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Interaction potentials for multiquark states from instantons and other background gauge field configurations

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We present a simple rule for calculating the contributions to the interaction potentials between constituent particles for a family of multiquark states, due to the presence of a semiclassical gauge field configuration which exists in a single SU(2) subgroup of color SU(3). In multiquark states beyond the baryon we find many-body potential terms. The static (Wilson-loop) limit is sufficient to elucidate the dependence of the potential on the color structure of the multiquark state.

I. INTRODUCTION

In recent years interest has grown in the study of the physical effects of instanton solutions of quantum chromodynamics (QCD),[1] as these objects provide the basis for a semiclassical treatment and a different partial view of the complex phenomena contained in the theory—one which emphasizes nonperturbative effects.

Recently, several authors[2-4] have considered the contribution of a dilute gas of instantons to the interaction energies of quarks and antiquarks arranged in meson and baryon configurations. These calculations have derived, at a first level, the static spin-independent potential from a Wilson-loop analysis, and have also shown how a systematic expansion for the effective Hamiltonian of a system composed of heavy quarks, in powers of $1/m_q$, can be constructed.

The emergence of a QCD-based theory of hadronic structure, the bag model of Callan, Dashen, and Gross,[5] which indicates that the instanton size cutoff provided by the bag makes dilute-gas calculations appropriate, suggests that once the instanton size distributions in the bag and the linkage of the bag to its contents are clarified, serious spectroscopic calculations will be possible.

This state of affairs, combined with the increasing evidence for the existence of multiquark states,[6] such as baryonium and dibaryon states, prompts us to investigate the nature of instanton contributions to the interparticle potentials in multiquark hadrons.

From the work of Callan *et al.*[3] and Aragão de Carvalho[4] on the heavy-quark expansion of the quark-antiquark potential due to a dilute gas of instantons, we see that color configuration dependence of the potential may be determined by analyses in the static limit. The terms in the $1/m_q$ expansion of the Hamiltonian result from instanton position, size, and color orientation integrations over the potential due to a single instanton, which itself consists of single-quark operators containing various m_q^{-n} terms acting on the color-structure-dependent effective Hamiltonian of the static limit. Accordingly, the generalization of the Wilson-loop expression to a multiquark system influenced by a single instanton suffices to elucidate any complexities associated with the more intricate color structure. Callan *et al.*[3] discovered a simple expression for the static spin-independent potential existing in a baryon due to a dilute gas of instantons in terms of the analogous potential generated in the meson cases:

$$V_{qqq}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{2} \sum_{\text{pairs}} V_{q\bar{q}}(\vec{x}_i, \vec{x}_j)\,. \tag{1}$$

They pointed out that this is a result of the semiclassical configuration involved (the instanton) lying in a single SU(2) subgroup of color SU(3). The three quarks in a baryon are all differently colored at any moment, and any given background instanton field, being of SU(2) nature, only affects two colors at a time, leading to this separation into two-body potentials once the summing over instanton color orientations is performed. If it is necessary to go beyong the dilute-gas approximation and consider group-orientation-dependent instanton–anti-instanton forces, then many-body forces are to be expected. Callan *et al.*[3] showed that this separation mechanism applies to (a dilute gas of) any semiclassical configuration which lies in a single SU(2) subgroup of color SU(3).

Besides the growing experimental[6] and theoretical interest[7,8] in multiquark states it seemed worthwhile to see the significance of the SU(2) nature of the instanton for these more complicated color structures. Once more than three quarks are present, their colors are not all simultaneously different and so many-body potentials are expected and these should give some insights into the spatial structure of multiquark states.

We present a simple general rule for calculating the interaction potentials due to a semiclassical background field contained in a SU(2) subgroup of $SU(3)_c$ for a family of multiquark states. Many-body potentials are found, and after some general features have been examined we consider the first many-body potential, which is the only one encountered in the configurations of immediate experimental interest and its contribution at the static spin-independent level is considered for a dilute gas of instantons.

In Sec. II we display the family of multiquark states for which we have developed the simple potential rule. Then in Sec. III we present an outline of the effective-Hamiltonian approach and the heavy-quark expansion of the potential. In Sec. IV the rule is proved inductively for the general member of the multiquark family, while in Sec. V the form of the many-body potentials encountered is examined. Section VI focuses on the first of the many-body potentials—the four-body potential—analyzing it for a dilute gas of instantons.

II. A FAMILY OF MULTIQUARK STATES

The family of multiquark states for which we have formulated a rule for interaction potentials includes T baryonium and also the dibaryon. For a given number of quarks and antiquarks various "color configurations" arise from the various ways in which the color direct product

$$\underline{1}\subset\underline{3}\otimes\underline{3}\otimes\cdots\otimes\underline{\overline{3}}\otimes\underline{\overline{3}} \tag{2}$$

of the color-triplet quarks ($\underline{3}$) and antiquarks ($\underline{\overline{3}}$) can form a color singlet. For the static-limit calculation we only require the color-structural part of the multiquark system wave function. (Naturally the color part, when taken together with space, spin, and flavor structure, must be antisymmetric.) The family of multiquarks we consider consists of those which are obtained by always taking the triplets from

$$\underline{3}\otimes\underline{3}=\underline{\overline{3}}\oplus\underline{6},\quad \underline{\overline{3}}\otimes\underline{\overline{3}}=\underline{3}\otimes\underline{\overline{6}}$$

and finally taking the singlet when expression (2) has been reduced to a final $\underline{3}\otimes\underline{\overline{3}}=\underline{1}\oplus\underline{8}$.

Clearly the meson is the simplest member of the family and the baryon is the next. The usual multiquark diagrams for this family,[8] with quarks and antiquarks at link ends and junctions representing Levi-Civita symbols, resemble Cayley trees with connectivity $K=2$, i.e., three "nearest"-neighbor junctions for each junction. Figure 1 shows the first few members of the family. The triplet nature of the decomposition of the direct product at all stages (except of course the final singlet from $\underline{3}\otimes\underline{\overline{3}}$) shows that the class is built up from the meson by the prescription of removing a quark (say) and replacing it with a color-antisymmetrized pair of antiquarks. It should be noted that each link connects a $\underline{3}$ (quarklike) to a $\underline{\overline{3}}$ (antiquarklike) object, or equivalently that each junction connects three like-natured objects. For comparison Fig. 2 shows the M-baryonium state drawn in the same system. In this state the quark pair is in a $\underline{6}$ representation with the antiquark pair in a $\underline{\overline{6}}$ representation yielding a color singlet.

For the effective Hamiltonian we shall require

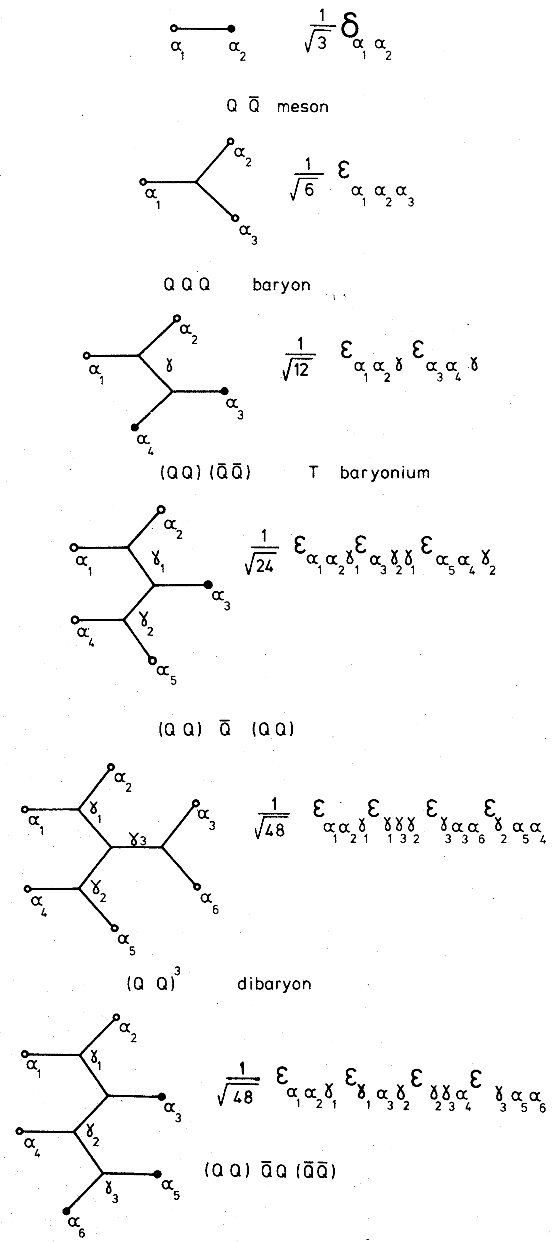
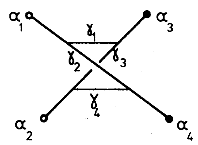


