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Kahler geometry and the renormalization of supersymmetric σ models

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It is shown using general arguments based on Kahler geometry that supersymmetric σ models defined on manifolds which are (A) Ricci-flat or (B) locally symmetric have the following ultraviolet properties. Class A theories are finite at one- and two-loop order. Class B theories, which include the O(n) and CP^{n-1} models, (i) are one-loop divergent, (ii) but have no two-loop divergences. The geometrical argument can be extended to higher order with new complications which are discussed. It seems probable that class B theories have no higher-loop ultraviolet divergences and quite possible that class A theories are entirely ultraviolet finite.

The nonlinear σ model in two space-time dimensions has been widely studied. The geometric nature of the bosonic model is evident from the action

$$I=\frac{1}{2}\int d^2x\, g_{ij}(\phi)\partial_\mu\phi^i(x)\partial_\mu\phi^j(x)\,, \tag{1}$$

where the fields $\phi^i(x)$, $i=1,\ldots,n$, are maps from flat Euclidean or Minkowski space to an n-dimensional manifold M with Riemannian metric $g_{ij}(\phi)$. One can obtain an $N=1$ supersymmetric extension[1] with n two-component Majorana spinor fields $\psi^i(x)$ which transform as a contravariant vector on M. There is also an important theorem due to Zumino[2] concerning extended $N=2$ supersymmetry. Given a complex manifold M^c with coordinate fields $z^\alpha(x)$ and conjugates $\bar z^\beta(x)$, $\alpha,\beta=1,\ldots,n$, and Hermitian metric $g_{\alpha\bar\beta}(z,\bar z)$, then the bosonic model with action

$$I=\int d^2x\, g_{\alpha\bar\beta}(z,\bar z)\partial_\mu z^\alpha\partial_\mu\bar z^\beta \tag{2}$$

has an $N=2$ supersymmetric extension, with n complex spinor fields, if and only if the metric satisfies the Kahler condition

$$\partial_\gamma g_{\alpha\bar\beta}(z,\bar z)=\partial_\alpha g_{\gamma\bar\beta}(z,\bar z)\,. \tag{3}$$

Thus Kahler geometry enters the subject in connection with $N=2$ extended supersymmetry. It is also possible to have $N=4$ supersymmetry in nonlinear models,[3] and it has recently been shown[4] that the simplest $N=4$ models involve Kahler manifolds with vanishing Ricci tensor.

In this paper we show that Kahler geometry has very strong implications for the renormalizability of supersymmetric σ models defined on either Riemannian or Kahler manifolds. We now summarize the results.

Strict renormalizability does not hold for general metric σ models in two space-time dimensions because there are typically an infinite number of counterterms allowed by invariance and power counting. The one-loop counterterm of the bosonic σ model involves the Ricci tensor of the manifold[5] together with a noncovariant field renormalization. The latter vanishes for all Kahler manifolds, and the former vanishes for all Ricci-flat manifolds whether Riemannian or Kahler. One-loop bosonic counterterms do not change when fermions are added with supersymmetric coupling, so that supersymmetric models on Ricci-flat manifolds are also one-loop ultraviolet finite.

At two-loop order the situation is rather different. All bosonic σ models have two-loop divergences.[6] However, using Kahler geometry one can argue without calculation that invariant divergences cancel in two classes of supersymmetric models, those defined on Ricci-flat manifolds ($R_{ij}=0$) or on locally symmetric manifolds ($D_iR_{jklm}=0$). The latter class includes the well-known supersymmetric extensions of the[7] O(n) and[8] CP^{n-1} models and all other models with instanton solutions.[9] The general argument allows one two-loop ultraviolet counterterm of specific tensor form which is nonvanishing on manifolds with $D_iR_{jk}\neq 0$. Explicit two-loop calculations confirm the absence of divergences for Ricci-flat and locally symmetric spaces and the presence of divergences associated with the one allowed tensor for general manifolds. This divergence appears to be a consequence of the generalized renormalization-group[6] pole equations.

