Does the equation of motion determine commutation relations?

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Some examples in both quantum mechanics and quantum field theory are used to demonstrate the fact that a given equation of motion can have more than one inequivalent Lagrangian and hence can admit two different quantization procedures.

I. INTRODUCTION

The Lagrangian formulation of dynamical systems is one of the most important principles in physics. It is a general belief that a Lagrangian is more fundamental than the resulting equation of motion, as we may see, for example, from Feynman's path-integral formulation of quantum-mechanical systems. However, since a Lagrangian itself is not a physically measurable quantity in the usual sense, one may adopt a somewhat unconventional view that the equation of motion is what really counts and that a Lagrangian, if it exists, is simply a convenient mathematical tool of no *a priori* physical significance. Such a view may be justifiable perhaps for classical mechanics and for some classes of quantum field theories where the Yang-Feldman formalism[1-3] is applicable to enable us to compute the S matrix directly from the equation of motion itself without any recourse to a Lagrangian at all. If we adopt this point of view, then we may raise many interesting questions of the following nature: For simplicity, let us label dynamical variables collectively as q, and write the equation of motion as a second-order differential equation of the form

$$\ddot{q}=f(q,\dot{q}), \tag{1.1}$$

where $f(q,\dot{q})$ is a function of q and its time derivative $\dot{q}\equiv dq/dt$.

(1) Given an equation of motion of the form of Eq. (1.1), can we always find a Lagrangian $L=L(q,\dot{q})$ which leads to Eq. (1.1)? If the reply is yes, is the Lagrangian unique or can we find more than one inequivalent Lagrangian giving rise to the same Eq. (1.1)? In this note, we call two Lagrangian L and L' inequivalent if any nontrivial linear combination of L and L' cannot be written as a total time derivative of a function.

(2) If a Lagrangian (or Lagrangians) exists, does it possess all the symmetry properties satisfied by the original equation of motion?

(3) Does the equation of motion determine canonical commutation relations uniquely? Are there many different ways of quantizing the equation of motion? Here, we are, of course, implicitly assuming that a quantization of the system is possible. Note that a consistent quantization may not always be possible[3-5] even for a dynamical system with Lagrangians.

The first question is called the inverse problem and has been answered by Santilli.[6] He finds that a Lagrangian always exists for any classical system with only one degree of freedom but that it is not always so for more complicated systems. Moreover, if a Lagrangian exists, it is not in general the only one.

The second question as well as some related problems have been investigated by many authors.[6-9] It is now known that a Lagrangian is, in general, invariant only under symmetries smaller than those enjoyed by the equation of motion. The third problem was originally raised by Wigner[10] some years ago in a paper with the same title as the present one. He has shown there that the answer is negative at least for a one-dimensional harmonic oscillator with the standard Hamiltonian. Also, a quantization based upon parastatistics[11] may be regarded as another unconventional quantization procedure for a harmonic-oscillator system.

The purpose of this note is to show that all these problems are mutually interrelated on the basis of some new examples where a given equation of motion admits more than one Lagrangian. We are especially interested in quantizing a system in several different ways for both quantum mechanics and quantum field theory, although examples may be rather unconventional and somewhat unphysical. In Sec. II, we shall consider cases of one-dimensional quantum-mechanical systems. A case of relativistic quantum field theory will be discussed in Sec. III.

II. EXAMPLES IN QUANTUM MECHANICS

Here, we restrict ourselves to discussions of systems with one degree of freedom for simplicity. First, let us consider an equation of motion

$$\ddot{q}+\gamma\dot{q}=0 \tag{2.1}$$

for a constant γ. Classical solutions of (2.1) are

$$q=q(t)=\begin{cases}q_0+c\exp(-\gamma t), & \gamma\neq 0\\ q_0+v_0t, & \gamma=0\end{cases} \tag{2.2}$$

for some constants q_0, c, and v_0. Therefore, Eq. (2.1) represents a decay state for $\gamma>0$, but a free particle for $\gamma=0$. Santilli[6] has discovered the following two inequivalent Lagrangians L and L' which give Eq. (2.1):

$$L=\tfrac{1}{2}\,e^{\gamma t}(\dot q)^2\,, \tag{2.3}$$

$$L'=\tfrac{1}{2}\,\dot q\,\ln(\dot q)^2-\gamma q+\dot q f(q)\,, \tag{2.4}$$

where $f(q)$ in Eq. (2.4) is an arbitrary function of q. Note that the last term in (2.4) is actually a total time derivative of a function of q. The first Lagrangian L in Eq. (2.3) depends explicitly upon the time t for $\gamma\neq 0$, so that it is unsuitable for the standard quantization procedure except for the free case of $\gamma=0$. Hence, we shall consider the second Lagrangian L' of Eq. (2.4). First, we note the equation of motion (2.1) is invariant under a scale transformation (λ being a constant)

$$q\to q'=\lambda q\,, \tag{2.5}$$

while the Lagrangian L' of (2.4) does not transform covariantly under the transformation, but $L\to\lambda^2 L$ for L of Eq. (2.3).