FIG. 1. The members of the "color tree" multiquark family for two to six particles, listed together with their normalized color-singlet projection operators. (Greek color indices take values 1, 2, 3.) Each junction represents a Levi-Civita antisymmetric tensor in the projection operator.



$$\frac{1}{\sqrt{24}}\,\epsilon_{\alpha_1\gamma_1\gamma_2}\,\epsilon_{\alpha_2\gamma_3\gamma_4}\,\epsilon_{\alpha_3\gamma_3\gamma_1}\,\epsilon_{\alpha_4\gamma_4\gamma_2} = \frac{1}{\sqrt{24}}\left(\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}+\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}\right)$$

M baryonium

FIG. 2. The color diagram for M baryonium. It has a closed-loop construction reflecting the fact that the two quarks (antiquarks) are in a symmetric $\underline{6}$ ($\underline{\bar{6}}$) representation and not the natural antisymmetric representation which the link-diagram trees characterize.

the color-singlet projection operators for the states. These are directly reflected by the color states, e.g.,

$$P_{\alpha_1\alpha_2\alpha_3}=\frac{1}{(3\times 2)^{1/2}}\epsilon_{\alpha_1\alpha_2\alpha_3}, \tag{3}$$

and are listed with their states in Fig. 1. The replacement of a particle color α_n by a pair of color-antisymmetrized antiparticles (colors β_1, β_2) to step up the family one level simply involves the modification

$$P'_{\alpha_1\cdots\alpha_{n-1}\beta_1\beta_2}=P_{\alpha_1\cdots\alpha_{n-1}\gamma}\frac{\epsilon_{\gamma\beta_1\beta_2}}{\sqrt{2}}, \tag{4}$$

i.e., the old particle label becomes a link label and the new antisymmetrized pair is added with the appropriate addition to the normalization.

III. EFFECTIVE HAMILTONIAN AND THE HEAVY-QUARK POTENTIALS

We present in this section an outline of the approach used by Callan[3] *et al.* and Aragão de Carvalho[4] in developing the large-mass expansion for the quark-antiquark potential. The extension to multiquark states is also considered.

To calculate the net effect of a dilute gas of instantons on a multiquark configuration, i.e., neglecting correlations and interactions between successive instantons, we need to find the effect of a single instanton on the evolution of the system and then integrate over the position, scale size, and color orientation degrees of freedom of the instanton with the appropriate density distribution of instanton sizes.

The time evolution of the multiquark system is obtained by consideration of the propagation of a single quark or antiquark through the field of an instanton or anti-instanton. The systematic large-mass expansion of the effective Hamiltonian for the multiquark is obtained by calculating the quark propagators in the external gauge field using the systematic $1/m_q$ expansion of the quark's Dirac Hamiltonian, provided by the Foldy-Wouthuysen transformation combined with standard nonrelativistic path-integral methods.

Consider the evolution in Euclidean space, using the gauge $A_0=0$ of a quark at $\vec{x}$ influenced by an instanton at $\vec{r}$. [$A_\mu = A_\mu{}^a\lambda^a/2$, using the usual Gell-Mann SU(3) matrices.] We assume that the quark is initially in a region where the gauge field is zero ($A_\mu=0$). It advances through a region where it interacts with the full instanton field and finally reaches the inequivalent pure gauge field region where

$$\vec{A}=\tfrac{1}{2}U^{-1}\vec{\nabla}U(\vec{x}-\vec{r}), \tag{5}$$

with

$$U(\vec{x}-\vec{r})=\exp\{i\pi\vec{\lambda}'\cdot(\vec{x}-\vec{r})/[(\vec{x}-\vec{r})^2+\rho^2]^{1/2}\}, \tag{6}$$

where the three components of $\vec{\lambda}'$ generate an SU(2) subgroup of SU(3). $U(\vec{x}-\vec{r})$ can also be written as

$$U(\vec{x}-\vec{r})=R\begin{bmatrix}\Omega & 0\\ 0 & 1\end{bmatrix}R^{-1}, \tag{7}$$

where the 2×2 matrix

$$\Omega(\vec{x}-\vec{r})=\exp\{i\pi\vec{\tau}\cdot(\vec{x}-\vec{r})/[(\vec{x}-\vec{r})^2+\rho^2]^{1/2}\}, \tag{8}$$

R is a rotation in color space, and the nature of the instanton as an embedding of an SU(2) object in color SU(3) is clearly demonstrated. The results of such calculation show that the quark wave function $\psi_{s_i\alpha_i}(\vec{x})$ with spinor label s_i and color label α_i is transformed by propagation through the region of influence of an instanton of scale size ρ located at the origin of spatial coordinates, to

$$\tilde{\psi}(\vec{x})=A(\vec{x},\vec{p},\vec{\sigma},\rho)\psi(\vec{x}), \tag{9}$$

where

$$A(\vec{x},\vec{p},\vec{\sigma},\rho)=O(\vec{x},\vec{p},\vec{\sigma},\rho)U^{-1}(\vec{x}). \tag{10}$$

The operator O, to order m^{-2} and neglecting some terms which appear only in $\theta\neq 0$ vacuums, i.e., in P- or T-violating worlds, has the form

$$\begin{aligned}O(\vec{x},\vec{p},\vec{\sigma},\rho)=1&-\frac{i}{m}\vec{L}\cdot\vec{\nabla}-\frac{i}{2m}\vec{\sigma}\cdot\vec{\nabla}\\ &-\frac{1}{2m^2}\vec{L}\cdot\vec{\nabla}\vec{\sigma}\cdot\vec{\nabla}+\frac{1}{4m^2}\vec{\sigma}\cdot(\vec{p}\times\vec{\nabla})\\ &-\frac{1}{5m^2}\left\{p_j,\frac{x_j\rho^2}{(\vec{x}^2+\rho^2)^2}(\vec{\sigma}\times\vec{x})_k\right\}_+\nabla_k\end{aligned} \tag{11}$$

[from Eqs. (18) and (2.14) of Callan *et al.*[3] and Aragão de Carvalho,[4] respectively], where $\vec{L}=\vec{x}\times\vec{p}$, $\vec{p}$ operates on the $\vec{x}$ of $\psi(\vec{x})$, while $\vec{\nabla}$ acts only on the $U^{-1}(\vec{x})$, and the repeated indices in the anticommutator term denote summation. For an antiquark, $U^{-1}(\vec{x})$ is replaced by $U^T(\vec{x})$, while the contribution from an anti-instanton reverses the signs of the $1/m$ terms. As shown by Callan *et al.*, the expression for the time-evolved multiquark wave function can be constructed as a tensor product of the single-quark operators and can be used to construct the effective Hamiltonian for any color-singlet system.

For a configuration of m quarks and $\bar{m}$ antiquarks at positions $\vec{x}_i$, with colors α_i, the evolution through the field of one instanton positioned at the spatial origin is given by

$$\tilde{\psi}_{\{\alpha_i\}}(\{\vec{x}_i\}) = \prod_{i=1}^{m+\bar{m}} O(\{i\}) \prod_{i=1}^{m} U^{-1}{}_{\alpha_i\alpha_i'}(\vec{x}_i) \prod_{i=m+1}^{m+\bar{m}} U^T_{\alpha_i\alpha_i'}(\vec{x}_i)\psi_{\{\alpha_i'\}}(\{\vec{x}_i\}). \tag{12}$$

The effective Hamiltonian for the one-instanton background field is then extracted as

$$-H\psi = \tilde{\psi} - \psi. \tag{13}$$

As we are interested in the color-singlet states we select that sector of the Hamiltonian by including in it projection operators for the type of multiquark state under consideration. This effective Hamiltonian would require integration over instanton center position, scale size, and color-space orientation, i.e., over all R rotations in Eq. (7), to yield the dilute-gas result.

One can see that the color structure of the one-instanton effective Hamiltonian is determined by the static-limit term in the multiquark evolution Eq. (9), i.e., the leading term in $O(\vec{x}, \vec{p}, \vec{\sigma}, \rho)$.

The rest of the effective Hamiltonian is generated by a complicated set of single-particle operators, the other terms of the $O(\{i\})$, acting on the static term, followed by the integration over the dilute-gas degrees of freedom.

