t_{j_n}\}+\mathrm{Tr}\{t_{j_1}[t_{j_2},t_e]\cdots t_{j_n}\}$$
$$+\cdots+\mathrm{Tr}\{t_{j_1}\cdots t_{j_{n-1}}[t_{j_n},t_e]\}] \tag{D5b}$$
$$=Q_{j_1}(\vec{q}_1)\cdots Q_{j_n}(\vec{q}_n)(-i)[-\mathrm{Tr}\{t_e t_{j_1}\cdots t_{j_n}\}+\mathrm{Tr}\{t_{j_1}\cdots t_{j_n}t_e\}]$$
$$=0\,. \tag{D5c}$$

Hence from (D2),

$$h\Psi_0[Q_{j_1}(\vec{q}_1),\ldots,Q_{j_n}(\vec{q}_n)]=0\,. \tag{D6}$$

Next, we shall show the following property of Ω:

$$f_{j_1'j_1e}\Omega^{i_1,\ldots,i_n}_{j_1',\ldots,j_n}+f_{j_2'j_2e}\Omega^{i_1,\ldots,i_n}_{j_1,j_2',\ldots,j_n}+\cdots+f_{j_n'j_ne}\Omega^{i_1,\ldots,i_n}_{j_1,\ldots,j_{n-1},j_n'} \tag{D7a}$$
$$=\Omega^{i_1',\ldots,i_n}_{j_1,\ldots,j_n}f_{i_1i_1'e}+\Omega^{i_1,i_2',\ldots,i_n}_{j_1,\ldots,j_n}f_{i_2i_2'e}+\cdots+\Omega^{i_1,\ldots,i_{n-1},i_n'}_{j_1,\ldots,j_n}f_{i_ni_n'e} \tag{D7b}$$

from which we conclude $h\Psi_\Omega=0$, as

$$\int\frac{d^2\vec{q}_2'}{(2\pi)^2}f_{bde}Q_d(\vec{q}_2')\frac{\delta}{\delta Q_b(\vec{q}_2')}\Psi_\Omega[Q_{j_1}(\vec{q}_1),\ldots,Q_{j_n}(\vec{q}_n)]$$
$$=Q_{j_1}(\vec{q}_1)\cdots Q_{j_n}(\vec{q}_n)\times\mathrm{Tr}(t_{i_1}\cdots t_{i_n})\times\{\Omega^{i_1',\ldots,i_n}_{j_1,\ldots,j_n}f_{i_1i_1'e}+\cdots+\Omega^{i_1,\ldots,i_{n-1},i_n'}_{j_1,\ldots,j_n}f_{i_ni_n'e}\}$$
$$=0 \tag{D8}$$

in view of (D7) and (D5).

We make use of a property of Ω. From the initial condition (6.1) and the form of H of (4.23), we see that Ω is formed from a product of structure constants $f_{\alpha\beta\gamma}$. We may write

$$\Omega^{i_1,\ldots,i_n}_{j_1,\ldots,j_n}=f_{j_lj_l'a}f_{j_mj_m'a}\bar{\Omega}^{i_1,\ldots,i_n}_{j_1,\ldots,j_l',\ldots,j_m',\ldots,j_n} \tag{D9}$$

so that (D7a) is

$$\sum_{\alpha\neq l,m}f_{j_lj_l'a}f_{j_mj_m'a}f_{j_\alpha'j_\alpha e}\bar{\Omega}^{i_1,\ldots,i_n}_{j_1,\ldots,j_\alpha',\ldots,j_l',\ldots,j_m',\ldots,j_n}+f_{j_l'j_l''a}f_{j_mj_m'a}f_{j_l'j_le}\bar{\Omega}^{i_1,\ldots,i_n}_{j_1,\ldots,j_l'',\ldots,j_m',\ldots,j_n}$$
$$+f_{j_lj_l'a}f_{j_m'j_m''a}f_{j_m'j_me}\bar{\Omega}^{i_1,\ldots,i_n}_{j_1,\ldots,j_l',\ldots,j_m'',\ldots,j_n} \tag{D10a}$$