Our argument can be applied to the study of ultraviolet divergences at three-loop and higher order, but there are new complications. At present the conclusions are more speculative because they are based on mathematical results which seem plausible but are not yet established. It appears very probable that all higher-loop invariant ultraviolet divergences cancel in supersymmetric σ models on locally symmetric spaces. This means that these models are field theories of a new type. They have ultraviolet divergences only at one-loop order, even though power counting and general invariance considerations permit higher-loop divergences. It is also possible that higher-loop divergences cancel for Ricci-flat manifolds so that supersymmetric models on Ricci-flat spaces are entirely ultraviolet finite.

In this paper the general line of argument leading to the conclusions above is developed. At one or two points we must use detailed properties of the background-field expansion. These will be more fully explained and detailed two-loop calculations presented in another planned paper.[10]

LOCAL KAHLER GEOMETRY

The local geometry of Kahler manifolds is important for our considerations.[11] An n-dimensional complex manifold can always be regarded as a $2n$-dimensional real manifold with coordinates z^A, where the index A runs through the n holomorphic indices $\alpha=1,2,\ldots,n$ and n antiholomorphic indices $\bar{\alpha}=1,2,\ldots,n$, and we have $z^{\bar{\alpha}}=\bar{z}^{\alpha}$. A vector V_A then has $2n$ components V_α and $V_{\bar{\alpha}}$. All formulas of Riemannian geometry, such as definitions of connection and curvature, are valid for complex manifolds when expressed in terms of $2n$-valued indices $A,B,C,\ldots$. In particular there is a real line element given by $ds^2=g_{AB}dz^A dz^B$ with $g_{AB}=g_{BA}$. Reality is ensured by the conditions $g_{\alpha\beta}=\bar{g}_{\bar{\alpha}\bar{\beta}}$ and $g_{\alpha\bar{\beta}}=\bar{g}_{\beta\bar{\alpha}}$. So far we have made no restriction but have simply chosen to describe an even-dimensional Riemannian manifold using complex coordinates.

Now we make the restriction that the manifold is Hermitian. This means that there is a preferred class of coordinate systems in which unmixed components of the metric tensor vanish ($g_{\alpha\beta}=g_{\bar{\alpha}\bar{\beta}}=0$) leaving the line element in the Hermitian form $ds^2=2g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta$.

Finally we make the restriction to Kahler manifolds by requiring that the Hermitian metric satisfy (3). There are then simplifications in the standard formulas for connections and curvature, and many components vanish. For example, all components of the connection Γ^A_{BC} vanish except those of the form $\Gamma^\alpha_{\beta\gamma}=\overline{\Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}}$. The nonvanishing components of the curvature tensor are of mixed form $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ together with components obtained using the usual symmetries. There is also a new symmetry, $R_{\alpha\bar{\beta}\gamma\bar{\delta}}=R_{\gamma\bar{\beta}\alpha\bar{\delta}}$, following from the usual cyclicity property.

We shall need the concept of a "Kahler tensor" which is a second-rank tensor T_{AB} with vanishing unmixed components and mixed components $T_{\alpha\bar{\beta}}$ satisfying the covariant condition $\partial_\gamma T_{\alpha\bar{\beta}}=\partial_\alpha T_{\gamma\bar{\beta}}$. A Kahler tensor defines a closed two-form $T=T_{\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta$, and is locally derivable from a scalar potential, i.e., $T_{\alpha\bar{\beta}}=\partial_\alpha\partial_{\bar{\beta}}S(z,\bar{z})$. The Ricci tensor $R_{\alpha\bar{\beta}}=g^{\delta\bar{\gamma}}R_{\alpha\bar{\gamma}\beta\bar{\delta}}$ is a Kahler tensor; it satisfies $R_{\alpha\beta}=0=R_{\bar{\alpha}\bar{\beta}}$ and $R_{\alpha\bar{\beta}}=\partial_\alpha\partial_{\bar{\beta}}\ln\det(g_{\gamma\bar{\delta}})$.