The canonical momentum p is given by

$$p=\frac{\partial L'}{\partial\dot q}=\tfrac{1}{2}\ln(\dot q)^2+f(q)+1\,, \tag{2.6}$$

so that the Hamiltonian H is determined by

$$H=\dot q p-L'=\dot q+\gamma q\,. \tag{2.7}$$

Note that the conservation law $dH/dt=0$ is automatically satisfied in view of the equation of motion (2.1). Eliminating $\dot q$ in terms of p and q, we find

$$\dot q=\pm\exp[p-f(q)-1]\,, \tag{2.8}$$

$$H=\gamma q\pm\exp[p-f(q)-1]\,. \tag{2.9}$$

Actually, two solutions corresponding to two signs in Eqs. (2.8) and (2.9) represent two disjoint solutions so that we may choose any one of two possible signs.

In order to check that our system is quantizable, we impose the standard canonical commutation relation

$$[p,q]=-i\hbar \tag{2.10}$$

and demand the validity of Hamilton's equation

$$\frac{d}{dt}\,Q=\frac{i}{\hbar}\,[H,Q] \tag{2.11}$$

for any observable Q. Choosing Q to be q and/or $\dot q$ in Eq. (2.11), we find

$$\dot q=\frac{i}{\hbar}\,[H,q]=\pm\exp[p-f(q)-1]\,,$$

$$\ddot q=\frac{i}{\hbar}\,[H,\dot q]=-\gamma\dot q$$

from Eqs. (2.9), (2.10), and (2.11). These are precisely Eqs. (2.8) and (2.1), respectively. Since Hamilton's equation reproduces the original Lagrange equation, we may say[10] that our quantization is consistent.[12] Of course, our demonstration is mathematically *ad hoc*, since we ignore questions of domains and ranges of unbounded operators.

At any rate, the special case $\gamma=0$ is interesting, since this example offers an alternative quantization of a free particle in comparison to the standard procedure based upon the free Lagrangian (2.3) with $\gamma=0$. Let us now choose the upper-sign solution in Eq. (2.9) and set

$$\tilde p=\sqrt{2}\,\exp\{\tfrac{1}{2}[p-f(q)-1]\}\,. \tag{2.12}$$

Then, Eqs. (2.9) and (2.10) become

$$H=\gamma q+\tfrac{1}{2}(\tilde p)^2\,, \tag{2.13a}$$

$$[\ln(\tilde p)^2,q]=-i\hbar\,. \tag{2.13b}$$

Again for the special case $\gamma=0$, this may be interpreted to offer another example of the Wigner's problem of finding nonstandard commutation relations for the free equation of motion

$$\ddot q=0 \tag{2.14}$$

with Hamiltonian $H=\tfrac{1}{2}(\tilde p)^2$.

For free-particle equation (2.14), we can find infinite numbers of Lagrangians as follows. Let $f(x)$ be any real function of the real variable x such that

$$\frac{d^2}{dx^2}f(x)\neq 0\,. \tag{2.15}$$

In other words, $f(x)$ is either a convex or concave function of x. Then, any Lagrangian of form

$$L=f(\dot q) \tag{2.16}$$

is easily seen to give the desired free equation of motion (2.14). A special choice of

$$L=\tfrac{1}{2}\,\alpha\,\ln(\dot q)^2 \tag{2.17}$$

for any nonzero constant α is of some interest, the reason of which will be explained shortly. Note that the Lagrangian (2.17) is not invariant again under the scale transformation (2.5). The canonical momentum p and the Hamiltonian H are now given by

$$p=\frac{\partial L}{\partial\dot q}=\frac{\alpha}{\dot q}\,, \tag{2.18}$$

$$H=p\dot q-L=\text{constant}+\tfrac{1}{2}\,\alpha\,\ln p^2\,. \tag{2.19}$$

We can again easily check that our system offers a consistent canonical quantization with canonical commutation relation (2.10) and Hamilton's equation (2.11). Changing the variable p into $\tilde{p}$ by

$$\tilde{p} = \frac{1}{\beta p} \qquad (2.20)$$

for any nonzero real constant β, Eqs. (2.10) and (2.19) are transformed into

$$[\tilde{p}, q] = i\hbar\beta(\tilde{p})^2 , \qquad (2.21)$$

$$H = -\tfrac{1}{2}\alpha \ln(\tilde{p})^2 + \text{constant.} \qquad (2.22)$$

In this form, we recognize Eqs. (2.21) and (2.22) as the quantum-mechanical analog of the classical Poisson-bracket relations which have been proposed by Mukunda *et al.*[9] Note that the right-hand side of Eq. (2.21) is now a q number.