As we are presently interested in the way potential terms arise through the color structure of the multiquark state, we therefore consider the static-limit one-instanton effective Hamiltonian

$$H_{\text{static},1} = -P_{\alpha_i\cdots\alpha_n}P_{\alpha_i'\cdots\alpha_n'}\left[\prod_{i=1}^{m} U^{-1}{}_{\alpha_i\alpha_i'}(\vec{x}_i) \prod_{i=m+1}^{m+\bar{m}} U^T_{\alpha_i\alpha_i'}(\vec{x}_i) - \prod_{i=1}^{m+\bar{m}} \delta_{\alpha_i\alpha_i'}\right]. \tag{14}$$

In Sec. IV we derive a simple rule for the construction of $H_{\text{static},1}$ for the family of multiquark states of Sec. II. From that we can, of course, also investigate the structure of the static spin-independent potential formed by use of $H_{\text{static},1}$ in the dilute-gas approximation. Equation (14) is just the multiquark generalization of the Wilson-loop argument.

IV. INDUCTION OF THE STATIC ONE-INSTANTON EFFECTIVE HAMILTONIAN FOR THE "TREE" MULTIQUARK FAMILY

The static one-instanton effective Hamiltonian for the meson is well known:

$$H^{q\bar{q}}_{\text{static},1} = -\tfrac{1}{3}\,\text{tr}[\Omega(\vec{x}_1 - \vec{r})\Omega^{-1}(\vec{x}_2 - \vec{r}) - I], \tag{15}$$

where the instanton has been placed at $\vec{r}$. We now proceed to show by induction how this can be generalized to the higher members of the family of multiquark states defined in Sec. II—the color "tree" family. As before, we focus attention on the effect of replacing a particle with a pair of color-antisymmetrized antiparticles. For definiteness consider the Hamiltonian Eq. (14) with the quark j, color α_j singled out for eventual replacement by antiquarks with colors β_1, β_2 at $\vec{y}_1$, $\vec{y}_2$, respectively:

$$H_{\text{static},1} = -P_{\alpha_1\cdots\alpha_j\cdots\alpha_n}P_{\alpha_1'\cdots\alpha_j'\cdots\alpha_n'}\left[\prod_{\substack{i=1\\ i\neq j}}^{m} U^{-1}{}_{\alpha_i\alpha_i'}(\vec{x}_i) \prod_{i=m+1}^{m+\bar{m}} U^T_{\alpha_i\alpha_i'}(\vec{x}_i)U^{-1}{}_{\alpha_j\alpha_j'}(\vec{x}_j) - \prod_{i=1}^{m+\bar{m}} \delta_{\alpha_i\alpha_i'}\right] \tag{16}$$

$$= -[X_{\alpha_j\alpha_j'}U^{-1}{}_{\alpha_j\alpha_j'}(\vec{x}_j) - 1] \tag{17}$$

$$= -[\text{tr}(X^T U^{-1}(\vec{x}_j)) - 1], \tag{18}$$

with

$$X_{\alpha_j\alpha_j'} = P_{\alpha_1\cdots\alpha_j\cdots\alpha_n}P_{\alpha_1'\cdots\alpha_j'\cdots\alpha_n'}\prod_{\substack{i=1\\ i\neq j}}^{m} U^{-1}{}_{\alpha_i\alpha_i'}(\vec{x}_i) \prod_{i=m+1}^{m+\bar{m}} U^T_{\alpha_i\alpha_i'}(\vec{x}_i). \tag{19}$$

Now the Hamiltonian of the extended state with the antiquarks at $\vec{y}_1, \vec{y}_2$ is

$$H_{\text{static},1} = -\left(P_{\alpha_1 \cdots \gamma \cdots \alpha_n} P_{\alpha_1' \cdots \gamma' \cdots \alpha_n} \tfrac{1}{2}\epsilon_{\gamma\beta_1\beta_2}\epsilon_{\gamma'\beta_1'\beta_2'} \prod_{\substack{i=1\\ i\neq j}}^{m} U^{-1}{}_{\alpha_i\alpha_i'}(\vec{x}_1) \prod_{i=m+1}^{m+\bar{m}} U^T_{\alpha_i\alpha_i'}(\vec{x}_1) U^T_{\beta_1\beta_1'}(\vec{y}_1) U^T_{\beta_2\beta_2'}(\vec{y}_2) - 1\right) \tag{20}$$

$$= -[\tfrac{1}{2}X_{\gamma\gamma'}\epsilon_{\gamma\beta_1\beta_2}\epsilon_{\gamma'\beta_1'\beta_2'} U^T_{\beta_1\beta_1'}(\vec{y}_1) U^T_{\beta_2\beta_2'}(\vec{y}_2) - 1]\,. \tag{21}$$

To proceed further, the nature of the instanton as an SU(2) object embedded in SU(3) is crucial. Given two SU(3) matrices A, B which consist of SU(2) matrices $\mathcal{A}, \mathcal{B}$ embedded into SU(3) with the same rotation matrix R,

$$A = R \begin{bmatrix} \mathcal{A} & 0 \\ 0 & 1 \end{bmatrix} R^{-1}, \quad B = R \begin{bmatrix} \mathcal{B} & 0 \\ 0 & 1 \end{bmatrix} R^{-1}, \tag{22}$$

then defining a matrix T by

$$T_{\alpha\alpha'} = \tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha'\beta'\gamma'} A_{\beta\beta'} B_{\gamma\gamma'}, \tag{23}$$

it is shown in Appendix A that

$$T = \frac{1}{2}\left\{ R \begin{bmatrix} \mathcal{A}^{-1} + \mathcal{B}^{-1} & 0 \\ 0 & \mathrm{tr}(\mathcal{A}\mathcal{B}^{-1}) \end{bmatrix} R^{-1} \right\}^T \tag{24}$$

$$= \frac{1}{2}\left(\left\{ R \begin{bmatrix} \mathcal{A}^{-1} & 0 \\ 0 & \frac{1}{2}\mathrm{tr}(\mathcal{A}\mathcal{B}^{-1}) \end{bmatrix} R^{-1} \right\}^T + \left\{ R \begin{bmatrix} \mathcal{B}^{-1} & 0 \\ 0 & \frac{1}{2}\mathrm{tr}(\mathcal{A}\mathcal{B}^{-1}) \end{bmatrix} R^{-1} \right\}^T \right) \tag{25}$$

so the matrix T is the average of two matrices which are the transposes of matrices rotated from block diagonal by the matrix R. The element T_{33} is also somewhat more involved than was the case in A and B.

Returning to our effective Hamiltonians, Eqs. (18) and (21), we note that if

$$U(\vec{x}) = R \begin{bmatrix} \Omega(\vec{x}) & 0 \\ 0 & 1 \end{bmatrix} R^{-1}, \tag{26}$$

then

$$U^{-1}(\vec{x}) = R \begin{bmatrix} \Omega^{-1}(\vec{x}) & 0 \\ 0 & 1 \end{bmatrix} R^{-1} \tag{27}$$

and

$$U^T(\vec{x}) = R^{-1T} \begin{bmatrix} \Omega^T(\vec{x}) & 0 \\ 0 & 1 \end{bmatrix} R^T. \tag{28}$$

Then it follows that

$$H_{\text{static},1} = -[\mathrm{tr}(X^T U^{-1}(\vec{x}_j)) - 1] \tag{29}$$

$$= -\left(\mathrm{tr}\left\{ X^T R \begin{bmatrix} \Omega^{-1}(\vec{x}) & 0 \\ 0 & 1 \end{bmatrix} R^{-1} \right\} - 1 \right) \tag{30}$$

and

$$H'_{\text{static},1} = -\left(\tfrac{1}{2} X_{\gamma\gamma'} \left\{ R^{-T} \begin{bmatrix} \Omega^{-1T}(\vec{y}_1) + \Omega^{-1T}(\vec{y}_2) & 0 \\ 0 & \mathrm{tr}(\Omega(\vec{y}_1)\Omega^{-1}(\vec{y}_2)) \end{bmatrix} R^T \right\}^T_{\gamma\gamma'} - 1 \right)$$

$$= -\left(\tfrac{1}{2} X_{\gamma\gamma'} \left\{ R \begin{bmatrix} \Omega^{-1}(\vec{y}_1) + \Omega^{-1}(\vec{y}_2) & 0 \\ 0 & \mathrm{tr}(\Omega(\vec{y}_1)\Omega^{-1}(\vec{y}_2)) \end{bmatrix} R^{-1} \right\}_{\gamma\gamma'} - 1 \right) \tag{31}$$

$$= -\left(\tfrac{1}{2}\,\mathrm{tr}\left\{ X^T R \begin{bmatrix} \Omega^{-1}(\vec{y}_1) + \Omega^{-1}(\vec{y}_2) & 0 \\ 0 & \mathrm{tr}(\Omega(\vec{y}_1)\Omega^{-1}(\vec{y}_2)) \end{bmatrix} R^{-1} \right\} - 1 \right). \tag{32}$$