GENERAL RENORMALIZABILITY CONSIDERATIONS

We consider the bosonic σ model with classical action (1). Power counting implies that the counter terms which correspond to ultraviolet divergences of one-particle-irreducible (1PI) Green's functions are operators of dimension two. There are two classes of counterterms. The class of major interest in this paper are the invariant counterterms which represent the "on-shell" divergences of the theory and take the form

$$I_{\rm ct}=\tfrac{1}{2}\int d^2x\,\partial_\mu\phi^i\partial_\mu\phi^j T_{ij}(\phi)\,, \tag{4}$$

where T_{ij} is a symmetric second-rank tensor algebraically constructed from the curvature tensor and covariant derivatives of the curvature tensor. For example, $T_{ij}(\phi)$ might involve R_{ij}, $R_{iklm}R_j^{\ klm}$, $D_iR_{klmn}D_jR^{klmn}$, or many other possibilities. In addition there are noncovariant counterterms which vanish if the classical field equations

$$\frac{\delta I}{\delta\phi^i}=-(\Box\phi^i+\Gamma^i{}_{jk}\partial_\mu\phi^j\partial_\mu\phi^k)=0 \tag{5}$$

are used. Except at one-loop order we will not discuss such terms which are compensated by field redefinitions and are of secondary interest.

In a general Riemannian metric infinitely many of the counterterm tensors T_{ij} will not be of the same functional form as the metric $g_{ij}(\phi)$. Therefore strict renormalizability does not hold and one cannot usually give meaning to a nonlinear σ model on a given manifold M with metric $g_{ij}(\phi)$. There seem to be the following possibilities for defining quantum field theories:

(a) Exceptional cases where at most finitely many counterterms differ in form from $g_{ij}(\phi)$. Such an exception occurs for spaces of constant curvature where the curvature tensor takes the form $R_{ijkl}=c(g_{ik}g_{jl}-g_{il}g_{jk})$ and covariant derivatives vanish. All tensors T_{ij} algebraically constructed from the curvature must be proportional to $g_{ij}(\phi)$. Such is the case for the O(n) models which are well known[12,13] to be renormalizable in the traditional sense.

(b) Cases where all but a finite number of counterterms vanish. This case is unlikely for bosonic σ models, but in this paper we show that it is very likely for a broad class of supersymmetric σ models, due to cancellations of boson and fermion divergences required by supersymmetry.

(c) Renormalization-group interpretation. Using dimensional regularization and minimal subtraction, Friedan[6] has derived renormalization-group equations of the form

$$\mu\frac{\partial}{\partial\mu}g_{ij}=-\beta_{ij}(g)\,. \tag{6}$$

Through two-loop order explicit calculations give

$$\beta_{ij}\left(\frac{1}{\hbar}g\right)=R_{ij}+\tfrac{1}{2}\hbar R_{iklm}R_j{}^{klm}+O(\hbar^2) \tag{7}$$

for bosonic σ models. In these equations μ is the renormalization scale parameter and $\hbar$ is Planck's constant. Integration of these equations yields trajectories in the space of all Riemannian metrics, so that the initial geometry changes under renormalization.

GENERAL STRATEGY

Our approach to the ultraviolet structure of supersymmetric σ models is to combine the universality of the background-field method[5] of calculation of invariant counterterms with the theorem[2] connecting Kahler manifolds and $N=2$ models. In the background-field method applied to a bosonic σ model the field $\phi^i(x)$ is split, $\phi^i(x)=\phi^i_{\rm cl}(x)+\xi^i(x)$, into a term satisfying the classical field equations (5) and a remainder $\xi^i(x)$. The remainder does not transform simply under reparametrizations of the manifold, but using Riemannian normal coordinates it can be expressed[5,6] in terms of the quantum field $\zeta^i(x)$ which transforms as a contravariant vector. After expansion of the action (1) as a functional Taylor series in $\zeta^i(x)$, the calculation of a relatively small number of Feynman graphs will give the counterterms up to any given order in perturbation theory in the form

