PHYSICAL REVIEW D VOLUME 22, NUMBER 4 15 AUGUST 1980

Quantization and the noncausality of massive spin-one particles in an external symmetrical tensor field

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(Received 24 January 1980)

The retarded and advanced Green's functions for massive spin-one particles in an external symmetrical tensor field are discussed. A method of determining the retarded Green's function for a special case is outlined. Then, by a limiting process, it is explicitly shown that the equal-time commutators derived from the equation of motion are in agreement with the canonical commutators.

I. INTRODUCTION

We will begin our discussion with a brief review of a method by which one can calculate the retarded and the advanced Green's functions for massive spin-one particles in a symmetrical tensor field. The method is that of the propagation of shock waves along the characteristic surfaces. When a certain range for the external field is assumed, the wave equation exhibits noncausal behavior, that is, it possesses a characteristic surface which lies outside the light cone. This ill effect becomes clearer when the retarded and advanced Green's functions do not vanish for all spacelike separations. This contradicts the canonical quantization where the field commutator is assumed to vanish for all spacelike separations.

Therefore, one is inclined to believe that for the noncausal equations canonical quantization is inconsistent with the equation of motion. We will show that on the plane $t=0$ the canonical quantization is consistent with the equation of motion but that the generalization of this result to all spacelike surfaces contradicts the equation of motion.

Our notation is that of Bjorken and Drell.[1] The metric tensor $g^{\mu\nu}$ is defined as $g^{00}=1$, $g^{11}=g^{22}=g^{33}=-1$. The Greek letters are used as Lorentz indices that range from 0 to 3, while the Latin letters range from 1 to 3. The Einstein summation rule for repeated indices is used throughout the paper.

II. RETARDED AND ADVANCED GREEN'S FUNCTIONS

The Lagrangian density for massive spin-one particles in an external symmetrical field is given by

$$\mathcal{L}=-\tfrac{1}{4}G^{\mu\nu}G_{\mu\nu}+\tfrac{1}{2}m^2\phi\cdot\phi+\tfrac{1}{2}\lambda\phi\cdot T\cdot\phi, \tag{2.1}$$

where ϕ^μ is the vector field, $G^{\mu\nu}$ is defined by $\partial^\mu\phi^\nu-\partial^\nu\phi^\mu$, and $T^{\mu\nu}=T^{\nu\mu}$ is a symmetrical tensor. By $\phi\cdot\phi$ we mean $\phi^\alpha\phi_\alpha$ and ∂^α is $\partial/\partial x_\alpha$. The field equation derived from Eq. (2.1) is

$$[(\partial^2+m^2)g^{\mu\alpha}-\partial^\mu\partial^\alpha+\lambda T^{\mu\alpha}]\phi_\alpha=0\ , \tag{2.2}$$

with $\partial_\mu G^{\mu 0}+m^2\phi^0+\lambda T^0\cdot\phi=0$ as the primary constraint and $(m^2\partial^\alpha+\lambda\partial\cdot T^\alpha)\phi_\alpha=0$ as the secondary constraint, which is obtained by contracting Eq. (2.2) with ∂_μ.

The retarded Green's function $D^R_{\mu\nu}(x,y)$ satisfies Eq. (2.2) with the δ function on the right-hand side,[2] i.e.,

$$[(\partial^2+m^2)g^{\mu\alpha}-\partial^\mu\partial^\alpha+\lambda T^{\mu\alpha}]D^R_{\alpha\nu}(x,y)=g^\mu{}_\nu\delta^4(x-y), \tag{2.3}$$

with the condition that

$$D^R_{\alpha\nu}(x,y)=0\quad\text{for}\quad(x^0-y^0)<0\ . \tag{2.3$'$}$$

Equation (2.3) does not have well-defined characteristic surfaces since the determinant of the characteristic matrix is identically zero.[3] To eliminate this problem, we take the divergence of Eq. (2.3) and substitute the result back into Eq. (2.3). The result is

$$\left[(\partial^2+m^2)g^{\mu\alpha}+\frac{\lambda}{m^2}\partial^\mu\partial\cdot T^\alpha+\lambda T^{\mu\alpha}\right]D^R_{\alpha\nu}(x,y)=\left(g^\mu{}_\nu+\frac{1}{m^2}\partial^\mu\partial_\nu\right)\delta^4(x-y)\ . \tag{2.4}$$