This analysis shows that, as required, the operator associated with the antisymmetrized pair of antiquarks transforms like the quark operator it replaced i.e., with R matrices. The replacement also retains the block-diagonal form of the matrix being rotated. We also consider the matrix X. It is at the other end of the link which originally terminated at the quark α_j. Accordingly, X is a $\underline{3}$ object (antiquarklike) and will have the form

$$X = R^{-1T} \begin{bmatrix} \Xi^T & 0 \\ 0 & x' \end{bmatrix} R^T, \tag{33}$$

e.g., if X is just an antiquark then

$$X = \tfrac{1}{3} R^{-1T} \begin{bmatrix} \Omega^T & 0 \\ 0 & 1 \end{bmatrix} R^T, \tag{34}$$

so that finally

$$H = -[\mathrm{tr}(\Xi\Omega^{-1}(\vec{x}_j)) + x' - 1], \tag{35}$$

while

$$H' = -\{\tfrac{1}{2}[\mathrm{tr}(\Xi\Omega^{-1}(\vec{y}_1)) + \mathrm{tr}(\Xi\Omega^{-1}(\vec{y}_2)) + x'\,\mathrm{tr}(\Omega(\vec{y}_1)\Omega^{-1}(\vec{y}_2))] - 1\}. \tag{36}$$

We now have our rule. When increasing the complexity of the multiquark state by replacing a particle by a pair of color-antisymmetrized antiparticles, the following procedure is performed on the expression for $H-1$:

(i) In all trace terms containing the matrix of the removed particle, that matrix is replaced by the average of traces involving the matrices of the two added particles (keeping the same type of matrix as originally), i.e., $\Omega(\vec{x}_j) \to \frac{1}{2}(\Omega(\vec{y}_1) + \Omega(\vec{y}_2))$.

(ii) All terms not containing the matrix of the removed particle are multiplied by the factor $\mathrm{tr}(\Omega(\vec{y}_1)\Omega^{-1}(\vec{y}_2))/2$.

For an initial meson,

$$X = \tfrac{1}{3} R^{-1T} \begin{bmatrix} \Omega^T(\vec{x}_1) & 0 \\ 0 & 1 \end{bmatrix} R^T \tag{37}$$

as X contains the normalization factors of the projection operators, so

$$H_{\mathrm{meson}}(\vec{x}_1, \vec{x}_2) = -\tfrac{1}{3}[\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2)) + 1] - 1 \tag{38}$$

or as more usually written

$$H_{\mathrm{meson}}(\vec{x}_1, \vec{x}_2) = -\{\tfrac{1}{3}[\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2)) - 2]\} = -\tfrac{1}{3}\,\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2) - I). \tag{39}$$

Equation (38) is obviously in the form to which the rule is applied. By use of the rule or by directly from Eq. (36) for H',

$$H_{\mathrm{baryon}}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = -\{\tfrac{1}{3}[\mathrm{tr}(\Omega(\vec{x}_1)\tfrac{1}{2}[\Omega^{-1}(\vec{x}_2) + \Omega^{-1}(\vec{x}_3)]) + 1\times\mathrm{tr}(\tfrac{1}{2}\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_3))] - 1\} \tag{40}$$

$$= -\tfrac{1}{6}[\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2)) + \mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_3)) + \mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_3)) - 6], \tag{41}$$

which reproduces the result Eq. (1) of Callan *et al.*, in this case for an antibaryon. (We replaced the quark by antiquarks.)

One can immediately construct the static effective Hamiltonians for the higher members of the color tree family. For T baryonium with quarks at $\vec{x}_1, \vec{x}_2$ and antiquarks at $\vec{x}_3, \vec{x}_4$,

$$H = -\tfrac{1}{12}[\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_4)) + \mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_3)) + \mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_4)) + \mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_3)) + \mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_4)) - 12], \tag{42}$$

TABLE I. The one-instanton static-limit effective Hamiltonians for the "color tree" states shown in Figs. 1(a)–1(d). The particle at the position $\vec{x}_i$ is of the type indicated by the figure and the order of the terms in each expression is that expected when systematically expanding, using the inductive rule presented in Sec. IV. For compactness, the Hamiltonians listed have not had the many-body terms broken down by the method of Sec. V, and have the instanton at the spatial origin.

State	H_{eff}
Meson	$-\frac{1}{3}[\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2)) - 2]$
Baryon	$-\frac{1}{6}[\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2)) + \mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_3)) + \mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_3)) - 6]$
T baryonium	$-\frac{1}{12}[\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_4)) + \mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_3))$
	$+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_4)) + \mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_3)) + \mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_4)) - 12]$
$(QQ)\bar{Q}(QQ)$	$\frac{1}{24}[\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))\,\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_4)) + \mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))$
	$\times\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_5)) + \mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_3))\,\mathrm{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_5))$
	$+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_4)) + \mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_5)) + \mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_3))$
	$\times\mathrm{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_5)) + \mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_4)) + \mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_5)) - 24]$

TABLE II. The one-instanton static-limit effective Hamiltonians for the six-particle states of Figs. 1(e) and 1(f). The terms in each expression are in the order obtained by systematic use of the induction rule of Sec. V. The instanton is at the spatial origin.

State	H_{eff}
$(QQ)^3$ (dibaryon)	$-\frac{1}{48}[\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))\,\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_4))+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))$ $\times\mathrm{tr}(\Omega(\vec{x}_6)\Omega^{-1}(\vec{x}_4))+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))\,\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_5))$ $+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))\,\mathrm{tr}(\Omega(\vec{x}_6)\Omega^{-1}(\vec{x}_5))+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_3))$ $\times\mathrm{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_5))+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_6))\,\mathrm{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_5))$ $+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_4))\,\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_6))+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_5))$ $\times\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_6))+\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_3))\,\mathrm{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_5))$ $+\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_6))\,\mathrm{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_5))+\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_4))$ $\times\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_6))+\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_5))\,\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_6))-48]$
$(QQ)\bar{Q}Q(\overline{QQ})$	$-\frac{1}{48}[\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))\,\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_4))\,\mathrm{tr}(\Omega(\vec{x}_5)\Omega^{-1}(\vec{x}_6))$ $+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))\,\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_5))+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))$ $\times\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_6))+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_3))\,\mathrm{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_5))$ $+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_3))\,\mathrm{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_6))+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_4))$ $\times\mathrm{tr}(\Omega(\vec{x}_5)\Omega^{-1}(\vec{x}_6))+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_5))+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_6))$ $+\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_3))\,\mathrm{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_5))+\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_3))$ $\times\mathrm{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_6))+\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_4))\,\mathrm{tr}(\Omega(\vec{x}_5)\Omega^{-1}(\vec{x}_6))$ $+\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_5))+\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_6))-48]$

showing how the multiquark's diquark and antidiquark elements (which are each color triplets) are involved in a many-body potential term as expected. It is to be noted that one can, of course, superpose the two particles in one of the color-antisymmetrized pairs $[\mathrm{tr}(\Omega(\vec{x})\Omega^{-1}(\vec{x}))=2]$ and so reverse down the multiquark family.

So, the effective Hamiltonian has many-body terms and these for static quarks are really just potentials for the multiquarks of Fig. 1. They are displayed in Tables I and II. Again for comparison the effective Hamiltonian for M baryonium, Fig. 2, is presented in Table III. It is interesting to note that only the four-body potential appears in both T and M baryonium, in the $4(Q)\bar{Q}$ state, and in the dibaryon, which are the multiquark states currently the subject of the most experimentation and theoretical speculation.

Alternatively, instead of building up to the complexity of the multiquark state of interest, one can work through the color tree of the multiquark building up the potential, starting from an antisymmetrized pair. We saw earlier that two matrices of the form

$$R\begin{bmatrix}\Omega^{-1}(\vec{x}_i) & 0\\ 0 & 1\end{bmatrix}R^{-1},$$

where Ω is an SU(2) matrix, can be combined at a vertex using the antisymmetric tensors in the projection operators to yield

TABLE III. One-instanton static-limit effective Hamiltonian for M baryonium, presented for comparison with that of T baryonium in Table I. The particle at $\vec{x}_i$ has the nature (quark or antiquark) indicated by Fig. 2.