$$I_{\rm ct}=\tfrac{1}{2}\int d^2x\,\partial_\mu\phi^i\partial_\mu\phi^j(a_1R_{ij}+a_2R_{iklm}R_j{}^{klm}+a_3R_{ik}R_j{}^k+\cdots)\,, \tag{8}$$

where the coefficients a_i are each a series of poles in the dimensional-regularization parameter $\epsilon=d-2$.

The universality of this method is most important; to wit, the calculations can be done for the most general Riemannian manifold M with metric $g_{ij}(\phi)$ unspecified, so that the coefficients $a_1, a_2, a_3, \ldots$ are universal and correct for all geometries. In particular results for a general Riemannian manifold apply immediately to σ models on general Kahler metrics $g_{\alpha\bar\beta}(z,\bar z)$ provided only that we incorporate the algebraic and differential restrictions implied by the fact that we are regarding the Riemannian case as the $2n$-dimensional real description of an n-dimensional Kahler metric. For example, the counterterm (8) would become

$$I_{\rm ct}=\int d^2x\,\partial_\mu z^\alpha\partial_\mu{}^\beta(a_1R_{\alpha\bar\beta}+2a_2R_{\alpha\bar\gamma\delta\bar\epsilon}R_{\bar\beta}{}^{\bar\gamma\delta\bar\epsilon}+a_3R_{\alpha\bar\gamma}R^{\bar\gamma}{}_{\bar\beta}+\cdots) \tag{9}$$

with the *same* coefficients $a_1, a_2, a_3, \ldots$ as in (8).

A further implication of the geometric structure of (1) [and of (10) below] is that the tensors T_{ij} which can appear as counterterms at l-loop order in perturbation theory can be characterized as contractions of products of the curvature which scale as $T_{ij}\to\Lambda^{l-1}T_{ij}$ under the constant scaling of the metric $g_{ij}\to\Lambda^{-1}g_{ij}$. The reason for this is that the l-loop counterterm has $\hbar^{l-1}$ as a factor, and a change in $\hbar$ is equivalent to the scaling of the metric above. Thus the tensor R_{ij} can only appear as a one-loop counterterm, while the tensors quadratic in the curvature in (8) are possible only at two-loop order.

We now turn to supersymmetric models with classical action

$$I_s=\tfrac{1}{2}\int d^2x\,[g_{ij}(\phi)\partial_\mu\phi^i\partial_\mu\phi^j+\tfrac{1}{2}ig_{ij}(\phi)\bar\psi^i\gamma^\mu(D_\mu\psi)^j-\tfrac{1}{12}R_{ijkl}(\bar\psi^i\psi^k)(\bar\psi^j\psi^l)] \tag{10}$$

which is the supersymmetric extension[1] of (1) for a general Riemannian manifold. The covariant derivative is $(D_\mu\psi)^j=\partial_\mu\psi^j+\Gamma^j{}_{kl}\partial_\mu\phi^k\psi^l$, and R_{ijkl} is the curvature tensor of the manifold.

The background-field method requires splitting of both $\phi^i(x)$ and $\psi^i(x)$. However, our general argument requires only implicit knowledge of the bosonic counterterms of the form (4) because supersymmetry requires that these are accompanied by fermion terms so that the whole structure is a generalization of (10) with g_{ij} replaced by T_{ij}. Strictly speaking this is true only before auxiliary fields are eliminated.