The characteristic surfaces are obtained by setting the determinant of the characteristic matrix equal to zero, i.e.,

$$\det\left|n^2 g^{\mu\alpha}+\frac{\lambda}{m^2}n^\mu n\cdot T^\alpha\right|=0\ , \tag{2.5}$$

which yields

$$-(n^2)^3\left(n^2+\frac{\lambda}{m^2}n\cdot T\cdot n\right)=0\ . \tag{2.6}$$

n^μ is the normal vector to the characteristic surfaces. There are two distinct surfaces

$$(n^2)^3=0 \text{ and } \left(n^2+\frac{\lambda}{m^2}n\cdot T\cdot n\right)=0\ . \tag{2.7}$$

The first factor is satisfied by the $u_1=t-r=0$ surface which is the future cone, where $t=x^0-y^0$ and $r=|\vec{x}-\vec{y}|$. To determine the surface $u_2=0$ satisfying the second factor, we consider a special case where $T^{\mu\nu}$ has only one nonzero constant component, T^{00}. Then we have $u_2=Vt-r$, where V is the maximum speed of the propagation of signals given by

$$V=\frac{1}{\left(1+\frac{\lambda}{m^2}T^{00}\right)^{1/2}}\ . \tag{2.8}$$

If we assume that T^{00} satisfies

$$-1<\frac{\lambda}{m^2}T^{00}<0\ , \tag{2.9}$$

the signals propagate faster than the speed of light, and $u_2=0$ lies outside the light cone in the spacelike region.

Equation (2.4) with T^{00}, the only nonzero component of $T^{\mu\nu}$, takes the form

$$\left[(\partial^2+m^2)g^{\mu\alpha}+\frac{\lambda}{m^2}T^{00}\delta^{0\alpha}\partial^\mu\partial^0+\lambda T^{00}\delta^{0\alpha}\delta^{0\mu}\right]D^R_{\alpha\nu}(x,y)=\left(g^\mu{}_\nu+\frac{1}{m^2}\partial^\mu\partial_\nu\right)\delta^4(x-y)\ , \tag{2.10}$$

where $\delta^{\mu\alpha}$ is the Kronecker δ.

Instead of solving Eq. (2.10), we will consider a closely related equation

$$\left[(\partial^2+m^2)g^{\mu\alpha}+\frac{\lambda}{m^2}T^{00}\delta^{0\alpha}\partial^\mu\partial^0+\lambda T^{00}\delta^{0\alpha}\delta^{0\mu}\right]\Delta^R_{\alpha\nu}(x,y)=g^\mu{}_\nu\delta^4(x-y)\ , \tag{2.11}$$

and then relate $\Delta^R_{\alpha\nu}(x,y)$ to $D^R_{\alpha\nu}(x,y)$ by

$$D^R_{\alpha\nu}(x,y)=\left(g^\lambda{}_\nu+\frac{1}{m^2}\partial^\lambda\partial_\nu\right)\Delta^R_{\lambda\alpha}(x,y)\ . \tag{2.12}$$

The solution to Eq. (2.11) has the form[4,5]

$$\Delta^R_{\alpha\nu}(x,y)=\sum_{j=1}^{2}\left[\sum_{k=\nu}^{0}\delta^k(u_j)E^{jk}_{\alpha\nu}(r)+\theta(u_j)\sum_{l=0}^{\infty}\frac{u_j{}^l}{l!}G^{lj}(r)\right]\ , \tag{2.13}$$

where $\delta^k(u_j)$ is the kth derivative of the Dirac δ function, and $\theta(u_j)$ is the step function satisfying

$$\theta(u_j)=0,\quad u_j<0$$
$$\theta(u_j)=1,\quad u_j\geq 0. \tag{2.14}$$

Substituting (2.13) into (2.11) and separating the coefficients of the different singularities, one obtains the unknown functions, E's and G's, on a particular surface. Note that we use spherical coordinates and by x we mean (t,r).

