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Quantum theory of a strong electromagnetic field: Semiclassical representation

E. P. Gordov and S. D. Tvorogov
The Institute of Atmospheric Optics, Tomsk Affiliate, U.S.S.R. Academy of Sciences, Siberian Branch, Tomsk, 634055, U.S.S.R.
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The operators of the classical amplitudes of an electromagnetic field and the complete orthonormal system of their eigenstates are introduced. The properties of these operators allow us to transform from a quantum-mechanical description of a strong field to a c-number semiclassical description, both in the case of a free field and in the case of a field interacting with a quantum system. The latter allows one to use equations of a semiclassical theory as a calculating apparatus of quantum theory.

I. INTRODUCTION

The mathematical foundations of the semiclassical representation method whose contents are the derivation from the quantum theory of the semiclassical approximation equations and their use for calculating the quantum averages have been introduced in Refs. 1–5 and described in detail in Ref. 6. The method of semiclassical representation uses the commuting operators of a classical field and their eigenstates (eigenfunctions of the vector-potential operator[1]) and the operator of quantum fluctuations introduced in Ref. 2. However, the version of the method used in these papers does not allow one to consider the situation with a finite number of modes and does not permit direct calculations of average values of contributions due to the operators of the first and second type. Therefore, to verify the correctness of the interpretation of the classical field and fluctuation operators, it is necessary to take into account additional physical considerations.

In this paper we give the generalization of the approach developed earlier, free from the limitations mentioned above. This generalization allows one to simplify essentially the calculation of quantum averages, for the case of a strong field interacting with a quantum system.

The mathematical contents of this paper can be defined as the use of the canonical conjugate operators of reducible, instead of irreducible, representations resulting, for the case of a strong field, in relations between commuting operators, i.e., in the description by equations for ordinary functions.

In this paper we do not specify the Hamiltonian of the quantum system and do not confine ourselves to linear interactions between the field and the system. Therefore one can assume that the results obtained can be used in the case of quantum electrodynamics of a strong field and in the case of a strong boson field, and in statistical physics.

In Sec. II, the classical amplitude operators and their eigenstates, which form the complete orthonormal system, are introduced. These operators allow transformation in a simple way from a quantum to a classical description. The equivalence of the use, in the problem of the transition from the quantum to the classical description of the field, of the classical amplitude operators and the procedures of normal ordering in the coherent-state representation is shown.

In Sec. III, we consider the description of the evolution of a free field, and a comparison is made with the use of coherent states. Section IV is devoted to consideration of a field interacting with a quantum system and its semiclassical limit. The problem of the semiclassical limit assumes a solution without a concrete definition of the Hamiltonian of the quantum system and the interaction Hamiltonian. This section shows that in the case of a strong field, given by an appropriate choice of initial density matrix, the creation and annihilation operators in the Heisenberg representation are determined by relations for commuting operators of classical field amplitudes. These relations result naturally in equations for ordinary c functions and it becomes possible to use equations of the semiclassical theory as a calculational apparatus in quantum theory.

In Sec. V we introduce and study the semiclassical representation which is characterized by the fact that the time dependence of operators in it is determined by the formulas of an appropriate semiclassical theory. Time evolution of states is defined by the Schrödinger equation, describing, strictly speaking, differences of quantum theory from semiclassical theory.

Section VI is devoted to the methods of solution of the quantum problem, using the algebra of operators of classical field amplitudes and quantum fluctuation operators, as well as the operator relations for the evolution of a strong field.

In Sec. VII we discuss the connection of the suggested approach and our preceding results, obtained for quantum electrodynamics.[1,6] It is shown that when satisfying the condition of real fields, an approach suggested in the paper leads to the results obtained previously. In Sec. VIII we discuss briefly the main results of the paper and possibilities of using the semiclassical representation method in the problems of statistical physics of intermolecular interactions.

II. OPERATORS FOR CLASSICAL AMPLITUDES

Let us introduce the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, operating in the Hilbert space H, for which the Bose-Einstein commutation relation is satisfied,

$$[\hat{a}, \hat{a}^\dagger] = 1\,. \tag{1}$$

We write, still formally,

$$\hat{a} = a_0 + \Delta a, \quad \hat{a}^\dagger = a_0^\dagger + \Delta a^\dagger\,, \tag{2}$$

where

$$[a_0, a_0^\dagger] = 0, \quad (a_0)^\dagger = a_0^\dagger\,. \tag{3}$$

From Eq. (3) it follows that the operators a_0 and $a_0^\dagger$ should have the same system of eigenstates:

$$a_0 |\psi_a\rangle = a |\psi_a\rangle, \quad a_0^\dagger |\psi_a\rangle = \bar{a} |\psi_a\rangle\,. \tag{4}$$

Here and in the following an overbar denotes complex conjugation. It is easy to make certain that from Eqs. (4) and (3) it follows that the states $|\psi_a\rangle$ must be orthogonal.

Further, we want to use a_0 and $a_0^\dagger$ for the description of a classical field. Therefore, we add one more condition on the eigenvalues a and $\bar{a}$, whose realization allows us to call the operators a_0 and $a_0^\dagger$ the classical field amplitude operators. We require that

$$a = \langle a | \hat{a} | a\rangle, \quad \bar{a} = \langle a | \hat{a}^\dagger | a\rangle\,, \tag{5}$$

where $|a\rangle$ is a coherent state in H: $\hat{a}|a\rangle = a|a\rangle$. After making simple calculations one can verify that the conditions (1)–(5) cannot be satisfied in H. Therefore, we proceed to the extended Hilbert space $W = H \otimes H$, where $\otimes$ is the direct-product sign. In the space W the operators $\hat{a}$ and $\hat{a}^\dagger$ are written as[7]

$$a_w = \hat{a} \otimes I, \quad a_w^\dagger = \hat{a}^\dagger \otimes I\,, \tag{6}$$

where I is the unit operator in H.

If the state in H is given by the density matrix ρ, it is natural to determine the state in W by the density matrix R:

$$R = \rho \otimes |0\rangle\langle 0|\,, \tag{7}$$

where $|0\rangle$ is the vacuum vector in H. Such a definition of the state keeps all the properties of density matrix in R (Hermitian character, positive definiteness, finiteness of trace, etc.), and makes it possible to calculate average values both in H and in W. In this case, if $f(\hat{a}^\dagger, \hat{a})$ is an arbitrary function of $\hat{a}^\dagger$, $\hat{a}$, then we have

$$\mathrm{Tr}\rho f(\hat{a}^\dagger, \hat{a}) = \mathrm{Tr} R f(a_w^\dagger, a_w)\,. \tag{8}$$

Now we rewrite (2) in W:

$$a_w = a_0 + \Delta a, \quad a_w^\dagger = a_0^\dagger + \Delta a^\dagger\,, \tag{2'}$$

where a_0, $a_0^\dagger$, Δa, and $\Delta a^\dagger$ are now operators in W. It is clear that $|\psi_a\rangle$ are now the states in W.

In W the relations (1) and (2′)–(5) are consistent, and after calculations analogous to those made in Refs. 2 and 6 one can determine the operators a_0, $a_0^\dagger$, Δa, $\Delta a^\dagger$, and the states $|\psi_a\rangle$ in the following way[8]:

$$a_0 = \hat{a} \otimes I + I \otimes \hat{a}^\dagger, \quad a_0^\dagger = \hat{a}^\dagger \otimes I + I \otimes \hat{a}\,, \tag{9}$$

$$\Delta a = -I \otimes \hat{a}^\dagger, \quad \Delta a^\dagger = -I \otimes \hat{a}\,, \tag{10}$$

$$|\psi_a\rangle = \frac{1}{\sqrt{\pi}} e^{-|a|^2/2} \int e^{\bar{a}\bar{\beta} + a\bar{\alpha} - \bar{\alpha}\bar{\beta}} |\alpha\rangle \otimes |\beta\rangle \pi^{-2} \times e^{-(|\alpha|^2 + |\beta|^2)/2} d^2\alpha\, d^2\beta\,. \tag{11}$$

In Eq. (11) $|\alpha\rangle$ and $|\beta\rangle$ are the coherent states, $|\alpha\rangle$ is from the left H, and $|\beta\rangle$ is from the right H.