$$=f_{j_lj_l'a}f_{j_mj_m'a}\Bigg[\sum_{\substack{\alpha\\(\alpha\neq l,m)}}f_{j_\alpha'j_\alpha e}\bar{\Omega}^{i_1,\ldots,i_n}_{j_1,\ldots,j_\alpha',\ldots,j_l',\ldots,j_m',\ldots,j_n}$$
$$+f_{j_l''j_l'e}\bar{\Omega}^{i_1,\ldots,i_n}_{j_1,\ldots,j_l'',\ldots,j_m',\ldots,j_n}+f_{j_m''j_m'e}\bar{\Omega}^{i_1,\ldots,i_n}_{j_1,\ldots,j_l',\ldots,j_m'',\ldots,j_n}\Bigg]\,. \tag{D10b}$$

(D10b) is obtained from (D10a) by use of the Jacobi identity on the last two terms of (D10a). The sum of terms inside the brackets in (D10b) is just that in (D7a), with Ω replaced by $\bar{\Omega}$. Hence we can define $\bar{\Omega}$ with two structure constants explicitly shown as in (D9) for Ω. We would arrive at (D10b) with $\bar{\Omega}$ replaced by $\bar{\bar{\Omega}}$. This procedure may be repeated as many times as is required to exhaust all the structure constants present in Ω, so that all Ω stands on the left of $f_{j_\alpha j_\alpha' e}$. ($\alpha=1,\ldots n$) as in (D7b). This is illustrated diagrammatically in Fig. 8.

The physical meaning of $h\Psi=0$ is readily seen. With h defined as in (D2), $(-2h)$ in fact is the invariant quadratic Casmir operator in functional space. For example, in SU(2) in the case that $\vec{q}$