Let us suppose that the background-field method applied to the supersymmetric σ model on a general Riemannian manifold M gave the bosonic invariant counterterm

$$I_{\rm ct}=\tfrac{1}{2}\int d^2x\,\partial_\mu\phi^i\partial_\mu\phi^j(b_1R_{ij}+b_2g_{ij}R+b_3R_{iklm}R_j{}^{klm}+\cdots)\,. \tag{11}$$

Universality then implies that upon restriction to a general Kahler manifold, one would obtain

$$I_{\rm ct}=\int d^2x\,\partial_\mu z^\alpha\partial_\mu\bar z^\beta(b_1R_{\alpha\bar\beta}+b_2g_{\alpha\bar\beta}R+2b_3R_{\alpha\bar\gamma\delta\bar\epsilon}R_{\bar\beta}{}^{\bar\gamma\delta\bar\epsilon}+\cdots) \tag{12}$$

with the same coefficients $b_1, b_2, b_3, \ldots$. Since the classical action would have $N=2$ supersymmetry, the bosonic counterterm must be accompanied by fermion terms with overall $N=2$ supersymmetry. However, by Zumino's theorem[2] this can happen only for tensors $T_{\alpha\bar\beta}$ which are Kahler tensors. Any linear combination of the possible tensor counterterms for a given order in perturbation theory which does not have vanishing curl cannot occur, and its coefficient must vanish universally both for general Kahler and Riemannian manifold models. For example, the tensors $R_{\alpha\bar{\beta}}$ and $g_{\alpha\bar{\beta}}R$ are the only possible invariant one-loop counterterms. The first has vanishing curl but the second does not for general Kahler manifolds. Therefore the coefficient b_2 must vanish for all supersymmetric σ models.

Thus our general strategy is to show that the Kahler condition is sufficiently restrictive that many possible counterterms allowed by Riemannian geometry cannot actually be present in supersymmetric models. However, there are several qualifications. First, some tensors T_{ij} acquire unmixed components upon specialization to the Kahler geometry. As discussed below some of these can be compensated by field redefinitions, but others cannot, causing complications in the general arguments at three-loop order and beyond. Second, the interplay between Riemannian and Kahler manifolds implied in our strategy surely fails for Riemannian manifolds of odd dimension. However, the universality of the background-field method is so strong that the coefficient of any particular tensor counterterm is actually independent of the dimension of the manifold. Dimension dependence could only arise from contractions $g^{ij}g_{ij}=n$ and these cannot occur as can be seen by cursory examination of the action obtained by the general background-field expansion.[10]

Finally our strategy requires a regularization procedure which preserves $N=1$ supersymmetry of Riemannian models and automatically gives $N=2$ supersymmetry when restriction to a Kahler metric is made. Dimensional regularization can be made consistent with $N=1$ supersymmetry,[14,15] and $N=2$ supersymmetry is simply a matter of an additional SO(2) internal symmetry which is also preserved. Note that in general an $N=2$ supersymmetric theory does not necessarily have SO(2) internal symmetry, but that this symmetry is present here because all Kahler-metric supersymmetric σ models on two-dimensional space-time can be obtained by reduction from four space-time dimensions.[2] It is part of our assumptions that supersymmetric dimensional regularization can be performed.

ONE-LOOP ORDER

The one-loop counterterm of the general bosonic σ model is obtained from the background-field expansion through second order in quantum fields $\xi^i(x)$. The divergent graphs of the expanded Lagrangian yield the one-loop counterterm for (1) and are easily calculated. The result is

$$\Delta I^{(1)}=\frac{1}{4\pi\epsilon}\int d^2x\left[R_{ij}(\phi)\partial_\mu\phi^i\partial_\mu\phi^j+g^{jk}\Gamma^i{}_{jk}\frac{\delta I}{\delta\phi^i}\right]. \tag{13}$$

The invariant term has been known for some time[5] [and is obviously related to (7)]. The noncovariant additional term was found[10] using a slightly different method[16] so that $\Delta I^{(1)}$ actually gives all divergences, both on-shell and off-shell, of 1PI Green's functions. Because it is proportional to the classical field equation the noncovariant term represents a field renormalization.