It should be noted that the states obtained are similar to the eigenstates of the vector potential operator.[1,2,6] It is known that such states appear when discussing the problem of quasimeasurement of the amplitude and phase of an electromagnetic field.[9] The spectrum of operators a_0 and $a_0^\dagger$ is continuous. Direct calculations show that the states $|\psi_a\rangle$ form a complete orthornormal system,

$$\begin{gathered} \langle \psi_a | \psi_b\rangle = \delta(\mathrm{Re}a - \mathrm{Re}b)\delta(\mathrm{Im}a - \mathrm{Im}b)\,, \\ \int |\psi_a\rangle\langle\psi_a| da = I \otimes I, \quad da \equiv d\,\mathrm{Re}a\,d\,\mathrm{Im}a\,. \end{gathered} \tag{12}$$

Simple calculations show that for the operators a_0 and $a_0^\dagger$ and for the states $|\psi_a\rangle$ determined in Eqs. (9)–(11), the relations (3)–(5) are fulfilled. It should be noted that the fulfillment of condition (5) is due to a choice of the state in W, Eq. (7), and to the following property of $|\psi_a\rangle$:

$$\langle 0 | \psi_a\rangle = |a\rangle\,, \tag{13}$$

where $|0\rangle$ is the vacuum state from the right H, and $|a\rangle$ is the coherent state from the left H.

Owing to the completeness of states $|\psi_a\rangle$, an average value of an arbitrary function of a_0 and $a_0^\dagger$ is of the form

$$\begin{aligned} \mathrm{Tr} R f(a_0^\dagger, a_0) &= \int da\, R_{aa} f(\bar{a}, a) \\ &= \int d^2 a \langle a | \rho | a\rangle f(\bar{a}, a)\,, \end{aligned} \tag{14}$$

$$R_{aa} \equiv \langle \psi_a | \rho | \psi_a\rangle\,.$$

Equation (14) is an average value for a classical random process.

Based on Eq. (8), the determinations (2′) and (7), one can obtain the following expression:

$$\langle f(\hat{a}^\dagger, \hat{a})\rangle = \mathrm{Tr}R\left[f(a_0^\dagger, a_0) + \frac{\partial f(a_0^\dagger, a_0)}{\partial a_0}\Delta a + \cdots\right]$$
$$= \mathrm{Tr}R\left[f(a_0^\dagger, a_0) + \frac{\partial^2 f(a_0^\dagger, a_0)}{\partial a_0 \partial a_0^\dagger}\langle 0|\hat{a}\hat{a}^\dagger|0\rangle + \cdots\right]. \tag{15}$$

When deriving this relation we took into account that the definition (7) resulted in the following form for an average of the operator $A\otimes B$,

$$\mathrm{Tr}RA\otimes B = \mathrm{Tr}\rho A\langle 0|B|0\rangle, \tag{16}$$

and we sometimes transferred from averaging in H by ρ to averaging in W by R and vice versa. Writing the trace in Eq. (15) in an obvious form, we obtain

$$\langle f\rangle = \int da\, R_{aa}[f(\bar{a}, a) + \partial^2 f(\bar{a}, a)/\partial a\, \partial\bar{a} + \cdots]$$
$$= \int d^2a\langle a|\rho|a\rangle[f(\bar{a}, a) + \partial^2 f(\bar{a}, a)/\partial a\,\partial\bar{a} + \cdots]. \tag{17}$$

The right-hand side of Eq. (17) is the ordinary quantum average value of the operator f, calculated by means of subsequent normal ordering and calculation of appropriate intermediate integrations. It is known[10-12] that in the case where ρ is the density matrix of a strong field with sufficiently large amplitudes ($|a|\gg 1$), then on the right-hand side and consequently on the left-hand side of Eq. (17), one can be restricted only by the first term, and we obtain Eq. (14) as the definition of the average value. Now it is clear that the operators a_0 and $a_0^\dagger$ are really connected with the classical description of the field and, according to Eq. (4), they can be called operators of classical amplitudes.

III. FREE FIELD

Now we consider the problem of description of a free field in the representation introduced. The Hamiltonian of the field has the usual form

$$H_R = \hbar\omega\hat{a}^\dagger\hat{a}. \tag{18}$$

The creation and annihilation operators in the Heisenberg representation are written as

$$\hat{a}(t) = e^{-i\omega t}\hat{a}, \quad \hat{a}^\dagger(t) = e^{i\omega t}\hat{a}^\dagger. \tag{19}$$

Converting the Hamiltonian H_R in the space W it is easy to calculate the operators a_0 and Δa in the Heisenberg representation. We have

$$a_{0H} = e^{(i/\hbar)H_R t}\otimes I a_0 e^{-(i/\hbar)H_R t}\otimes I$$
$$= a_0 e^{-i\omega t} + \Delta a(e^{-i\omega t} - 1), \tag{20}$$

$$\Delta a_H = e^{(i/\hbar)H_R t}\otimes I\Delta a e^{(i/\hbar)H_R t}\otimes I = \Delta a, \tag{21}$$

and the Hermitian conjugate relations for $a_0^\dagger$ and $\Delta a^\dagger$. It should be noted that the equations of motion for $\hat{a}(t)$ and a_{0H} coincide with those for the amplitudes of a classical field:

$$i\frac{\partial a_{0H}}{\partial t} = \omega a_{0H}, \quad a_{0H}(0) = a_0. \tag{22}$$

From Eqs. (19) and (2′) it follows that in W, one can formally represent $a_w(t)$ as

$$a_w(t) = a_0(t) + \Delta a(t), \quad a_0(t) = a_0 e^{-i\omega t}, \quad \Delta a(t) = \Delta a e^{-i\omega t}. \tag{23}$$

It is clear that in this case $a_0(t)$ and $\Delta a(t)$ are not the operators a_0 and Δa in the Heisenberg representation. However, such a representation will be useful below. The latter is due to the fact that the operators $a_0(t)$ and $a_0^\dagger(t)$ commute, and their eigenvalues in the basis $|\psi_a\rangle$ determine the amplitudes of a free classical field, depending on time:

$$a_0(t)|\psi_a\rangle = a(t)|\psi_a\rangle = ae^{-i\omega t}|\psi_a\rangle, \tag{24}$$

$$a(t) = \langle a|\hat{a}(t)|a\rangle. \tag{25}$$

In this case, one can obtain an analog of Eqs. (15) and (17) for an average value:

$$\langle f\rangle_t = \mathrm{Tr}R\left[f(a_0^\dagger(t), a_0(t)) + \frac{\partial^2 f(a_0^\dagger(t), a_0(t))}{\partial a_0(t)\partial a_0^\dagger(t)} + \cdots\right]$$
$$= \int da\, R_{aa}\left[f(\bar{a}(t), a(t)) + \frac{\partial^2 f(\bar{a}(t), a(t))}{\partial a(t)\partial\bar{a}(t)} + \cdots\right]$$
$$= \int d^2a\langle a|\rho|a\rangle\left[f(\bar{a}(t), a(t)) + \frac{\partial^2 f(\bar{a}(t), a(t))}{\partial a(t)\partial\bar{a}(t)} + \cdots\right]. \tag{26}$$

It is clear that, as before the case of a strong field (classical limit) results in consideration of the first term only in Eq. (26). Simplicity of transition to the classical limit in the case of a strong field is due to the form of the Hamiltonian (18).