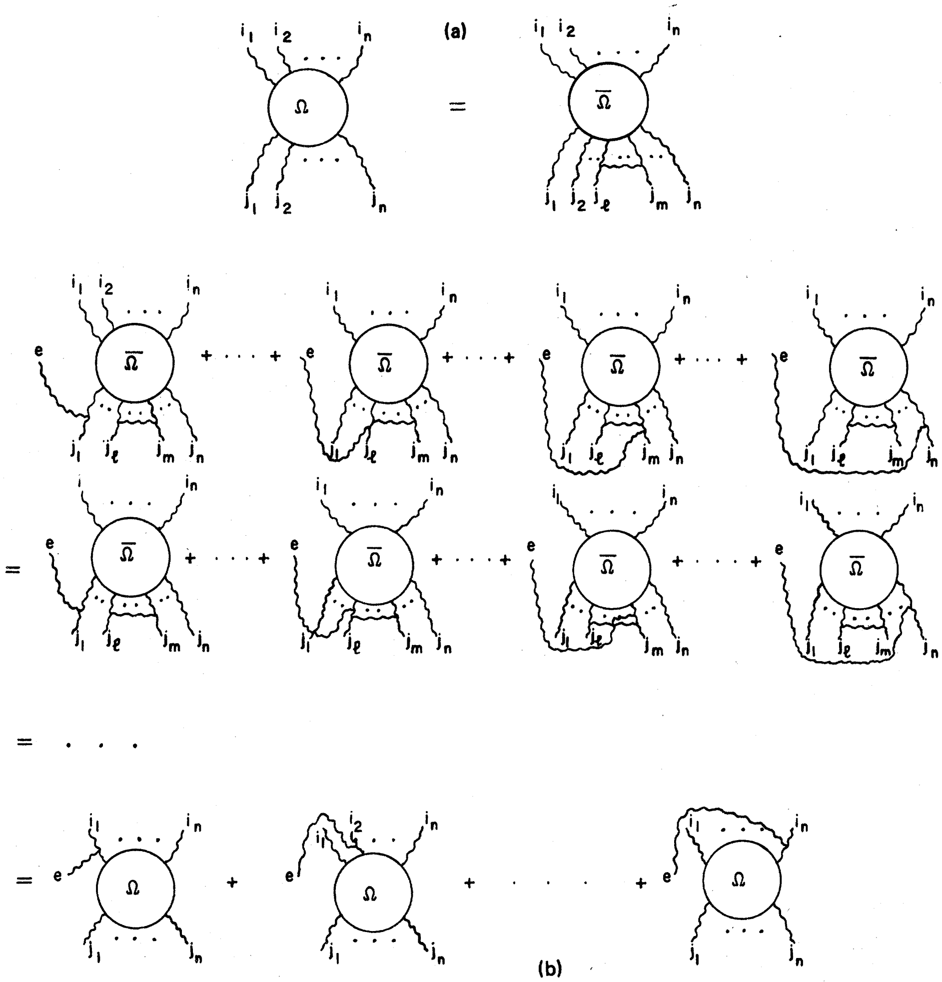


FIG. 8. (a) Diagrammatic representation of Eq. (D9). (b) Diagrammatic representation of Eqs. (D10) and (D7).

space consists of only one point (as in the solvable model of Sec. V), $-2h=\vec{L}_Q^2$, the square of the angular momentum operator in $\vec{Q}$ space $\vec{L}_Q=(1/i)\vec{Q}\times\vec{\nabla}_Q$. If we consider $\vec{q}$ space as a discrete lattice, then in SU(2), $-2h=\vec{L}_{tot}^2$, the square of the *total* angular momentum operator $\vec{L}_{tot}$ ($\vec{L}_{tot}$ is the sum of all $\vec{L}_{Q_j}$ defined at each lattice point j in $\vec{q}$ space.) Since Ψ is a singlet state, just as $S^{(1)}$ of (6.2a) is, we must have $h\Psi=0$.

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[4]H. Cheng and T. T. Wu, Phys. Rev. D 6, 2637 (1972) and references therein. A random selection of the literature on incorporation of hadronic wave functions in scattering amplitudes are S. J. Brodsky and G. R. Farrar, Phys. Rev. D 11, 1309 (1975); S. J. Brodsky and G. P. Lepage, SLAC Report No. SLAC-PUB-2294, 1979 (unpublished).

[5]R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Wiley-Interscience, New York, 1959), Vol. I. Also see Refs. 1, 2, and 3.

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[7]H. Cheng, in *Proceedings of the 19th International Conference on High Energy Physics, Tokyo, 1978,*

edited by S. Homma, M. Kawaguchi, and H. Miyazawa (Phys. Soc. Japan, Tokyo, 1979).

[8]H. Cheng and J. Dickinson have informed the author that they have independently verified the eikonal approximation up to 6th perturbative order for scattering of two on-mass-shell quarks of large plus momenta from two on-mass-shell quarks of large minus momentum. Hadronic wave functions were not used in their calculation.

[9]J. Mandula, Phys. Rev. D 14, 3497 (1976).

[10]The same situation arises when e^{A+B} is expressed in terms of e^A by the Campbell-Hausdorf series, if A and B are highly noncommuting.

[11]An analogous relation appears in evaluating the generating functional for Green's functions $W[J]$ [see, for instance, E. S. Abers and B. W. Lee, Phys. Rep. 9C, 5 (1973), Eq. (12.16)]:

$$W[J] \sim \int [d\phi] \exp\left(i\int d^4x\,[\mathcal{L}_O + \mathcal{L}_I(\phi) + J\phi]\right)$$
$$= \exp\left[i\int d^4x\,\mathcal{L}_I\left(\frac{l}{i}\,\frac{\delta}{\delta J(x)}\right)\right]$$
$$\times \int [d\phi]\exp\left(i\int d^4x[\mathcal{L}_O + J\phi]\right).$$

We observe that this relation holds even if ϕ is a matrix, provided $J(x)$ is not. A normal-ordering prescription is required if $\mathcal{L}_I$ is a function of both J and ϕ, as in the case of (4.17).