State	H_{eff}
M baryonium	$-\frac{1}{24}\{\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_3))+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_4))+\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_3))$ $+\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_4))-\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))\,\mathrm{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_4))$ $+2[\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_3))\,\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_4))+\mathrm{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_4))$ $\times\mathrm{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_3))]-20\}$

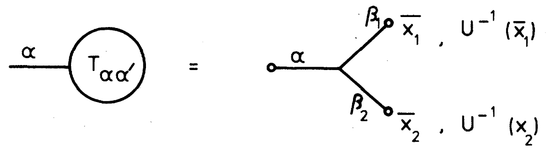


FIG. 3. A pair of color-antisymmetrized quarks attached to the link $\alpha\alpha'$.

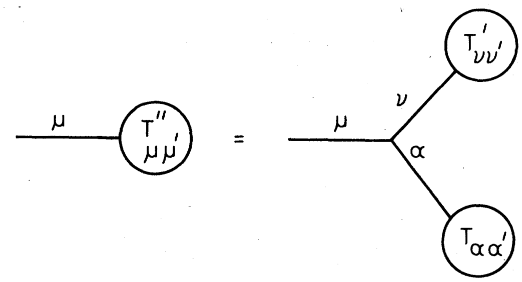


FIG. 4. Combining T, T' matrices at a junction.

$$T_{\alpha\alpha'}=\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha'\beta'\gamma'}\left\{R\begin{bmatrix}\Omega^{-1}(\bar{x}_1) & 0\\ 0 & 1\end{bmatrix}R^{-1}\right\}_{\beta\beta'}\left\{R\begin{bmatrix}\Omega^{-1}(\bar{x}_2) & 0\\ 0 & 1\end{bmatrix}R^{-1}\right\}_{\gamma\gamma'}$$
$$=\tfrac{1}{2}\left\{R\begin{bmatrix}\Omega^{T}(\bar{x}_1)+\Omega^{T}(\bar{x}_2) & 0\\ 0 & \mathrm{tr}(\Omega(\bar{x}_1)\Omega^{-1}(\bar{x}_2))\end{bmatrix}R^{-1}\right\}^{T}_{\alpha\alpha'}, \qquad (43)$$

where T has the expected U^T-type color-space rotation reflecting our selection of $\underline{\bar{3}}$ from $\underline{3}\otimes\underline{3}$. In dividing up T as before, we see that we have a sum of two matrices of antiquark nature which, on proceeding along the color tree, we find are either summed against a quark ending the chain, or else enter a junction with like-natured ($\underline{\bar{3}}$) objects. To continue combining these T matrices we need a generalization of Eq. (43). In Appendix A we show that for block-diagonal matrices

$$A=R\begin{bmatrix}\mathcal{A} & 0\\ 0 & a\end{bmatrix}R^{-1},\quad B=R\begin{bmatrix}\mathcal{B} & 0\\ 0 & b\end{bmatrix}R^{-1} \qquad (44)$$

with $\mathcal{A},\mathcal{B}$ SU(2) matrices, we find that

$$T_{\alpha\alpha'}=\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha'\beta'\gamma'}A_{\beta\beta'}B_{\alpha\alpha'} \qquad (45)$$

implies

$$T^{T}=\tfrac{1}{2}R\begin{bmatrix}b\mathcal{A}^{-1}+a\mathcal{B}^{-1} & 0\\ 0 & \mathrm{tr}(\mathcal{A}\mathcal{B}^{-1})\end{bmatrix}R^{-1}, \qquad (46)$$

or

$$T=\tfrac{1}{2}bR^{-1T}\begin{bmatrix}\mathcal{A}^{-1T} & 0\\ 0 & \dfrac{\frac{1}{2}\mathrm{tr}(\mathcal{A}\mathcal{B}^{-1})}{b}\end{bmatrix}R^{T}$$
$$+\tfrac{1}{2}aR^{-1T}\begin{bmatrix}\mathcal{B}^{-1T} & 0\\ 0 & \dfrac{\frac{1}{2}\mathrm{tr}(\mathcal{A}\mathcal{B}^{-1})}{a}\end{bmatrix}R^{T}, \qquad (47)$$

when we have regained a sum of two block-diagonal matrices transforming with the transpose-rotation matrices, and with SU(2) upper blocks.

With this generalization we can combine matrices from our first pair of antisymmetrized quarks with one of the two like-natured $\underline{\bar{3}}$ matrices (or sums of matrices) at the next junction. The meson normalization factor of $\frac{1}{3}$ for the projection operators must be inserted at the first step. For example, the portion of a multiquark in Fig. 3 is easily dealt with by our previous matrix result yielding

$$T_{\alpha\alpha'}=\frac{1}{3}\times\frac{1}{2}\times\left\{R\begin{bmatrix}\Omega(\bar{x}_1)+\Omega(\bar{x}_2) & 0\\ 0 & \mathrm{tr}(\Omega(\bar{x}_1)\Omega^{-1}(\bar{x}_2))\end{bmatrix}R^{-1}\right\}^{T}_{\alpha\alpha'}, \qquad (48)$$

and now with the above generalization we can combine this with the $\underline{\bar{3}}$ object T' of Fig. 4 as

$$T''_{\mu\mu'}=\tfrac{1}{2}\epsilon_{\mu\alpha\gamma}\epsilon_{\mu'\alpha'\nu'}T_{\alpha\alpha'}T'_{\nu\nu'}. \qquad (49)$$

For example, if T' is a simple antiquark matrix $U^{T}_{\nu\nu'}(\bar{x}_3)$ as in Fig. 5, then

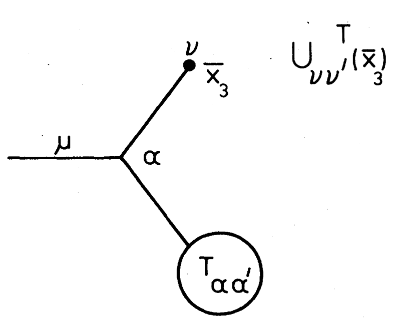


FIG. 5. A particular case of Fig. 4 when $\bar{x}_3$ particle is an antiquark.

$$T''_{\mu\mu'}=\tfrac{1}{6}\epsilon_{\mu\alpha\nu}\epsilon_{\mu'\alpha\gamma'}\frac{1}{2}\left\{R^{-1T}\begin{bmatrix}\Omega^T(\vec{x}_1) & 0\\ 0 & \tfrac{1}{2}\operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))\end{bmatrix}R^T\right\}_{\alpha\alpha'}\left\{R^{-1T}\begin{bmatrix}\Omega^T(\vec{x}_3) & 0\\ 0 & 1\end{bmatrix}R^T\right\}_{\nu\nu'}+(\vec{x}_1\leftrightarrow\vec{x}_2)\,. \tag{50}$$

So, using our rule from Eq. (46),

$$\begin{aligned}T''=&\tfrac{1}{12}R\begin{bmatrix}\Omega^{-1}(\vec{x}_3)\tfrac{1}{2}\operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))+1\Omega^{-1}(\vec{x}_1) & 0\\ 0 & \operatorname{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_1))\end{bmatrix}R^{-1}\\ &+\tfrac{1}{12}R\begin{bmatrix}\Omega^{-1}(\vec{x}_3)\tfrac{1}{2}\operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))+1\Omega^{-1}(\vec{x}_2) & 0\\ 0 & \operatorname{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_2))\end{bmatrix}R^{-1}\,.\end{aligned} \tag{51}$$

Note that we can continue to restore these matrices to the form of a sum of multiples of matrices with the block-diagonal form of A or B [Eq. (44)] and an SU(2) matrix as the upper block and so we can continue to combine them at vertices all the way through the color tree using Eq. (46) until we meet a single particle. We complete the above example by attaching an antiquark to $T''_{\mu\mu'}$ to generate again the T-baryonium result

$$H-1=-\tfrac{1}{12}[U^T_{\mu\mu'}(\vec{x}_4)T''_{\mu\mu'}] \tag{52}$$

$$=-\tfrac{1}{12}[\operatorname{tr}(U(\vec{x}_4)T'')] \tag{53}$$

$$=-\tfrac{1}{12}\left(\operatorname{tr}\left\{\begin{bmatrix}\Omega(\vec{x}_4) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\Omega^{-1}(\vec{x}_3)\operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))+\Omega^{-1}(\vec{x}_2)+\Omega^{-1}(\vec{x}_1) & 0\\ 0 & \operatorname{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_1))+\operatorname{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_2))\end{bmatrix}\right\}\right), \tag{54}$$

$$H-1=-\tfrac{1}{12}[\operatorname{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_3))\operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))+\operatorname{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_2))+\operatorname{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_1))+\operatorname{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_2))+\operatorname{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_1))]\,, \tag{55}$$

regaining the result of Table I. The increasing number of matrices into which the $T^{(n)}$ must be divided to regain the form required for combination at vertices using Eq. (46) directly shows the large number of terms which appear for multiquark states.