It is evident that the transition to the classical limit by means of the operators a_0 and $a_0^\dagger$, analyzed here and in Sec. III, is merely illustrative and can be made, as usual, in the representation of coherent states without extending the Hilbert space and introducting new operators. However, in the case of the field interacting with the quantum system, the use of the operators a_0 and $a_0^\dagger$ and the orthonormal basis $|\psi_a\rangle$ leads to the appearance of simple calculational schemes for a quantum problem, especially for the case of a strong field.

IV. THE FIELD INTERACTING WITH THE QUANTUM SYSTEM

Let the Hamiltonian of the problem be of the form

$$H = H_R + H_2(x) + H_{12}(\hat{a}^\dagger, \hat{a}, x), \tag{27}$$

where $H_2(x)$ is the Hamiltonian of the quantum system, H_{12} is its interaction with the field, and x denotes a necessary set of the quantum system operators. Let the total density matrix in the initial moment of time be given by

$$\rho_0 = \rho T \,, \tag{28}$$

where T is the initial density matrix of the quantum system and ρ, as before is the initial density matrix of a field.

The Schrödinger equation for the evolution operator U in the interaction representation over the field is

$$i\hbar \frac{\partial U}{\partial t} = [H_2(x) + H_{12}(\hat{a}^\dagger(t), \hat{a}(t), x)]U, \quad U(t_0) = 1 \,. \tag{29}$$

In the space W the density matrix is rewritten as

$$R = \rho_0 \otimes |0\rangle\langle 0| \,. \tag{30}$$

The operators a_0 and Δa in the Heisenberg representation are

$$\Delta a_H = \Delta a \,,$$

$$(U^{-1} e^{(i/\hbar)H_R t})_w a_0 (e^{-(i/\hbar)H_R t})_w = U_w^{-1} a_0(t) U(t) + \Delta a(e^{-i\omega t} - 1) \,. \tag{31}$$

Here we use Eqs. (2′), (10), (20), and (21). Now, using the designations of Eq. (23), the operators $a_w^\dagger$ and a_w in the Heisenberg representation can be written as

$$U_w^{-1} a_w^\dagger(t) U_w = U_w^{-1} a_0^\dagger(t) U_w + \Delta a^\dagger(t) \,,$$
$$U_w^{-1} a_w(t) U_w = U_w^{-1} a_0(t) U_w + \Delta a(t) \,. \tag{32}$$

Thus, the main problem of quantum electrodynamics, i.e., the calculation of time evolution of the operators $\hat{a}^\dagger$ and $\hat{a}$, is the problem of time evolution of operators $a_0^\dagger(t)$ and $a_0(t)$. For solving this problem it is necessary to calculate the operator U_w. After transition into the space W the Hamiltonian H_u in Eq. (29) is written as

$$H_u = H_2(x) + H_{12}(a_0^\dagger(t), a_0(t), x) + \frac{\partial H_{12}(a_0^\dagger(t), a_0(t), x)}{\partial a_0(t)} \Delta a(t) + \frac{\partial H_{12}(a_0^\dagger(t), a_0(t), x)}{\partial a_0^\dagger(t)} \Delta a^\dagger(t) + \cdots \,. \tag{33}$$

Let us introduce the designation

$$a_{0H}(t) = U_w^{-1}(t) a_0(t) U_w(t) \tag{34}$$

and rewrite Eq. (34) using the Schrödinger equation (29):

$$a_{0H}(t) = a_0(t) + \frac{i}{\hbar} \int_{t_0}^{t} U_w^{-1}(\tau) [H_u(\tau), a_0(t)] U_w(\tau) d\tau \,. \tag{35}$$

Having now substituted the expansion (33) into Eq. (35), we obtain the following expressions:

$$a_{0H}(t) = a_0(t) + \frac{i}{\hbar} \int_{t_0}^{t} U_w^{-1}(\tau) \left[\frac{\partial H_{12}(a_0^\dagger(\tau), a_0(\tau), x)}{\partial a_0^\dagger(\tau)} + \frac{\partial^2 H_{12}(a_0^\dagger(\tau), a_0(\tau), x)}{\partial a_0^\dagger(\tau) \partial a_0(\tau)} \Delta a(\tau) + \frac{\partial^2 H_{12}(a_0^\dagger(\tau), a_0(\tau), x)}{\partial a_0^{\dagger 2}(\tau)} \Delta a^\dagger(\tau) + \cdots \right] U_w(\tau) e^{i\omega(t-\tau)} d\tau \,. \tag{36}$$

Here in brackets we wrote out the terms depending only on a_0, $a_0^\dagger$ and proportional to Δa and $\Delta a^\dagger$. It is clear that, furthermore, we should write out the terms proportional to $(\Delta a)^2$, $(\Delta a^\dagger)^2$, $\Delta a \Delta a^\dagger$, etc.

It should be noted that

$$U_w^{-1} \Delta a(t) U_w = \Delta a(t) \,. \tag{37}$$

Therefore Eq. (36) may now be rewritten as

$$a_{0H}(t) = a_0(t) + \frac{i}{\hbar} \int_{t_0}^{t} \left[\frac{\hat{\partial} H_{12}(a_{0H}^\dagger(\tau), a_{0H}(\tau) | x_H(\tau))}{\partial a_{0H}^\dagger(\tau)} + \frac{\hat{\partial}^2 H_{12}(a_{0H}^\dagger(\tau), a_{0H}(\tau) | x_H(\tau))}{\partial a_{0H}^\dagger(\tau) \partial a_{0H}(\tau)} \Delta a(\tau) + \frac{\hat{\partial}^2 H_{12}(a_{0H}^\dagger(\tau), a_{0H}(\tau) | x_H(\tau))}{\partial a_{0H}^\dagger(\tau)^2} \Delta a^\dagger(\tau) + \cdots \right] e^{i\omega(t-\tau)} d\tau \,, \tag{38}$$

where

$$\frac{\hat{\partial} H_{12}(\dagger)}{\partial a_{0H}^\dagger(\tau)} \equiv U_w^{-1}(\tau) \frac{\partial H_{12}(a_0^\dagger(\tau), a_0(\tau), x)}{\partial a_0^\dagger(\tau)} U_w(\tau)$$

and

$$\hat{\partial}^2 H_{12} / \partial a_{0H}^{\dagger\,2}$$

are determined similarly.

It should be mentioned that the presence of the operator x_H in the expansion (38) does not allow one to interpret it as a series in powers $\Delta a^\dagger$ and Δa. Therefore this operator should be considered in detail.

From the Schrödinger equation (29) and the expansion of the Hamiltonian H_u in (33) it follows that the evolution operator $U_w(t)$ can be represented as

$$U_w(t) = C(a_0^\dagger, a_0, t)Q(t)\,, \tag{39}$$

where C is the unitary operator depending only on the operators of the classical amplitudes $a_0^\dagger$ and a_0.

The operator C is required to be connected with the evolution operator in semiclassical theory, i.e.,

$$i\hbar\frac{\partial C}{\partial t} = [H_2(x) + H_{12}(a_c^\dagger(t), a_c(t), x)]C\,,\quad C(t_0) = 1\,, \tag{40}$$

where the operators $a_c^\dagger$ and a_c are the functionals of $a_0^\dagger$ and a_0 being undetermined.