For a bosonic Kahler manifold the one-loop counterterm obtained immediately from (13) is

$$\Delta I^{(1)}=\frac{1}{2\pi\epsilon}\int d^2x\,R_{\alpha\bar{\beta}}(z,\bar{z})\partial_\mu z^\alpha\partial_\mu\bar{z}^\beta. \tag{14}$$

The noncovariant term cancels because $g^{\alpha\beta}=0$ and $\Gamma^A{}_{\alpha\bar{\beta}}=0$ in any Kahler manifold. This curious result is confirmed by calculations in the O(n) model[13] where the field renormalization [when defined appropriately, see Eq. (A12) of Ref. 13] vanishes for $n=3$ because the O(3) and CP^1 models are equivalent and CP^1 is a Kahler manifold. Actually because of the noncovariance of $\Gamma^i{}_{jk}$ the vanishing of the O(3) model field renormalization is required only for real coordinates adapted to the Kahler structure[17] such as the conformal coordinates of Ref. 15. The invariant term in (13) also agrees with O(n) model calculations.[12,13]

We note also that the invariant counterterm vanishes for any Ricci-flat geometry, whether Riemannian or Kahler.

We now come to the supersymmetric model (10). This action must also be expanded to second order in quantum fields. We are only interested in the effect of fermions on the bosonic counterterms. Thus there are no fermionic background fields, and the $\psi^i(x)$ in (5) can be viewed as quantum fields. Only the quadratic term in (5) is relevant for one-loop effects. The fermion contribution is actually finite which can be seen by referring the Fermi fields to tangent frames on the manifold in which case the fermion Lagrangian becomes

$$\mathcal{L}=\tfrac{1}{2}i\bar{\psi}^a\gamma^\mu[\partial_\mu\psi^a+\omega_i^{ab}(\phi)\partial_\mu\phi^i\psi^b], \tag{15}$$

where $\omega_i^{ab}(\phi)$ is a spin connection on the manifold and the product $\omega_i^{ab}(\phi)\partial_\mu\phi^i$ transforms like a gauge potential under rotations of the tangent frame. Thus one-loop fermion effects are equivalent to those of a non-Abelian gauge field theory in two dimensions which are known to be ultraviolet finite.

Therefore to one-loop order the invariant counterterm of a Bose σ model involves the Ricci tensor of the underlying manifold and is unchanged by the supersymmetric addition of fermions. A noncovariant field renormalization vanishes for Kahler manifolds.

TWO-LOOP ORDER

For a bosonic σ model the two-loop ultraviolet counterterm cannot vanish. This is obvious from the trace $g^{ij}\beta_{ij}$ of the generalized β function (7). For supersymmetric models the situation improves dramatically.

The candidate two-loop invariant bosonic counterterm involves tensors T_{ij} of weight one under the scaling $g_{ij} \to \Lambda^{-1} g_{ij}$. After taking into account restrictions due to the Bianchi and Ricci identities one finds that the most general possible counterterm is

$$I_{\rm ct}^{(2)} = \tfrac{1}{2}\hbar \int d^2x\, \partial_\mu \phi^i \partial_\mu \phi^j (b_1 R_{iklm} R_j{}^{klm} + b_2 R_{ij} R + b_3 g_{ij} R^2 + b_4 R_{iklj} R^{kl} + b_5 R_{ik} R_j{}^k + b_6 D^k D_k R_{ij} + b_7 D_i D_j R) \quad (16)$$

which involves seven independent tensors. The same answer can also be obtained by inspection of the background-field expansion of the action (10) through fourth order in quantum Bose fields $\xi^i(x)$ and fermion fields $\psi^i(x)$ [and the expansion of the supersymmetric extension of the one-loop counterterm (13) through second order in quantum fields].