V. MANY-BODY POTENTIAL TERMS

The formation of the static effective Hamiltonian for a multiquark state of the color tree type due to the influence of an instanton showed the existence of many-body potential terms for states more complicated than the baryon. These terms have the form of products of two-particle trace factors, e.g.,

$$H_{\rm MB}(\{\vec{x}_i\})=\frac{-1}{N}[\operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2))\operatorname{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_5))\operatorname{tr}(\Omega(\vec{x}_4)\Omega^{-1}(\vec{x}_6))\cdots-2^n]\,, \tag{56}$$

where n is the number of trace factors, and $N=3\times 2^k$ denotes the normalization denominator factor for the projection operators, with k denoting the number of tree junctions. Clearly, if the particles were superposed in all but one trace factor, $H_{\rm MB}$ would take on the form

$$H_{\rm MB}(\vec{x}_1,\vec{x}_2)=\frac{-2^{n-1}}{N}[\operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2)-I)]\,, \tag{57}$$

i.e., there are two-body-like terms in $H_{\rm MB}$. In fact, when the two particles in a trace factor are close together compared with the distance $\vec{r}$ to an instanton from their neighborhood, i.e., $|\vec{x}_1-\vec{x}_2|\ll|\tfrac{1}{2}(\vec{x}_1+\vec{x}_2)-\vec{r}|$,

$$\operatorname{tr}(\Omega(\vec{x}_1-\vec{r})\Omega^{-1}(\vec{x}_2-\vec{r}))\sim 2[1+O(r^{-6})]\,. \tag{58}$$

We extract these two-body contributions leaving many-body potentials which will be less sensitive to distant instantons.

The breakup into two-body terms and further many-body potentials is achieved using the decomposition

$$\begin{aligned}\prod_{i=1}^{n}x_i-2^n&=\prod_{i=1}^{n}(x_i-2+2)-2^n\\ &=2^{n-1}\prod_i(x_i-2)+2^{n-2}\prod_{i\neq j}(x_i-2)(x_j-2)+\cdots+2^{n-j}\sum_{\{\sigma_j\}}\prod_{k\in\sigma_j}(x_k-2)+\cdots+\prod_{i=1}^{n}(x_i-2)\,,\end{aligned} \tag{59}$$

where σ_j is a selection of j elements from $\{1,\ldots,n\}$. Using this for the trace-factor product allows the extraction of the two-body potentials [the (x_i-2)'s] and generates the various many-body terms. The factors in the many-body terms will now behave for $r\gg|\vec{x}_1|$, $|\vec{x}_2|$, as $O(r^{-6})$, and so the more factors present, the less the effect of distant instantons will be felt. This property should prove very useful in calculating their contributions in a dilute-gas approximation.

As an example, consider the T-baryonium effective Hamiltonian Eq. (42) and Table I with quarks at

$\vec{x}_1, \vec{x}_2$, and antiquarks at $\vec{x}_3, \vec{x}_4$. Making this modification to the many-body term and casting two-body terms into mesonlike form

$$H_T = -\tfrac{1}{12}[\operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_3)-I)+\operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_4)-I)+\operatorname{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_3)-I)+\operatorname{tr}(\Omega(\vec{x}_2)\Omega^{-1}(\vec{x}_4)-I)$$
$$+2\operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2)-I)+2\operatorname{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_4)-I)+\operatorname{tr}(\Omega(\vec{x}_1)\Omega(\vec{x}_2)-I)\operatorname{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_4)-I)] \tag{60}$$

$$=\tfrac{1}{4}\sum_{\substack{i=1,2\\ j=3,4}} H_{q\bar{q}}(\vec{x}_i,\vec{x}_j)+\tfrac{1}{2}H_{q\bar{q}}(\vec{x}_1,\vec{x}_2)+\tfrac{1}{2}H_{q\bar{q}}(\vec{x}_3,\vec{x}_4)-\tfrac{1}{12}\operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2)-I)\operatorname{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_4)-I)\,, \tag{61}$$

where

$$H_{q\bar{q}}(\vec{x}_1,\vec{x}_2) = -\tfrac{1}{3}\operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2)-I)\,.$$

We see that the particles in each diquark of the T-baryonium are bound by a two-body potential twice the strength of the quark-antiquark potentials in the system. The color-structural nature of these potentials is reflected in the fact that this relative weighting of the two-body potentials between the quark pairs and the quark-antiquark pairs is just that found from the gluon-exchange-motivated interactions of the form $\operatorname{tr}(\lambda_i\lambda_j)V(\vec{x}_i-\vec{x}_j)$ when the color trace is evaluated between T-baryonium color states. In Sec. VI we present some remarks concerning the asymptotic behavior of the four-body potential contribution in the dilute-gas approximation.

VI. THE FOUR-BODY POTENTIAL

Inspection of Tables I and II shows that the only many-body potential to present itself in the one-instanton effective Hamiltonian for the T and M baryoniums, the $(QQ)\bar{Q}(QQ)$ and the $(QQ)^3$ dibaryon, is the four-body term

$$H_4 = \operatorname{tr}(\Omega(\vec{x}_1)\Omega^{-1}(\vec{x}_2)-I)\operatorname{tr}(\Omega(\vec{x}_3)\Omega^{-1}(\vec{x}_4)-I)\,. \tag{62}$$

As this term will also be found in the Hamiltonian of any system which has more complicated many-body terms, due to the decomposition [Eq. (59)], we consider it worthwhile to examine the static-spin-independent dilute-gas approximation effective Hamiltonian, which could be obtained from H_4 by integrating over instanton size with the appropriate density distribution sizes, integrating over color orientation, and integrating over position of the instanton center. The size integration would be cut off at a scale size determined by the size of the abnormal bag of dilute instanton gas existing around the quarks in the normal strong-coupling phase. At present we concentrate on the instanton position integration. As the color-orientation matices have disappeared, the integration over color orientation is trivial.

Following Callan *et al.*, we consider a dimensionless potential $W_{(4)}(\{\vec{x}_i/\rho\})$ for an instanton size ρ,

$$H_4 = +2\int \frac{d\rho}{\rho^2}D(\rho)W_4(\{\vec{x}_i/\rho\})\,, \tag{63}$$

where the factor of 2 takes account of anti-instantons and $D(\rho)$ is the density of instanton of size ρ,

$$D(\rho)=x^6e^{-x} \quad \text{for } x=8\pi^2/g^2(\rho),$$

and

$$W_4\left(\frac{\vec{x}_1}{\rho},\frac{\vec{x}_2}{\rho},\frac{\vec{x}_3}{\rho},\frac{\vec{x}_4}{\rho}\right) = -\frac{1}{\rho^3}\frac{1}{N}\int d^3\vec{r}\,\operatorname{tr}(\Omega(\vec{x}_1-\vec{r})\Omega^{-1}(\vec{x}_2-\vec{r})-I)\operatorname{tr}(\Omega(\vec{x}_3-\vec{r})\Omega^{-1}(\vec{x}_4-\vec{r})-I) \tag{64}$$

$$= -\frac{4}{N\rho^3}\int d^3\vec{r}\left[\cos\left(\frac{\pi|\vec{x}_1-\vec{r}|}{[(\vec{x}_1-\vec{r})^2+\rho]^{1/2}}\right)\cos\left(\frac{\pi|\vec{x}_2-\vec{r}|}{[(\vec{x}_1-\vec{r})^2+\rho^2]^{1/2}}\right)\right.$$
$$\left.+\frac{(\vec{x}_1-\vec{r})\cdot(\vec{x}_2-\vec{r})}{|\vec{x}_1-\vec{r}|\,|\vec{x}_2-\vec{r}|}\sin\left(\frac{\pi|\vec{x}_2-\vec{r}|}{[(\vec{x}_1-\vec{r})^2+\rho^2]^{1/2}}\right)\sin\left(\frac{\pi|\vec{x}_2-\vec{r}|}{[(\vec{x}_2-\vec{r})^2+\rho^2]^{1/2}}\right)-1\right]\times\begin{bmatrix}x_1\to x_3\\ x_2\to x_4\end{bmatrix}. \tag{65}$$