The equation for the operator Q follows from Eqs. (29), (33), (39), and (40):

$$i\hbar\frac{\partial Q}{\partial t} = \left\{H_{12}(a_c^\dagger(t), a_c(t)\,|\,x_c(t)) - H_{12}(a_0^\dagger(t), a_0(t)\,|\,x_c) - \frac{\partial H_{12}(a_0^\dagger(t), a_0(t)\,|\,x_c)}{\partial a_0(t)}C^{-1}\Delta a(t)C \right.$$
$$\left. - \frac{\partial H_{12}(a_0^\dagger(t), a_0(t)\,|\,x_c)}{\partial a_0^\dagger(t)}C^{-1}\Delta a^\dagger(t)C + \cdots\right\}Q = H_Q Q\,,\quad x_c \equiv C^{-1}xC\,. \tag{41}$$

We represent the operators $C^{-1}\Delta aC$ as

$$C^{-1}\Delta aC = (\langle C^{-1}\Delta aC\rangle_x - \Delta a) + (C^{-1}\Delta aC - \langle C^{-1}\Delta aC\rangle_x) + \Delta a\,,$$

where

$$\langle C^{-1}\Delta aC\rangle_x = \mathrm{Tr}\,TC^{-1}\Delta aC$$

and we can see that the operator $\langle C^{-1}\Delta aC\rangle - \Delta a$ is the functional based on the operators $a_0^\dagger$ and a_0. Now Eq. (41) can be rewritten as

$$i\hbar\frac{\partial Q}{\partial t} = \left\{H_{12}(a_c^\dagger, a_c\,|\,x_c) - H_{12}(a_0^\dagger + \langle C^{-1}\Delta a^\dagger C\rangle_x - \Delta a, a_0 + \langle C^{-1}\Delta aC\rangle_x - \Delta a\,|\,x_c)\right.$$
$$\left. - \frac{\partial H_{12}(a_0^\dagger(t), a_0(t)\,|\,x_c)}{\partial a_0(t)}[\delta a_c + \Delta a(t)] - \frac{\partial H_{12}(a_0^\dagger(t), a_0(t)\,|\,x_c)}{\partial a_0^\dagger(t)}[\delta a_c^\dagger + \Delta a^\dagger(t)] + \cdots\right\}Q\,, \tag{42}$$

where

$$\delta a_c = C^{-1}\Delta a(t)C - \langle C^{-1}\Delta a(t)C\rangle_x\,.$$

Assuming

$$a_c(t) = a_0(t) + \langle C^{-1}\Delta a(t)C\rangle_x - \Delta a(t)\,, \tag{43}$$

we obtain from Eq. (42) the following expressions:

$$i\hbar\frac{\partial Q}{\partial t} = -\left\{\frac{\partial H_{12}(a_0^\dagger(t), a_0(t)\,|\,x_c)}{\partial a_0(t)}[\delta a_c + \Delta a(t)] + \frac{\partial H_{12}(a_0^\dagger(t), a_0(t)\,|\,x_c)}{\partial a_0^\dagger(t)}[\delta a_c^\dagger + \Delta a^\dagger(t)] + \cdots\right\}Q\,. \tag{44}$$

The form of Eq. (44) shows that the choice of a_c, namely, Eq. (43), results in the field fluctuation structure in the Hamiltonian of the Schrödinger equation for Q, i.e., in fast series convergence of the perturbation theory for Q.

Now, using Eqs. (39), (43), and (44), we obtain for x_H

$$x_H = x_c + \frac{i}{\hbar}\int_{t_0}^{t} Q^{-1}(\tau)[H_Q(\tau), x_c(t)]Q(\tau)d\tau$$
$$= x_c + \frac{i}{\hbar}\int_{t_0}^{t}\left\{\left[\frac{\partial H_{12}(\,|\,x_c(\tau))}{\partial a_0(\tau)}, x_c(t)\right][\delta a_c + \Delta a(\tau)] + \left[\frac{\partial H_{12}(\,|\,x_c(\tau))}{\partial a_0^\dagger(\tau)}, x_c(t)\right][\delta a_c^\dagger + \Delta a^\dagger(\tau)]\right.$$
$$\left. + \frac{\partial H_{12}(\,|\,x_c(\tau))}{\partial a_0(\tau)}[\delta a_c + \Delta a(\tau), x_c(t)] + \frac{\partial H_{12}(\,|\,x_c(\tau))}{\partial a_0^\dagger(\tau)}[\delta a_c^\dagger + \Delta a^\dagger(\tau), x_c(t)]\right\}d\tau$$
$$= x_c + \varphi_1\{x_c\} + \varphi_2\{x_c\}\Delta a + \varphi_3\{x_c\}\Delta a^\dagger + \cdots\,. \tag{45}$$

In this case the sum of the terms, depending on a_0, $a_0^\dagger$ only, is denoted as φ_1, the coefficient at Δa is denoted as φ_2, and the coefficient at $\Delta a^\dagger$ is denoted as φ_3.

Substituting the expansions of the operator x_H in (45) into $\partial \hat{H}_{12}/\partial a_{0H}$ one can obtain for this operator the following expansion:

$$\frac{\partial \hat{H}_{12}(|x_H)}{\partial a_{0H}}=\frac{\partial \hat{H}_{12}(|x_c)}{\partial a_{0H}}+\frac{\partial \hat{H}_{12}^{(1)}(|x_c,\varphi_1)}{\partial a_{0H}}+\frac{\partial \hat{H}_{12}(|x_c,\varphi_1,\varphi_2)}{\partial a_{0H}}\Delta a+\frac{\partial \hat{H}_{12}(|x_c,\varphi_1,\varphi_3)}{\partial a_{0H}}\Delta a^\dagger+\cdots . \tag{46}$$

The meaning of the expansion terms is clear because in the first term only x_c is retained from x_H, in the second term φ_1 is taken into account, and the third and fourth terms are the coefficients of Δa and $\Delta a^\dagger$.

In this case the operator a_w in the Heisenberg representation can be written as

$$\begin{aligned}U_w^{-1}a_wU_w&=a_{0H}(t)+\Delta a(t)\\&=a_0(t)+\Delta a(t)+\frac{i}{\hbar}\int_{t_0}^{t}\left[\frac{\partial \hat{H}_{12}(a_{0H}^\dagger(\tau),a_{0H}(\tau)|x_c(\tau))}{\partial a_{0H}^\dagger(\tau)}+\frac{\partial \hat{H}_{12}^{(1)}(|x_c,\varphi_1)}{\partial a_{0H}^\dagger(\tau)}\right]e^{i\omega(t-\tau)}d\tau\\&\quad+\frac{i}{\hbar}\int_{t_0}^{t}\left\{\left[\frac{\partial \hat{H}_{12}^{(2)}(|x_c,\varphi_1,\varphi_2)}{\partial a_{0H}^\dagger}+\frac{\partial^2 \hat{H}_{12}(|x_c)}{\partial a_{0H}^{\dagger 2}}\right]\Delta a+\left[\frac{\partial \hat{H}_{12}^{(3)}(|x_c,\varphi_1,\varphi_3)}{\partial a_{0H}^\dagger}+\frac{\partial^2 \hat{H}_{12}(|x_c)}{\partial a_{0H}\partial a_{0H}^\dagger}\right]\Delta a^\dagger+\cdots\right\}e^{i\omega(t-\tau)}d\tau\\&=A(a_0,a_0^\dagger)+B(a_0,a_0^\dagger)\Delta a+D(a_0,a_0^\dagger)\Delta a^\dagger+\cdots ,\end{aligned} \tag{47}$$

$$A(a_0,a_0^\dagger)\equiv a_0(t)+\frac{i}{\hbar}\int_{t_0}^{t}\left[\frac{\partial \hat{H}_{12}(|x_c)}{\partial a_{0H}^\dagger}+\frac{\partial \hat{H}_{12}^{(1)}(|x_c,\varphi_1)}{\partial a_{0H}^\dagger}\right]e^{i\omega(t-\tau)}d\tau . \tag{48}$$