In ending this section, let us consider

$$\ddot{q} + \frac{d}{dq} v(q) = 0 . \qquad (2.23)$$

This equation admits the following unconventional (classical) Lagrangian:

$$L = F(\dot{q}) \exp[-\lambda v(q)] , \qquad (2.24)$$

where $F(x)$ is given by

$$F(x) = x \int^{x} \frac{dy}{y^2} \exp\left(-\frac{\lambda}{2} y^2\right) \qquad (2.25)$$

and λ is an arbitrary constant, $\lambda \neq 0$. However, quantization based upon this Lagrangian is involved in view of ordering problems of operators p and q, and will not be discussed here.

III. EXAMPLES IN QUANTUM FIELD THEORY

Let $\phi(x)$ be a spin-zero field in four-dimensional Minkowski spacetime. Suppose that $\phi(x)$ satisfies an equation of the form

$$\Box\, \phi(x) = \lambda \exp[\phi(x)] \qquad (3.1)$$

or

$$\Box\, \phi(x) = \lambda\, [\phi(x)]^3 \qquad (3.2)$$

for a nonzero real constant λ. Consider first the case of Eq. (3.1). The usual Lagrangian for it is evidently given by

$$L(x) = \tfrac{1}{2}\, \partial^\mu \phi(x) \partial_\mu \phi(x) + \lambda \exp[\phi(x)] . \qquad (3.3)$$

However, there is an alternative way of obtaining Eq. (3.1), as follows: We introduce an auxiliary vector field $A_\mu(x)$ and consider a Lagrangian of the form

$$L'(x) = \tfrac{1}{2}\partial^\mu A_\mu(x) \ln[\partial^\nu A_\nu(x)]^2 + \beta \partial^\mu A_\mu(x) + \tfrac{1}{2}\lambda A^\mu(x) A_\mu(x) , \qquad (3.4)$$

where the repeated indices on μ and ν imply an automatic summation on values 0, 1, 2, 3, as usual and where β and λ ($\lambda \neq 0$) are some real constants. Note a similarity of Lagrangian (3.4) to that of (2.4). The variational principle

$$\delta \int d^4x\, L'(x) = 0$$

gives an equation of motion

$$\tfrac{1}{2}\, \partial_\mu \ln[\partial^\nu A_\nu(x)]^2 = \lambda A_\mu(x) . \qquad (3.5)$$

Defining $\phi(x)$ by

$$\phi(x) = \tfrac{1}{2} \ln[\partial^\nu A_\nu(x)]^2 , \qquad (3.6)$$

Eq. (3.5) is then rewritten as

$$A_\mu(x) = \frac{1}{\lambda}\, \partial_\mu \phi(x) \qquad (3.7)$$

and hence we recover the original Eq. (3.1), i.e.,

$$\Box\, \phi(x) = \pm\, \lambda \exp[\phi(x)] , \qquad (3.8)$$

from Eqs. (3.5) and (3.6). Again, two possible sign cases in Eq. (3.8) represent two disjoint solutions and we may choose any one of them. Therefore, the new Lagrangian with new variable $A_\mu(x)$ effectively describes the equation of motion (3.1) for the scalar field $\phi(x)$. In this sense, the same equation of motion may be said to admit two inequivalent Lagrangians. Note that if we insert the expression (3.7) into L', then it will contain second-order derivatives of $\phi(x)$ so that $L'(x)$ is not identical with $L(x)$ given by (3.3).

The canonical variable $\pi_0(x)$ corresponding to $A_0(x)$ is given by

$$\pi_0(x) = \frac{\delta L'(x)}{\delta \dot{A}_0(x)} = \beta + 1 + \tfrac{1}{2} \ln(\partial^\nu A_\nu)^2 , \qquad (3.9)$$

which is equal to $\phi(x)$ given by (3.6), apart from some additive constant. The canonical commutation relations at equal time $x_0 = y_0$ are

$$[A_0(x), A_0(y)] = [\pi_0(x), \pi_0(y)] = 0 , \qquad (3.10a)$$

$$[A_0(x), \pi_0(y)] = i\delta^3(x - y) \qquad (3.10b)$$

in the natural unit $\hbar = c = 1$. In contrast, the spatial components $A_j(x)$ ($j = 1, 2, 3$) are not independent canonical variables, since Eq. (3.5) demands

$$A_j(x) = \frac{1}{\lambda}\, \partial_j \pi_0(x) \quad (j \neq 0) \qquad (3.11)$$

which may be imposed as a subsidiary constraint condition. The Hamiltonian is now calculated to be

$$H(x) = \dot{A}_0(x)\pi_0(x) - L(x) = \frac{1}{2\lambda}\, \partial^k \pi_0(x) \partial_k \pi_0(x) - \frac{\lambda}{2} A^0(x) A_0(x) \pm \exp[\pi_0(x) - (\beta + 1)] \qquad (3.12)$$