[12]This model turns out to be essentially the same as that considered by R. L. Sugar, Phys. Rev. D 9, 2474 (1974). (We consider the leading-unitary terms of Sugar's model.) Using our approach, we arrive at more quantitative results than obtained in the above paper, concerning the behavior of the scattering amplitudes in the model.

[13]It is not useful to discuss color-singlet scattering in this context as we would be considering a zero-charge particle positioned at this only point of $\vec{q}$ or $\vec{b}$ space, so that effectively $t_a = 0$.

[14]The derivation is straightforward. For example, the analog of (4.14) is

$$\chi_c(N+1) = \chi_c(N) + \sqrt{\epsilon}V^{ac}(N+1)\,\chi_a(N) + \epsilon(\alpha-1)\chi_c(N),$$

where $V^{ac}(N+1) = ig\epsilon_{acd}[a_d^\dagger(N+1) - a_d(N+1)]$, the lattice version of (5.4), with $[a_c(l), a_c^\dagger(l')] = \delta_{ll'}$. So the matrix element $\langle 0|V^{ac}(N+1)V^{bd}(N+1)|0\rangle$ appearing in H (see 4.20) is $g^2\epsilon_{ace}\epsilon_{bde}$.

[15]For example, see A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N.J., 1960) Ch. 5; see also E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1959).

[16]Here, r is a meaningful variable as H_R is a function of r ($[H_R, r] \neq 0$), unlike the variable Q in the solvable model discussed in Sec. V with $\alpha = 1 - g^2$ ($\beta = 1$), in which $H = -\frac{1}{2}g^2\vec{L}_Q^2$, so that Q is an extraneous variable ($[H, Q] = 0$).

[17]High-energy fixed-t color-singlet scattering has been discussed before using other approaches. For example, see C. D. Stockham, Phys. Rev. D 15, 1736 (1977). The Reggeon field theory approach to this problem is described in J. Bartels, DESY Report No. 80/54, 1980 (unpublished) and A. R. White, Nucl. Phys. B159, 77 (1979) and references therein.

[18]R. Blankenbecler and H. M. Fried, Phys. Rev. D 8, 678 (1973); S. Auerbach, R. Aviv, R. Sugar, and R. Blankenbecler, *ibid.* 6, 2216 (1972); R. Aviv, R. Sugar, and R. Blankenbecler, *ibid.* 5, 3252 (1972); G. Calucci, R. Jengo, and C. Rebbi, Nuovo Cimento 4A, 330 (1971); 6A, 601 (1971). Also see other references in Refs. 1 and 3 of this paper.

[19]For instance, there may be a simple physical explanation for the exact negative definiteness and Hermiticity of H_R of Eq. (6.14), which is a central ingredient in our proof of total absorption in Sec. VI. Mathematically, this property of H_R results from a curious interplay of the Reggeon propagator with the Reggeon-Reggeon-gluon vertex function in the eikonal [see the discussion following Eq. (5.8); the QCD case corresponds to the particular choice in Sec. V of $\beta = 1$ if $\vec{q}$ space consists of only one point, in which case H is Hermitian and semi-negative definite as in Eq. (5.13)].

[20]H. Cheng, J. Dickinson, K. Olaussen, and P. S. Yeung (unpublished).

[21]Research by D. Coon based on computer evaluation of $S(b, T)$ in a ϕ^3-theory model supports the totally absorptive behavior as T increases (unpublished). Research in the QCD case based on the eikonal form of Sec. IIC of this paper is in progress.

[22]See Ref. 20 and also the first two articles quoted in Ref. 18.

[23]H. Cheng and T. T. Wu, Phys. Rev. 186, 1611 (1978).

[24]Infinite-momentum techniques are discussed in H. Cheng and T. T. Wu, Phys. Rev. 182, 1899 (1969). See also S. J. Chang and S. K. Ma, *ibid.* 180, 1506 (1969); 188, 2385 (1969).