Our final aim is to calculate the average values according to the density matrix R, Eq. (30). For an arbitrary function of the operators $\hat{a}^\dagger$ and $\hat{a}$ one has, taking into account Eqs. (17) and (47), the following expression:

$$\begin{aligned}\langle f\rangle_t=\mathrm{Tr}R\bigg(&f\{A,A^\dagger\}+\frac{\hat{\partial}^2 f}{\partial A\,\partial A^\dagger}+\frac{\hat{\partial}^2 f}{\partial A\,\partial A^\dagger}(BB^\dagger+DD^\dagger)\\&+\frac{\hat{\partial}^2 f}{\partial A\,\partial A^\dagger}(DB+BD)\\&+\frac{\hat{\partial}^2 f}{\partial A^{\dagger 2}}(B^\dagger D^\dagger+D^\dagger B^\dagger)+\cdots\bigg).\end{aligned} \tag{49}$$

It is evident that if ρ is the density matrix of a strong field, then for a polynomial function f all the terms (except the first one) in Eq. (49) may be neglected and the average value for this case is

$$\langle f\rangle_t=\mathrm{Tr}Rf\{A,A^\dagger\}, \tag{50}$$

where A and $A^\dagger$ depend on the field operators a_0 and $a_0^\dagger$ only, i.e., they have classical analogs.

To determine A and $A^\dagger$ we have Eqs. (48), (43), and (40). Let us rewrite the determination (43) as an integral equation:

$$a_c(t)=a_0(t)+\frac{i}{\hbar}\int_{t_0}^{t}\left\langle\frac{\partial H_{12}(a_c^\dagger(\tau),a_c(\tau)|x_c)}{\partial a_0^\dagger(\tau)}\right\rangle_x e^{i\omega(t-\tau)}d\tau . \tag{51}$$

Thus, to calculate the operators a_w and $a_w^\dagger$ in the Heisenberg representation in the case of a strong external field we have a set of relations (40), (48), and (51), operating only with commuting field operators. Because we need matrix elements from A and $A^\dagger$ in the basis $|\psi_a\rangle$ when calculating the trace in Eq. (50), we rewrite these relations in the representation $\psi_a=\langle\psi_a|\,\rangle$:

$$\begin{aligned}A(\bar{a},a)=a(t)+\frac{i}{\hbar}\int_{t_0}^{t}\bigg[&\frac{\partial \hat{H}_{12}(\bar{A}(\tau),A(\tau)|x_c)}{\partial \bar{A}(\tau)}\\&+\frac{\partial \hat{H}_{12}^{(1)}(\bar{A}(\tau),A(\tau)|x_c,\varphi_1)}{\partial \bar{A}(\tau)}\bigg]\\&\times e^{i\omega(t-\tau)}d\tau ,\end{aligned} \tag{52}$$

$$i\hbar\frac{\partial C}{\partial t}=[H_2(x)+H_{12}(\bar{a}_c,a_c,x)]C , \tag{53}$$

$$a_c=a(t)+\frac{i}{\hbar}\int_{t_0}^{t}\left\langle\frac{\partial H_{12}(\bar{a}_c,a_c|x_c)}{\partial \bar{a}(\tau)}\right\rangle_x e^{i\omega(t-\tau)}d\tau . \tag{54}$$

In fact, the set of Eqs. (52)–(54) consists of a self-consistent set of nonlinear equations (53)–(54), which are similar to equations of semiclassical electrodynamics and Eq. (52) solved for the calculated $\bar{a}_c$ and a_c.

It is simple to transform this set of equations to the ordinary nonlinear Hamilton's equations.[13] For this purpose it is sufficient to suppose in the Schrödinger equation (40) that

$$a_c=a_M=a_0(t)+\frac{i}{\hbar}\int_{t_0}^{t}\left\langle\frac{\partial H_{12}(a_M^\dagger,a_M|x_c)}{\partial a_M^\dagger}\right\rangle_x e^{i\omega(t-\tau)}d\tau . \tag{55}$$

Then Eq. (52) can be rewritten as

$$A(\bar{a},a)=a_M+\frac{i}{\hbar}\int_{t_0}^{t}\frac{\partial H_{12}^{(1)}(\bar{a}_M,a_M|x_c,\varphi_1)}{\partial \bar{a}_M}\times e^{i\omega(t-\tau)}d\tau+\cdots , \tag{56}$$

i.e., the solution of the self-consistent problem a_M is the first approximation for A and corrections can be found by perturbation theory. It should be noted that with such a determination of a_c, Eq. (42) for Q will certainly be changed, i.e., in the expansions (45) and (47) the coefficients are to be changed. Relations of this type, written for the vector-potential operator, were used[6] for the determination of the applicability limits of semiclassical electrodynamics and the definition of quantum values by solutions of semiclassical equations.

V. SEMICLASSICAL REPRESENTATION

The evolution operator in the form of Eq. (39), in principle, introduces the semiclassical representation in quantum electrodynamics. In this case the evolution operators with time can be described by the operator C, and the state evolution by the operator Q. It should be noted that $C^{-1}a_0C=a_0$, i.e., in the semiclassical representation the time dependence of operators is varied only for the medium operators $x_c=C^{-1}xC$. For the state we have

$$R_Q=QRQ^{-1}\,. \tag{57}$$

The field density matrix is introduced in the usual way:

$$R^F=\mathrm{Tr}_x QRQ^{-1}\,. \tag{58}$$

The average value of an arbitrary field operator is now written as

$$\langle f\rangle_t=\mathrm{Tr}R^F(t)\left[f(a_0^\dagger(t),a_0(t))+\frac{\partial^2 f(a_0^\dagger(t),a_0(t))}{\partial a_0(t)\partial a_0^\dagger(t)}+\cdots\right]. \tag{59}$$

Equations for R^F can be found by the standard method for nonequilibrium statistical physics.[14] For this purpose we write $R_Q=R^FT+\Delta R$ and from Eqs. (57), (58), and (41) an equation is calculated for $R^F(t)$ with an accuracy to the second order of perturbation theory according to H_Q from Eq. (41). Converting to the matrix elements in the states $|\psi_a\rangle$ we obtain

$$\begin{aligned}\frac{\partial R^F_{aa}(t)}{\partial t}&=\frac{1}{i\hbar}\left[\frac{\partial}{\partial\bar{a}(t)}\left\langle\frac{\partial H_{12}(\bar{a}(t),a(t)\,|\,x_c)}{\partial a(t)}\right\rangle_x R^F_{aa}(t)-\frac{\partial}{\partial a(t)}\left\langle\frac{\partial H_{12}(\bar{a}(t),a(t)\,|\,x_c)}{\partial\bar{a}(t)}\right\rangle_x R^F_{aa}(t)\right]\\&\quad-\left(\frac{1}{i\hbar}\right)^2\int_{t_0}^{t}\left\langle\!\!\left\langle\frac{\partial}{\partial\bar{a}(t)}\frac{\partial H_{12}(t)}{\partial a(t)}-\frac{\partial}{\partial a(t)}\frac{\partial H_{12}(t)}{\partial\bar{a}(t)},\frac{\partial}{\partial\bar{a}(\tau)}\frac{\partial H_{12}(\tau)}{\partial a(\tau)}-\frac{\partial}{\partial a(\tau)}\frac{\partial H_{12}(\tau)}{\partial\bar{a}(\tau)}\right\rangle\!\!\right\rangle_x R^F_{aa}(\tau)d\tau\,,\end{aligned} \tag{60}$$

$$\langle\!\langle A,B\rangle\!\rangle_x\equiv\langle A\rangle_x\langle B\rangle_x-\langle[A,B]_+\rangle_x\,.$$