after dropping a totally spatial divergence term

$$+\frac{1}{2\lambda}\,\partial^k[\pi_0(x)\partial_k\pi^0(x)]\,.$$

We can easily verify that Hamilton's equations

$$\begin{aligned}\frac{d}{dt}A_0(x)&=i\left[\int d^3y\,H(y),A_0(x)\right]_{x_0=y_0},\\ \frac{d}{dt}\pi_0(x)&=i\left[\int d^3y\,H(y),\pi_0(x)\right]_{x_0=y_0}\end{aligned}\qquad(3.13)$$

become

$$\begin{aligned}\dot{A}_0(x)&=-\frac{1}{\lambda}\,\partial^k\partial_k\pi_0(x)\pm\exp[\pi_0(x)-(\beta+1)]\,,\\ \dot{\pi}_0(x)&=\lambda A_0(x)\,.\end{aligned}\qquad(3.14)$$

Therefore, noting Eq. (3.11), these reproduce the desired Eqs. (3.5)–(3.8). In this sense, our quantization can be said to be self-consistent, although the Hamiltonian (3.12) may not be positive definite.

Next, let us briefly discuss the equation of motion corresponding to Eq. (3.2). This can again be recoverable from another unconventional Lagrangian,

$$\begin{aligned}L'(x)=&\tfrac{3}{4}[\partial^\mu A_\mu(x)]^{4/3}+\beta\partial^\mu A_\mu(x)\\ &+\tfrac{1}{2}\lambda A^\mu(x)A_\mu(x)\,,\end{aligned}\qquad(3.15)$$

which leads to

$$A_\mu(x)=\frac{1}{\lambda}\,\partial_\mu\,[\partial^\nu A_\nu(x)]^{1/3}\,.\qquad(3.16)$$

Setting

$$\phi(x)=[\partial^\nu A_\nu(x)]^{1/3}\,,\qquad(3.17)$$

these indeed reproduce the desired Eq. (3.2).

In ending this note, we remark on the following. Until now, it has been implicitly assumed that all operator algebras obey the associative law as usual. However, for the purpose of consistent quantization, this requirement may be relaxed[13] to a class of nonassociative operator algebras satisfying both flexibility and Lie-admissibility laws. This would enlarge the category of unconventional quantization procedures. However, since the reason behind this possible generalization has been explained elsewhere,[13] we will not go into details here.

Note added:

(1) Several of the expressions in our paper are not globally differentiable [Eqs. (2.4), (2.17), (3.6), and (3.15)]. This is likely to affect the topology of the tangent/cotangent bundles.

(2) A quantization of the time-dependent Lagrangian, Eq. (2.3), has been attempted by R. Santilli [in Had. J. 2, 1883 (1979)], who uses a nonassociative Lie-admissible algebra.

(3) The most general time-independent Lagrangain reproducing Eq. (2.23), i.e.,

$$\ddot{q}+\frac{d}{dq}v(q)=0\,,$$

is found to have a form

$$L=L(q,\dot{q})=\dot{q}\int^{\dot{q}}\frac{dp}{p^2}\,f\left(\tfrac{1}{2}p^2+v(q)\right),$$

where $f(x)$ is an arbitrary function of a real variable x such that $f'(x)\neq 0$. The usual canonical Lagrangian corresponds to a choice of $f(x)=x$, while Eq. (2.24) is a special case $f(x)=\exp(-\lambda x)$, $\lambda\neq 0$.

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[4]A well-known example is nonlocal field theory (see Ref. 3). However, after some modification of Feynman's path-integral method, a consistent quantization for some of such a system may be generalized. See S. Okubo, Prog. Theor. Phys. 51, 920 (1974).

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[12]This fact should not be confused with the impossibility of the so-called full (or strong) quantization for such systems, which has been proven in R. Abraham and J. E. Marsden, *Foundations of Mechanics* (Benjamin/Cummings, Reading, Mass., 1979). The validity of the theorem requires much stronger assumptions. The usual quantization procedure as is used in this paper

as well as in any physics literature may be termed as a weak (or partial) quantization in order to avoid possible confusion with the mathematician's usage.

[13]S. Okubo, in *Proceedings of the Third Workshop on Current Problems in High Energy Particle Theory, Florence, Italy, 1979,* edited by R. Casalbuoni, G. Domokos, and S. Kovesi-Domokos (Johns Hopkins Univ. Press, Baltimore, 1979), p. 103. Any nonassociative algebra is said to satisfy the flexibility law, if we have $(xy)x = x(yx)$. The Lie-admissibility law implies the validity of the Jacobi identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$. Any associative algebra of course satisfies both flexibility and Lie-admissibility laws, but the converse is not necessarily true. See also Ref. 6 in this connection.