Expanding the integrand factors for all $|\vec{x}_i/\rho|\ll 1$ we find that $\operatorname{tr}(\Omega(\vec{x}-\vec{r})\Omega^{-1}(\vec{x}-\vec{r})-I)$ for small $|\vec{x}_i/\rho|$ is

$$2\left\{\frac{(\vec{x}_1-\vec{x}_2)^2}{\rho^2}\left[-\sin^2\left(\frac{\pi r/\rho}{[1+r^2/\rho^2]^{1/2}}\right)\right]+\frac{[(\vec{x}_1-\vec{x}_2)\cdot\hat{r}]^2}{\rho^2}\left[\frac{\rho^2}{2r^2}\sin^2\left(\frac{(\pi r)}{(r^2+\rho^2)^{1/2}}\right)-\frac{\pi^2\rho^2}{2(\rho^2+r^2)^3}\right]\right\} \tag{66}$$

and, changing to dimensionless variables $r/\rho\to r$,

$$W=-\frac{4}{N}\left(\int d^3\vec{r}\left\{\vec{s}^2\left[-\frac{1}{2r^2}\sin^2\left(\frac{\pi r}{(1+\vec{r}^2)^{1/2}}\right)\right]+(\vec{s}\cdot\hat{r})^2\left[\frac{1}{2r^2}\sin^2\left(\frac{\pi r}{(1+r^2)^{1/2}}\right)-\frac{\pi^2}{2(1+r^2)^3}\right]\right\}\right.$$
$$\left.\times\left\{\vec{t}^2\left[-\frac{1}{2r^2}\sin^2\left(\frac{\pi r}{(1+r^2)^{1/2}}\right)\right]+(\vec{t}\cdot\hat{r})^2\left[\frac{1}{2r^2}\sin^2\left(\frac{\pi r}{(1+r^2)^{1/2}}\right)-\frac{\pi^2}{2(1+r^2)^3}\right]\right\}\right), \tag{67}$$

where $\vec{s}=(\vec{x}_1-\vec{x}_2)/\rho$ and $\vec{t}=(\vec{x}_3-\vec{x}_4)/\rho$, and

$$W=-\frac{4}{N}s^2t^2\Big\{4\pi\int r^2dr\left[\sin^4\left(\frac{\pi r}{(1+r^2)^{1/2}}\right)\frac{1}{4r^4}\right] + \int d\Omega(\hat{s}\cdot\hat{r})^2(\hat{t}\cdot\hat{r})^2\int r^2dr\left[\sin^2\left(\frac{\pi r}{(1+r^2)^{1/2}}\right)\frac{1}{2r^2}-\frac{\pi^2}{2(1+r^2)^3}\right]^2 + \int d\Omega[(\hat{t}\cdot\hat{r})^2+(\hat{s}\cdot\hat{r})^2]\int r^2dr\left[-\sin^4\left(\frac{(\pi r)}{(1+r^2)^{1/2}}\right)\frac{1}{4r^4}+\sin^2\left(\frac{\pi r}{(1+r^2)^{1/2}}\right)\frac{\pi^2}{4r^2(1+r^2)^3}\right]\Big\}. \tag{68}$$

The angular integrations can be done immediately:

$$\int d\Omega(\hat{t}\cdot\hat{r})^2=\frac{4\pi}{3}, \tag{69a}$$

$$\int d\Omega(\hat{t}\cdot\hat{r})^2(\hat{s}\cdot\hat{r})^2=\frac{4\pi}{15}(1+2\cos^2\theta_{st}), \tag{69b}$$

when θ_{st} is the angle between the two "diquark" vectors $\vec{s}$ and $\vec{t}$. The appearance of θ_{st} in the four-body potential suggests that many-body potentials can give us information on the spatial structures of multiquark states. Rearranging we find

$$W=-\frac{16\pi s^2t^2}{N}[A+(\hat{s}\cdot\hat{t})^2B], \tag{70}$$

$$A=\int_0^\infty r^2dr\left[\frac{1}{12r^4}\sin^4\left(\frac{\pi r}{(1+r^2)^{1/2}}\right)+\tfrac{2}{3}\sin^2\left(\frac{\pi r}{(1+r^2)^{1/2}}\right)\frac{\pi^2}{4r^2(1+r^2)^3}\right]+\tfrac{1}{2}B, \tag{71a}$$

$$B=\tfrac{2}{15}\int_0^\infty r^2dr\left[\frac{1}{2r^2}\sin^2\left(\frac{\pi^2}{(1+r^2)^{1/2}}\right)-\frac{\pi^2}{2(1+r^2)^3}\right]^2. \tag{71b}$$

We note that A and B are positive definite so that the four-body potential is repulsive for all four particles close together in T baryonium. Numerical evaluation finds $A=0.715$ and B much smaller, $B=0.013$, suggesting that spatial orientation may be a little elusive, although a "minimization" of W would suggest $\vec{s}$ and $\vec{t}$ parallel or antiparallel in T baryonium. M baryonium, which has three potential terms of this type with both signs, is more complicated.

The second limit which is easily accessible occurs when all interparticle separations are large. In this limit the integrand in Eq. (65) can be considered in two regions. First, if the $\vec{r}$ vector is near one of the particles, say $\vec{x}_1$, then taking that particle as the origin the integrand becomes

$$-\frac{4}{N}\left[-\cos\left(\frac{\pi r}{(1+r^2)^{1/2}}\right)\left(1-\frac{\pi^2}{8|\vec{x}_2-\vec{r}|^4}\right)+\frac{\hat{r}\cdot(\vec{x}_2-\vec{r})\pi}{2|\vec{x}_2-\vec{r}|^3}\sin\left(\frac{\pi r}{(1+r^2)^{1/2}}\right)+\text{higher-order terms}-1\right] \times\left[\frac{-\pi^2}{8|\vec{x}_3-\vec{r}|^4}+\frac{-\pi^2}{8|\vec{x}_4-\vec{r}|^4}+\frac{\pi^2}{4}\frac{(\vec{x}_3-\vec{r})\cdot(\vec{x}_4-\vec{r})}{|\vec{x}_3-r|^3|\vec{x}_4-r|^3}+\cdots\right]. \tag{72}$$

If $\vec{r}$ is in the vicinity of $\vec{x}$, then $|\vec{x}_i-\vec{r}|\simeq|\vec{x}_i-\vec{x}_1|=x_i$. So the integrand is given by

$$-\frac{4}{N}\left[1+\cos\left(\frac{\pi r}{(1+r^2)^{1/2}}\right)\right]\left(\frac{\pi^2}{8}\right)\left(\frac{\vec{x}_3}{x_3{}^3}-\frac{\vec{x}_4}{x_4{}^3}\right)^2 \tag{73}$$

near $\vec{x}_1$, which is taken as the origin.

Far from all the particles the integrand becomes

$$-\frac{\pi^4}{16N}\left[\frac{(\vec{x}_1-\vec{r})}{|\vec{x}_1-\vec{r}|^3}-\frac{(\vec{x}_2-\vec{r})}{|\vec{x}_2-\vec{r}|^3}\right]^2\left[\frac{(\vec{x}_3-\vec{r})}{|\vec{x}_3-\vec{r}|^3}-\frac{(\vec{x}_4-\vec{r})}{|\vec{x}_4-\vec{r}|^3}\right]^2, \tag{74}$$

to lowest order. To obtain a large separation limit for the potential we consider integrating around the vicinity of each particle in turn, such that

$$W_4\to-\frac{\pi^2}{N}\left\{\tfrac{1}{2}\int d^3\vec{r}\left[1+\cos\left(\frac{\pi r}{(1+r^2)^{1/2}}\right)\right]\right\}\left\{\left(\frac{(\vec{x}_3-\vec{x}_1)}{|\vec{x}_3-\vec{x}_1|^3}-\frac{(\vec{x}_4-\vec{x}_1)}{|\vec{x}_4-\vec{x}_1|^3}\right)^2+\left(\frac{(\vec{x}_1-\vec{x}_3)}{|\vec{x}_1-\vec{x}_3|^3}-\frac{(\vec{x}_2-\vec{x}_3)}{|\vec{x}_2-\vec{x}_3|^3}\right)^2 +\left(\frac{(\vec{x}_1-\vec{x}_4)}{|\vec{x}_1-\vec{x}_4|^3}-\frac{(\vec{x}_2-\vec{x}_4)}{|\vec{x}_2-\vec{x}_4|^3}\right)^2+\left(\frac{(\vec{x}_3-\vec{x}_2)}{|\vec{x}_3-\vec{x}_2|^3}-\frac{(\vec{x}_4-\vec{x}_2)}{|\vec{x}_4-\vec{x}_2|^3}\right)^2+O\left(\left(\frac{1}{\Delta\vec{x}}\right)^6\right)\right\}, \tag{75}$$

for all $|\vec{x}_i - \vec{x}_j|$ large. The potential is attractive here and the terms in Eq. (75) relate to $q\bar{q}$ separation, and not to qq separation within a diquark. In the limit of large separation for all quarks, i.e., $|x_i - x_j| \gg \rho$, the spatial integration over the neighorhood of each quark gives a leading contribution which blinds each quark to its diquark partner yielding the form of potential Eq. (75).