Equation (60) is a generalized Fokker-Planck equation for the distribution function of strong field amplitudes. In fact, instead of solving the quantum problem for a strong field or its semiclassical analog (52)–(54), the semiclassical representation allows one to solve the semiclassical self-consistent problem, Eqs. (53) and (54), to determine the coefficients in Eq. (60) and then to solve this equation. The semiclassical representation removes the direct dependence from the strong field of the Hamiltonian H_Q and results in the appearance of the coefficient's dependence in the Fokker-Planck equation on the solutions of a corresponding semiclassical problem. Therefore, the use of Eq. (60) for application is of greater interest than the use of the Fokker-Planck equation obtained using the coherent states.[15]

As was mentioned above, instead of Eqs. (53) and (54), one can use the analog of the nonlinear Maxwell equations.[2,3] In this case the semiclassical representation is introduced by the equations

$$U_w=GQ\,, \tag{61}$$

$$i\hbar\frac{\partial G}{\partial t}=[H_2(x)+H_{12}(a_M^\dagger,a_M,x)]G\,, \tag{62}$$

$$a_M=a_0(t)+\frac{i}{\hbar}\int_{t_0}^{t}\left\langle\frac{\partial H_{12}(a_M^\dagger(\tau),a_M(\tau)\,|\,x_G(\tau))}{\partial a_M^\dagger(\tau)}\right\rangle_x e^{i\omega(t-\tau)}d\tau. \tag{63}$$

The equation for Q follows from Eqs. (61)–(63) and (29). We shall not cite this equation and the corresponding Fokker-Planck equation for the density matrix and discuss them, because such consideration was given previously.[4,5,6]

VI. OPERATOR SOLUTIONS

The fact that the evolution of operators $\hat{a}$ and $\hat{a}^\dagger$ is determined by Eqs. (32) as well as the simplicity of the algebra of operators a_0, $a_0^\dagger$, Δa, $\Delta a^\dagger$ allow one to hope for a possibility of using the operator methods of solving the time evolution problem for a strong field. Here we give two methods which allow the solutions of the quantum problem to be obtained. First of all, let us represent the evolution operator in the form

$$U_w=C_0Q_0\,, \tag{64}$$

where

$$i\hbar\frac{\partial C_0}{\partial t}=[H_2(x)+H_{12}(a_0^\dagger(t),a_0(t),x)]C_0,\quad C_0(t_0)=1. \tag{65}$$

The equation for Q_0 is determined from Eqs. (29), (64), and (65):

$$i\hbar\frac{\partial Q_0}{\partial t}=\left[\frac{\partial H_{12}(\,|x_0)}{\partial a_0(t)}C_0^{-1}\Delta a(t)C_0+\text{H.c.}\right]Q_0,$$
$$x_0=C_0^{-1}xC_0. \qquad (66)$$

In Eq. (66) we use the Hamiltonian (33) with an accuracy up to terms proportional to Δa and $\Delta a^\dagger$. The latter can be obtained as we are interested in the strong field behavior. It is clear that consideration of the higher orders in Eq. (33), as follows from Eqs. (32) and (38), can result in the appearance of small corrections only.

Now we introduce the operator Q_0 as

$$Q_0=QQ_1\,, \qquad (67)$$

where

$$i\hbar\frac{\partial Q}{\partial t}=\left\{\left\langle\frac{\partial H_{12}(\,|x_0)}{\partial a_0(t)}\right\rangle_x\Delta a(t)+\frac{1}{2}\left[\Delta a(t),\left\langle\frac{\partial H_{12}(\,|x_0)}{\partial a_0(t)}\right\rangle_x\right]+\text{H.c.}\right\}Q\,, \qquad (68)$$

and consider the time dependence of the operators due to C_0 and Q. We have

$$\begin{aligned}\tilde{a}&=Q^{-1}a_0(t)Q\\&=a_0(t)+\frac{i}{\hbar}\int_{t_0}^{t}Q^{-1}(\tau)\left\langle\frac{\partial H_{12}(\tau)}{\partial a_0^\dagger(\tau)}\right\rangle_x Q(\tau)e^{i\omega(t-\tau)}d\tau\\&=a_0(t)+\frac{i}{\hbar}\int_{t_0}^{t}\left\langle\frac{\partial H_{12}(\tilde{a}^\dagger(\tau),\tilde{a}(\tau)\,|\tilde{x}_0)}{\partial\tilde{a}^\dagger(\tau)}\right\rangle_x e^{i\omega(t-\tau)}d\tau,\end{aligned} \qquad (69)$$

$$\tilde{x}_0=Q^{-1}C_0^{-1}xC_0Q=Q^{-1}x_0Q=x_0\{\tilde{a}^\dagger,\tilde{a}\}=\tilde{C}^{-1}x\tilde{C}, \qquad (70)$$

$$i\hbar\frac{\partial\tilde{C}}{\partial t}=[H_2(x)+H_{12}(\tilde{a}^\dagger,\tilde{a},x)]\tilde{C}+\left\langle\frac{\partial H_{12}(\,|\tilde{x}_0)}{\partial\tilde{a}^\dagger}\right\rangle_x Q^{-1}[\Delta a(t),C_0]Q+\text{H.c.} \qquad (71)$$

Equations (69) and (70) show that the evolution of a_0 in the introduced representation is described by the Hamilton equation, and the evolution of medium operators is described by the Schrödinger equation (71).

To make clear the connection of $\tilde{a}$ and $\tilde{x}_0$ with a_{0H} and x_H, it is necessary to obtain from Eqs. (66)–(68) an equation for Q_1. We have

$$i\hbar\frac{\partial Q_1}{\partial t}=\left\{\frac{\partial H_{12}(\tilde{a}^\dagger,\tilde{a}\,|\tilde{x}_0)}{\partial\tilde{a}}Q^{-1}C_0^{-1}\Delta a(t)C_0Q-\left\langle\frac{\partial H_{12}(\,|)}{\partial\tilde{a}}\right\rangle_x Q^{-1}\Delta a(t)Q-Q^{-1}\left[\Delta a(t),\left\langle\frac{\partial H_{12}(a_0^\dagger(t),a_0(t)\,|x_0)}{\partial a_0(t)}\right\rangle_x\right]Q+\text{H.c.}\right\}Q_1\,. \qquad (72)$$

Simple calculations show that

$$\begin{aligned}Q^{-1}\Delta a(t)Q&=\Delta a(t)-\frac{i}{\hbar}\int_{t_0}^{t}Q^{-1}(\tau)\left\langle\frac{\partial H_{12}(a_0^\dagger(\tau),a_0(\tau)\,|x_0)}{\partial a_0^\dagger(\tau)}\right\rangle_x Q(\tau)e^{i\omega(t-\tau)}d\tau\\&\quad+\frac{i}{\hbar}\int_{t_0}^{t}Q^{-1}(\tau)\left[\left[\Delta a(\tau),\left\langle\frac{\partial H_{12}(\tau)}{\partial a_0(\tau)}\right\rangle_x\right],\Delta a(t)\right]Q(\tau)d\tau\\&\quad+\frac{i}{\hbar}\int_{t_0}^{t}Q^{-1}(\tau)\left[\left[\Delta a^\dagger(\tau),\left\langle\frac{\partial H_{12}(\tau)}{\partial a_0^\dagger(\tau)}\right\rangle_x\right],\Delta a(t)\right]Q(\tau)d\tau\,,\\Q^{-1}C_0^{-1}\Delta a(t)C_0Q&=Q^{-1}\left(\Delta a(t)+\frac{i}{\hbar}\int_{t_0}^{t}\frac{\partial H_{12}(a_0^\dagger(\tau),a_0(\tau)\,|x_0)}{\partial a_0^\dagger(\tau)}e^{i\omega(t-\tau)}\right)Q\\&=Q^{-1}\Delta a(t)Q+\frac{i}{\hbar}\int_{t_0}^{t}Q^{-1}(t)\frac{\partial H_{12}(a_0^\dagger(\tau),a_0(\tau)\,|x_0)}{\partial a_0^\dagger(\tau)}Q(t)e^{i\omega(t-\tau)}d\tau\,.\end{aligned}$$