The final result is

$$D^R_{\alpha\nu}(x)=\frac{1}{4\pi}\left[\frac{V}{m^2}\partial_\alpha\partial_\nu\frac{\delta(u_2)}{r}+\partial_\alpha\frac{\theta(u_2)}{r}\delta_{0\nu}+\delta_{0\alpha}\partial_\nu\frac{\theta(u_2)}{r}-\partial_\alpha\frac{\theta(u_1)}{r}\delta_{0\nu}-\delta_{0\alpha}\partial_\nu\frac{\theta(u_1)}{r}-\frac{1}{2V}\partial_\alpha\partial_\nu\theta(u_2)\right.$$
$$+\partial_\alpha\partial_\nu\frac{u_1\theta(u_1)}{r}+g_{\alpha\nu}\frac{\delta(u_1)}{r}-\frac{1}{V}\partial_\alpha\partial_\nu\frac{u_2\theta(u_2)}{r}-\frac{m^2}{2}g_{\alpha\nu}\theta(u_1)$$
$$\left.+\frac{m^2}{2}\partial_\alpha\theta(u_1)\delta_{0\nu}+\frac{m^2}{8V^3}\partial_\alpha\partial_\nu u_2\theta(u_2)r+\cdots\right]\ . \tag{2.15}$$

From Eq. (2.15) it is clear that $D^R_{\alpha\nu}$ does not vanish in the spacelike region bounded by the $u_2=Vt-r=0$ surface and the light cone.

The advanced Green's function can be calculated in exactly the same way with

$$u_1=t+r \tag{2.16}$$

and

$$u_2=Vt+r \tag{2.17}$$

as the characteristic surfaces.

III. EQUAL-TIME COMMUTATION RELATIONS

According to Peierl's quantization,[6] we have

$$[\phi_\alpha(x),\ \phi_\beta(y)] = -iD_{\alpha\nu}(x,y)\ , \tag{3.1}$$

where $D_{\alpha\nu}$ is defined as

$$D_{\alpha\nu} = D^R_{\alpha\nu} - D^A_{\alpha\nu}\ . \tag{3.2}$$

To calculate the equal-time commutators, it suffices to calculate

$$\lim_{t\to+0} D_{\alpha\nu} = \lim_{t\to+0} D^R_{\alpha\nu}\ , \tag{3.3}$$

since $D^R_{\alpha\nu} = \theta(t)D_{\alpha\nu}$.

To compute this limit, first let us define the limit of the derivatives of the δ function $\delta^k(u)$ as t approaches $+0$.[7] For a small t, $\delta^k(u)$ is defined by

$$\int \delta^k(u) f(\vec{r}) r^2 dr\, d\Omega\ , \tag{3.4}$$

where $u = t - r$ or $u = Vt - r$, and $d\Omega$ is the angular part of the spherical volume element. The function f is a smooth test function. Using the definition of the derivatives of the δ function, we write the integral (3.4) as

$$(-1)^k \int \left[\frac{\partial^k}{\partial(-r)^k}(r^2 f)\right]_{r=t \text{ or } r=Vt} d\Omega\ . \tag{3.5}$$

To find the limit, we let t approach $+0$. We may generalize (3.5) to define the limit of $\delta^k(u)E(r)$ by the integral

$$\lim_{t\to+0}\left\{\int \frac{\partial^k}{\partial r^k}[E(r)r^2 f]\, d\Omega\right\}_{r=t \text{ or } r=Vt} \tag{3.6}$$

where $E(r)$ is a function defined on the $u=0$ surface.

In calculating the limit of $D^R_{\alpha\nu}$, we find that all but one component is nonvanishing, D^R_{0j}. Below, we will show the calculation of D^R_{0j}.

The highest singular term is the only term contributing to the limit. Therefore, the term which we are considering is

$$\lim_{t\to+0} \frac{V}{4\pi m^2}\partial_0\partial_j \frac{\delta(u_2)}{r}\ . \tag{3.7}$$

Carrying out the differentiation and using integral (3.6), we get

$$-\frac{V^2}{4\pi m^2}\lim_{t\to+0}\left\{\int \frac{\partial}{\partial r}[(\partial_j r)f]\, d\Omega - \int \frac{\partial^2}{\partial r^2}[(\partial_j r) r f]\, d\Omega\right\}_{r=Vt}\ . \tag{3.8}$$