VII. CONCLUSION

We have found a simple but general rule for calculating the contribution to interaction potentials for the color tree family of multiquark states, due to the presence of an instanton or any other semiclassical gauge field configuration which exists in a single SU(2) subgroup of SU(3). As expected, we found many-body potential terms appearing in the more complex states. We were then able to extract hidden two-body potentials from these many-body terms generating further, more tractable, many-body interactions. The static-limit effective Hamiltonian due to one instanton provides the basic building block for the construction of systematic expansions of the effective Hamiltonian in powers of $1/m_q$ and for dilute-gas calculations. Accordingly, we see that there will be spin-spin and spin-orbit corrections to our many-body potential terms. The four-body potential is the only many-body term which appears in T and in M baryonium, in the $(QQ)\bar{Q}(QQ)$ state and in the dibaryon, and therefore it seems worth some effort to investigate it—at least at the level of the dilute-gas approximation for the static spin-independent term. Detailed analysis extending the asymptotic comments of Sec. VI is in progress and will be presented in a companion paper. It is perhaps fitting to note at this point that an attempt to study instanton effects on baryonium states, via the effective determinanal four-fermion interaction related to the fermion zero mode of the background instanton field, and first introduced by 't Hooft, has been made by Hikosaka *et al.*[9] using MIT bag-model wave functions and perturbation theory. Finally, the most interesting feature of instanton effects in multiquark states *is* the ability of these nonperturbative objects to generate many-body potentials.

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APPENDIX A

We present here a proof of the results used in the paper as Eqs. (24) and (46). Consider

$$T_{\alpha\alpha'} = \epsilon_{\alpha ab}\epsilon_{\alpha' a' b'} A_{aa'} B_{bb'} . \tag{A1}$$

Then by writing the ϵ symbols in terms of Kronecker δ's we find

$$\begin{aligned} T_{\alpha\alpha'} = {} & \delta_{\alpha\alpha'}(\mathrm{tr}A\,\mathrm{tr}B - \mathrm{tr}AB) \\ & + A_{b'\alpha}B_{\alpha' b'} - A_{\alpha'\alpha}B_{bb'} \\ & + A_{\alpha' a'}B_{a'\alpha} - A_{aa}B_{\alpha'\alpha} . \end{aligned} \tag{A2a}$$

Using

$$\begin{aligned} \epsilon_{\alpha ab}\epsilon_{\alpha' a' b'} = {} & \delta_{\alpha\alpha'}(\delta_{bb'}\delta_{aa'} - \delta_{ba'}\delta_{b'a}) \\ & + \delta_{\alpha a'}(\delta_{ab'}\delta_{b\alpha'} - \delta_{a\alpha'}\delta_{bb'}) \\ & + \delta_{\alpha b'}(\delta_{a\alpha'}\delta_{ba'} - \delta_{aa'}\delta_{b\alpha'}) , \end{aligned} \tag{A2b}$$

we obtain

$$\begin{aligned} T_{\alpha\alpha'} = {} & \delta_{\alpha\alpha'}(\mathrm{tr}A\,\mathrm{tr}B - \mathrm{tr}AB) + (BA)_{\alpha'\alpha} \\ & - A_{\alpha'\alpha}\mathrm{tr}B + (AB)_{\alpha'\alpha} - \mathrm{tr}AB_{\alpha'\alpha} , \\ T = {} & I(\mathrm{tr}A\,\mathrm{tr}B - \mathrm{tr}AB) + (BA)^T \\ & - A^T(\mathrm{tr}B) + (AB)^T - (\mathrm{tr}A)B^T . \end{aligned} \tag{A3}$$

Now we are considering

$$A = R\begin{bmatrix} \mathcal{A} & 0 \\ 0 & a \end{bmatrix} R^{-1} , \quad B = R\begin{bmatrix} \mathcal{B} & 0 \\ 0 & b \end{bmatrix} R^{-1} , \tag{A4}$$

with $\mathcal{A}, \mathcal{B}$ SU(2) matrices, so

$$\mathrm{tr}A\,\mathrm{tr}B - \mathrm{tr}AB = \mathrm{tr}\mathcal{A}\,\mathrm{tr}\mathcal{B} + b\,\mathrm{tr}\mathcal{A} + a\,\mathrm{tr}\mathcal{B} - \mathrm{tr}(\mathcal{A}\mathcal{B}) . \tag{A5}$$

Now for the SU(2) matrices we know that

$$\mathrm{tr}\mathcal{A}\,\mathrm{tr}\mathcal{B} - \mathrm{tr}\mathcal{A}\mathcal{B} = \mathrm{tr}(\mathcal{A}\mathcal{B}^{-1}) , \tag{A6}$$

so

$$\begin{aligned} T = {} & I[b\,\mathrm{tr}\mathcal{A} + a\,\mathrm{tr}\mathcal{B} + \mathrm{tr}(\mathcal{A}\mathcal{B}^{-1})] \\ & + R\begin{bmatrix} \mathcal{B}\mathcal{A} & 0 \\ 0 & ab \end{bmatrix} R^{-1^T} - R\begin{bmatrix} \mathcal{A} & 0 \\ 0 & a \end{bmatrix} R^{-1^T}(b + \mathrm{tr}\mathcal{B}) \\ & + R\begin{bmatrix} \mathcal{A}\mathcal{B} & 0 \\ 0 & ab \end{bmatrix} R^{-1^T} - (a + \mathrm{tr}\mathcal{A})R\begin{bmatrix} \mathcal{B} & 0 \\ 0 & b \end{bmatrix} R^{-1^T} \end{aligned} \tag{A7}$$

and

$$\begin{aligned} T = (R^{-1})^T \Bigg\{ & I[b\,\mathrm{tr}\mathcal{A} + a\,\mathrm{tr}\mathcal{B} + \mathrm{tr}(\mathcal{A}\mathcal{B}^{-1})] \\ & + \begin{bmatrix} (\mathcal{B}\mathcal{A})^T & 0 \\ 0 & ab \end{bmatrix} + \begin{bmatrix} (\mathcal{A}\mathcal{B})^T & 0 \\ 0 & ab \end{bmatrix} \\ & - (b + \mathrm{tr}\mathcal{B})\begin{bmatrix} \mathcal{A}^T & 0 \\ 0 & a \end{bmatrix} - (a + \mathrm{tr}\mathcal{A})\begin{bmatrix} \mathcal{B}^T & 0 \\ 0 & b \end{bmatrix} \Bigg\} R^T . \end{aligned} \tag{A8}$$

Now we assemble all the elements of T. The result after much cancellation is

$$T^T = R\begin{bmatrix} b\mathfrak{A}_{22}+a\mathfrak{B}_{22} & -b\mathfrak{A}_{12}-a\mathfrak{B}_{12} & 0 \\ -b\mathfrak{A}_{21}-a\mathfrak{B}_{21} & -b\mathfrak{A}_{11}+a\mathfrak{B}_{11} & 0 \\ 0 & 0 & \mathrm{tr}(\mathfrak{A}\mathfrak{B}^{-1}) \end{bmatrix} R^{-1}, \tag{A9}$$

so

$$T = \left(R\begin{bmatrix} b\mathfrak{A}^{-1}+a\mathfrak{B}^{-1} & 0 \\ 0 & \mathrm{tr}(\mathfrak{A}\mathfrak{B}^{-1}) \end{bmatrix}R^{-1}\right)^T, \tag{A10}$$

using the definition of an inverse for a 2×2 unitary matrix. This is the result in the construction of the rule for multiquark effective Hamiltonians.

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