Omitting the terms of higher orders (of the type $\partial^2H_{12}/\partial a_0^{\,2}$, $\partial^3H_{12}/\partial a_0^{\,3}$) these expressions can be rewritten as

$$\begin{aligned}Q^{-1}\Delta a(t)Q&=\Delta a(t)-\frac{i}{\hbar}\int_{t_0}^{t}\left\langle\frac{\partial H_{12}(\tilde{a}^\dagger(\tau),\tilde{a}(\tau)\,|\tilde{x}_0)}{\partial\tilde{a}^\dagger(\tau)}\right\rangle_x e^{i\omega(t-\tau)}d\tau\,,\\Q^{-1}C_0^{-1}\Delta a(t)C_0Q&=\Delta a(t)-\frac{i}{\hbar}\int_{t_0}^{t}\left[\left\langle\frac{\partial H_{12}(\tilde{a}^\dagger(\tau),\tilde{a}(\tau)\,|\tilde{x}_0)}{\partial\tilde{a}^\dagger(\tau)}\right\rangle_x-\frac{\partial H_{12}(\tilde{a}^\dagger(\tau),\tilde{a}(\tau)\,|\tilde{x}_0)}{\partial\tilde{a}^\dagger(\tau)}\right]e^{i\omega(t-\tau)}d\tau\,.\end{aligned}$$

Now Eq. (72) is rewritten with the same degree of accuracy:

$$\begin{aligned}i\hbar\frac{\partial Q_1}{\partial t}=\Bigg[&\left(\frac{\partial H_{12}(\tilde{a}^\dagger,\tilde{a}\,|\tilde{x}_0)}{\partial\tilde{a}}-\left\langle\frac{\partial H_{12}(\tilde{a}^\dagger,\tilde{a}\,|\tilde{x}_0)}{\partial\tilde{a}}\right\rangle_x\right)\Delta a(t)+\frac{\partial H_{12}(\tilde{a}^\dagger,\tilde{a}\,|\tilde{x}_0)}{\partial\tilde{a}}\frac{i}{\hbar}\int_{t_0}^{t}\left(\frac{\partial H_{12}(\tau)}{\partial\tilde{a}^\dagger(\tau)}-\left\langle\frac{\partial H_{12}(\tau)}{\partial\tilde{a}^\dagger(\tau)}\right\rangle_x\right)e^{i\omega(t-\tau)}d\tau\\&+\left\langle\frac{\partial H_{12}(\tilde{a}^\dagger,\tilde{a}\,|\tilde{x}_0)}{\partial\tilde{a}}\right\rangle_x\frac{i}{\hbar}\int_{t_0}^{t}\left\langle\frac{\partial H_{12}(\tilde{a}^\dagger(\tau),\tilde{a}(\tau)\,|\tilde{x}_0)}{\partial\tilde{a}^\dagger(\tau)}\right\rangle_x e^{i\omega(t-\tau)}d\tau+\text{H.c.}\Bigg]Q_1\,.\end{aligned} \qquad (73)$$

The Hamiltonian form in Eq. (73) shows that Eqs. (65), (68), and (69) describe evolution quite well in the case of the interaction of a strong field with a quantum medium. If the density matrix R_Q is introduced in this representation,

$$R_Q = Q_1 R Q_1^{-1}, \tag{74}$$

then for average values the following formula is fulfilled:

$$\langle f(\hat{a}^\dagger, \hat{a}, x)\rangle_t = \mathrm{Tr} R_Q(t) f(\tilde{a}^\dagger(t), \tilde{a}(t), \tilde{x}_0(t)). \tag{75}$$

In this case if f is the field operator only, then while determining R_Q the first term only in Eq. (73) should be taken into account, as the contribution of subsequent terms will be negligibly small, and in this case Q converges very rapidly.

The above relations assume a simple calculation scheme. The Schrödinger equation (65) is by its meaning the Schrödinger equation for a quantum system in an external classical field, which in some cases can be solved. In the general case the perturbation theory solution can be used. The solution obtained allows one to determine the Hamiltonian in Eq. (68). The simplicity of the algebra of the operators permits the approximate operator solution of Eq. (68) to be calculated. Now calculation of the operators $\tilde{a}$ and $\tilde{x}_0$ presents no difficulties. More precise determination of the values obtained can be made using Eqs. (73) and (67) or calculating the density matrix (74) and averaging according to Eq. (75).

The second method of operator solution of the problem is based on the representation (61) and the Schrödinger equation (62).

The operator Θ is chosen as the solution of the equation

$$i\hbar\frac{\partial\Theta}{\partial t} = \left\{\left\langle\frac{\partial H_{12}(a_0^\dagger, a_0 \mid x_G)}{\partial a_0}\right\rangle_x \Delta a(t) + \frac{1}{2}\left[\Delta a(t), \left\langle\frac{\partial H_{12}(a_0^\dagger, a_0 \mid x_G)}{\partial a_0}\right\rangle_x\right] + \mathrm{H.c.}\right\}\Theta. \tag{76}$$

In the representation defined by G and Θ, the operators are of the form

$$\Theta^{-1} a_0(t)\Theta = \tilde{a}(t) = a_0(t) + \frac{i}{\hbar}\int_{t_0}^{t}\left\langle\frac{\partial H_{12}(\tilde{a}^\dagger(\tau), \tilde{a}(\tau) \mid \tilde{x}_G)}{\partial \tilde{a}^\dagger(\tau)}\right\rangle_x \times e^{i\omega(t-\tau)}d\tau, \tag{77}$$

$$\Theta^{-1}G^{-1}xG\Theta = \Theta^{-1}x_G\Theta = x_G\{\Theta^{-1}a_0^\dagger\Theta, \Theta^{-1}a_0\Theta\} \equiv \tilde{x}_G. \tag{78}$$

We now determine the operator Q from Eq. (61) as

$$Q = \Theta\Theta_1. \tag{79}$$

The equation for Θ_1 is obtained as in the case of Eq. (73):

$$\begin{aligned} i\hbar\frac{\partial\Theta_1}{\partial t} = \Bigg\{&\left(\frac{\partial H_{12}(\tilde{a}^\dagger, \tilde{a} \mid \tilde{x}_G)}{\partial \tilde{a}} - \left\langle\frac{\partial H_{12}(\tilde{a}^\dagger, \tilde{a} \mid \tilde{x}_G)}{\partial \tilde{a}}\right\rangle_x\right)\Delta a(t) \\ &+ \frac{\partial H_{12}(\tilde{a}^\dagger, \tilde{a} \mid \tilde{x}_G)}{\partial \tilde{a}}\frac{i}{\hbar}\int_{t_0}^{t}\left(\left\langle\frac{\partial H_{12}(\tilde{a}^\dagger, \tilde{a} \mid \tilde{x}_G)}{\partial \tilde{a}^\dagger}\right\rangle_x - \frac{\partial H_{12}(\tilde{a}^\dagger, \tilde{a} \mid \tilde{x}_G)}{\partial \tilde{a}^\dagger}\right)e^{i\omega(t-\tau)}d\tau \\ &+ \left\langle\frac{\partial H_{12}(\tilde{a}^\dagger, \tilde{a} \mid \tilde{x}_G)}{\partial \tilde{a}}\right\rangle_x \frac{i}{\hbar}\int_{t_0}^{t}\left\langle\frac{\partial H_{12}(\tilde{a}^\dagger, \tilde{a} \mid \tilde{x}_G)}{\partial \tilde{a}^\dagger}\right\rangle_x e^{i\omega(t-\tau)}d\tau + \mathrm{H.c.}\Bigg\}\Theta_1. \end{aligned} \tag{80}$$