Since the integrals are taken near the origin, we make a Taylor expansion of the test function f. The first integral in (3.8) becomes

$$-\frac{V^2}{4\pi m^2}\lim_{t\to+0}\left\{\int \frac{\partial}{\partial r}\left[\frac{x_i}{r}\left(f(0) + x^i \frac{\partial f(0)}{\partial x^i} + \frac{1}{2!}x^i x^k \frac{\partial^2 f(0)}{\partial x^i \partial x^k} + \cdots\right)\right] d\Omega\right\}_{r=Vt}\ , \tag{3.9}$$

where we have substituted x_j/r for $\partial_j r$. The terms with odd numbers of x's vanish when they are integrated over the angles, and the terms with r^n for $n \geq 2$ do not contribute when $r \to t \to 0$. Therefore, the remaining term is

$$-\frac{V^2}{4\pi m^2}\lim_{t\to+0}\left\{\int \frac{\partial}{\partial r}\left[\frac{x_i x_j}{r}\frac{\partial f(0)}{\partial x^i}\right] d\Omega\right\}_{r=Vt}\ , \tag{3.10}$$

which, after integration over the angles, becomes

$$-\frac{V^2}{4\pi m^2}\frac{4\pi}{3}\delta_{ij}\frac{\partial f(0)}{\partial x^i}\ . \tag{3.11}$$

The second integral in (3.8) can be written as

$$-\frac{V^2}{4\pi m^2}\lim_{t\to+0}\left\{\int r(\partial_j r)\frac{\partial^2 f}{\partial r^2}\, d\Omega + 2\int \frac{\partial}{\partial r}[(\partial_j r)f]\, d\Omega\right\}_{r=Vt}\ . \tag{3.12}$$

The first integral vanishes when $t \to 0$ and the second integral becomes

$$-\frac{2V^2}{4\pi m^2}\frac{4\pi}{3}\delta_{ij}\frac{\partial f(0)}{\partial x^i}\ . \tag{3.13}$$

Combining (3.13) and (3.11), we get

$$D_{0j}(x,y)_{x^0=y^0} = \lim_{t\to+0} D^R_{0j} = \frac{V^2}{m^2}\partial_j \delta^3(x-y)\ . \tag{3.14}$$

The time derivatives of $D_{\alpha\nu}$ on the $t=0$ plane can be calculated in a similar fashion, yielding

$$\frac{\partial}{\partial t} D^R_{\alpha\nu}(x,y)\bigg|_{x^0=y^0} = \frac{1}{4\pi}\left[\frac{V^4}{m^2}\delta_{0\alpha}\delta_{0\nu}\nabla^2\delta^3(x-y) + \left(\delta_{\alpha\nu} + \frac{V^2}{m^2}\partial_\alpha\partial_\nu\right)\delta_{\alpha i}\,\delta_{\nu j}\,\delta^3(x-y)\right], \tag{3.15}$$

where $\nabla^2 = \partial^i\partial^i$.

The above results agree with the canonical quantization on the $t=0$ plane, where one imposes the quantization

$$
\begin{aligned}
&[\phi^i(x),\ \phi^j(y)]_{x^0=y^0}=[\pi^i(x),\ \pi^j(y)]_{x^0=y^0}=0\ ,\\
&[\pi^i(x),\ \phi^j(y)]_{x^0=y^0}=-i\delta^{ij}\delta^3(x-y)\ ,
\end{aligned}
\tag{3.16}
$$

where π^i is defined by $\pi^i=G^{i0}=\partial^i\phi^0-\partial^0\phi^i$. From the equation of motion, Eq. (2.2), and the primary and the secondary constraints, we can calculate

$$
[\phi^0(x),\ \phi^j(y)]_{x^0=y^0}=-i\,\frac{V^2}{m^2}\,\partial^j\,\delta^3(x-y)\ , \tag{3.17}
$$

which is the same result as (3.14).

The above results show that even though the field commutator is nonvanishing for all spacelike separations, the equal-time commutator is canonical. Although we have shown this for a particular case, it is probably true for all noncausal theories of this nature.

IV. CONCLUSION

The canonical quantization is consistent with the equation of motion as long as it is done on the $t=0$ plane, as was shown explicitly in Sec. III. The examination of the Lorentz invariance of this theory by conventional methods shows no ill effects if the generators of the Poincaré group are integrals over the flat $t=0$ surface.

It seems that in field equations where one is faced with noncausal behavior, all spacelike surfaces are not equivalent and generalization of relations on the $t=0$ plane to all spacelike surfaces contradicts the equation of motion. For example, the commutation relations on $t=0$, when generalized on all spacelike surfaces, contradicts the results obtained in Sec. II, Eq. (2.15), which is obtained directly from the equation of motion. Therefore, the action principle which relies on spacelike surfaces does not seem to apply to these theories.

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