The relations (62) and (76)–(80) can also be used for calculating quantum operators. In this case it is necessary to know any solution of the nonlinear Hamilton equations, i.e., the operator G. Let it be calculated approximately by perturbation theory. Then it is necessary to find an approximate solution of Eqs. (76) for Θ and to calculate $\tilde{a} = \Theta^{-1}a_0\Theta$. The average values in this case can be calculated by the formula

$$\langle f\rangle_t = \mathrm{Tr} R_\Theta(t) f(\tilde{a}^\dagger, \tilde{a}, \tilde{x}_G), \quad R_\Theta = \Theta_1 R \Theta_1^{-1} \tag{81}$$

and calculation of R_Θ allows the average values to be defined more exactly.

VII. A MULTIMODE CASE

We consider the connection of the suggested approach to the description of a strong field interacting with a quantum medium and our preceding results.[1-6] It should be noted that previously we considered the case of an infinite number of modes. In this case the vector-potential operator of the field is as follows:

$$\vec{A}(\vec{r}) = \sum_{\vec{k},\lambda} \vec{\kappa}_{\vec{k},\lambda}(\hat{a}_{\vec{k},\lambda}e^{i\vec{k}\cdot\vec{r}} + \hat{a}^\dagger_{\vec{k},\lambda}e^{-i\vec{k}\cdot\vec{r}}), \quad \vec{\kappa}_{\vec{k},\lambda} \equiv \left(\frac{2\pi\hbar c}{kL^3}\right)^{1/2}\vec{e}_\lambda(\vec{k}). \tag{82}$$

Standard designations are used.

Generalization of the preceding results for a multimode case is trivial. It is sufficient to extend the Hilbert space connected with each mode and to introduce the operators $a_{0\vec{k},\lambda}$, $a^{\dagger}_{0\vec{k},\lambda}$, $\Delta a_{\vec{k},\lambda}$, $\Delta a^{\dagger}_{\vec{k}\lambda}$. It is clear that the operator $\vec{A}_0$, determined according to Eq. (82) as

$$\vec{A}_0(\vec{r}) = \sum_{\vec{k},\lambda} \vec{\kappa}_{\vec{k},\lambda}(a_{0\vec{k},\lambda}e^{i\vec{k}\cdot\vec{r}} + a^{\dagger}_{0\vec{k},\lambda}e^{-i\vec{k}\cdot\vec{r}}),$$

can be called the vector-potential operator of a classical field and the operator $\Delta\vec{A}$ is

$$\Delta\vec{A}(\vec{r}) = \sum_{\vec{k},\lambda} \vec{\kappa}_{\vec{k},\lambda}(\Delta a_{\vec{k},\lambda}e^{i\vec{k}\cdot\vec{r}} + \text{H.c.}).$$

It will be connected with quantum properties of the field and will be called the vector-potential operator of quantum fluctuations.

Converting to the interaction representation and taking into account that

$$\vec{A}(\vec{r},t) = \vec{A}_0(\vec{r},t) + \Delta\vec{A}(\vec{r},t), \tag{83}$$

we obtain

$$\vec{A}_0(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\kappa}_{\vec{k},\lambda}[a_{0\vec{k},\lambda}(t)e^{i\vec{k}\cdot\vec{r}} + \text{H.c.}], \tag{84}$$

$$\Delta\vec{A}(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{\kappa}_{\vec{k},\lambda}[a_{\vec{k},\lambda}(t)e^{i\vec{k}\cdot\vec{r}} + \text{H.c.}]. \tag{85}$$

It should be noted that the relation (83) was introduced previously,[2] but for $\vec{A}_0$ and $\Delta\vec{A}$ other definitions were suggested; in particular, $\vec{A}_0$ was represented as an expansion only by $\cos\omega_k t$. It is clear that these differences are due to the fact that in Ref. 2 in evident form the realness of the classical field vector potential was taken into account, which decreases the number of independent amplitudes by a factor of 2. The condition of realness $a_{\vec{k},\lambda} = \bar{a}_{-\vec{k},\lambda}$ results in coincidence of the matrix elements of operators (84) and (85) with the matrix elements of operators $\vec{A}_0$ and $\Delta\vec{A}$ from Ref. 2. This fact explains why in a general multimode case it became possible to make analogous analysis without extending the Hilbert space.

VIII. CONCLUSION

We have introduced the mathematical apparatus which allows one to obtain naturally and easily the derivation of the semiclassical schemes for the description of field and matter interaction from quantum theory. The fact that the suggested method allows one to consider the model problems of quantum electrodynamics, i.e., the interaction of the final number of modes with a quantum system, and the fact that such problems in semiclassical electrodynamics can be solved, has made it possible to use semiclassical electrodynamics as a calculational apparatus of quantum electrodynamics. Besides, the simplicity of the algebra of operators and the structure of relations for evolution of a strong field enables one to obtain the simple operator solutions of the quantum problem.

It should also be noted that the scheme of introduction of the operators a_0 and $a_0^{\dagger}$ allows a natural generalization for the case when a particle close to a classical system is considered. In this case the momentum p_0 and the coordinate q_0 operators are determined as follows:

$$p_0 = i\left(\frac{2m\Omega}{\hbar}\right)^{1/2}(a_0^{\dagger} - a_0),$$

$$q_0 = \left(\frac{2\hbar}{m\Omega}\right)^{1/2}(a_0^{\dagger} + a_0).$$

Here a_0 and $a_0^{\dagger}$ are defined by Eq. (9), m is the particle mass, and $\Omega = \langle 0|H|0\rangle/\hbar$, where $|0\rangle$ is the vacuum vector and H is the particle Hamiltonian. It should be noted that such a determination of Ω allows us to describe an arbitrary particle, but not to limit ourselves to the case of a harmonic oscillator.

It is clear that all the preceding results can be reformulated for the introduced operators, and we obtain a convenient apparatus for the problems of the statistical physics of intermolecular interaction. In this case the external degrees of freedom of molecules will be the system for which the transition to a classical limit takes place, and the internal degrees of freedom, interacting with them, will be the analog of the quantum system. Some preliminary results of such an approach are given in our work.[16] A more complete description will soon be published.

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[8]It should be noted that except for Eqs. (9)–(11) there is a possibility of defining these values:

$$a_0 = \hat{a}\otimes I - I\otimes\hat{a}^\dagger, \quad \Delta a = I\otimes\hat{a}^\dagger,$$

$$|\psi_a\rangle = \pi^{-1/2} e^{-|a|^2/2} \int \exp(a\bar{\alpha} - \bar{a}\beta + \bar{\alpha}\bar{\beta})|\alpha\rangle|\beta\rangle\pi^{-2} \times \exp[-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2)]d^2\alpha\, d^2\beta.$$

The operators and states of this type appear when using the eigenstates of the operator of electric field intensity $\vec{E}$. Analyzing this problem, formulated by means of the above relations, it is clear that no principal differences from the case using Eqs. (9)–(11) appear in the final results. It should be noted that the determinations given can be more convenient in the case where the interaction Hamiltonian is written in the form of $H_{0R} = \vec{d}\cdot\vec{E}$, where $\vec{d}$ is the dipole moment of the quantum system.

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