We establish a lemma below which shows that the tensor $D_i D_j R$ is compensated by a field redefinition and is not an on-shell divergence of the theory. Upon restriction to a general Kahler manifold, the first six tensors all have vanishing unmixed components. The remaining task is to examine whether any linear combinations of the six are Kahler tensors. The detailed calculation is presented in the Appendix. The result is that one linear combination has vanishing curl and is an allowed counterterm. This is the tensor

$$\begin{aligned} T_{ij} &= D^k D_k R_{ij} + 2R_{ikjl} R^k + 2R_{ik} R^k{}_j \\ &= D^k D_k R_{ij} + 2[D_i, D_k] R^k{}_j . \end{aligned} \quad (17)$$

Thus the two-loop ultraviolet counterterm in a supersymmetric σ model must be a universal multiple of this tensor. Since T_{ij} vanishes for Ricci-flat and locally symmetric manifolds we conclude that these two classes of σ models are two-loop ultraviolet finite. It is curious to note that T_{ij} coincides with the generalized Laplacian of Lichnerowicz[18] applied to second-rank symmetric tensors.

EXPLICIT TWO-LOOP CALCULATIONS

Several calculations of two-loop diagrams have been performed to check the results of the general argument of the previous section. There is a simple form of the background-field method called the moving frame method[19] which is applicable to the bosonic O(n) models[20] and which can be extended to include supersymmetrically coupled fermions. For the bosonic case one works with the Lagrangian and constraint

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi^i)^2 , \qquad (\phi^i)^2 = 1 . \quad (18)$$

Given a classical solution $\phi^i_{\rm c1}(x)$ one defines a set of $n-1$ unit vectors $e^i_a(x)$ which are mutually orthogonal and orthogonal to $\phi_{\rm c1}(x)$. An arbitrary field is then referred to this frame, i.e., we expand $\phi(x) = u_0 \phi_{\rm c1}(x) + u_a e_a(x)$. The constraint implies $u_0 = (1 - u_a u_a)^{1/2}$, which can be expanded to provide a simple expansion in quantum fields $u_a(x)$. The calculation further simplifies due to considerations of SO($n-1$) gauge invariance and power counting. By this method we have reproduced the results of Ref. 12 for the two-loop coupling-constant renormalization of the bosonic model and have then shown that fermion effects cause the two-loop divergence to cancel.

We have also completed the two-loop background-field calculation of ultraviolet divergences for σ models on general Riemannian manifolds. We first make the simplification to a Ricci-flat metric. Here we reproduce the result of Ref. 6 for the bosonic case and then show that fermions cancel the divergence found there. For a general supersymmetric σ model we find an additional divergence with tensor form given by (17) and a coefficient which is a second-order pole in ϵ. In a standard renormalizable field theory two-loop double poles in counterterms are determined from one-loop single poles by renormalization-group pole equations.[21] One would also expect such pole equations in the generalized renormalizable framework[6] for nonlinear σ models. Indeed a simplified analysis suggests that the tensor T_{ij} is the exact two-loop divergence predicted from the one-loop counterterm (13) by the generalized pole equations. We hope to discuss this situation further in Ref. 10.

A LEMMA

We now show that any invariant counterterm of the form

$$I_{\rm ct} = \frac{1}{2\epsilon} \int d^2x\, \partial_\mu \phi^i \partial_\mu \phi^j D_i V_j(\phi) , \quad (19)$$

where $V_j(\phi)$ is a vector field on the manifold can be canceled by a field redefinition of the form $\phi'^i(x) = \phi^i(x) - (1/2\epsilon) V^i(\phi(x))$. The infinitesimal effect of this redefinition on the initial bosonic action (1) is simply

$$\delta I = -\frac{1}{2\epsilon}\int d^2x\,(\partial_\mu V^i \partial_\mu \phi^j g_{ij} + \tfrac{1}{2}\partial_\mu \phi^i \partial_\mu \phi^j V^k \partial_k g_{ij})$$

$$= -\frac{1}{2\epsilon}\int d^2x\,\partial_\mu \phi^i \partial_\mu \phi^j D_i V_j \qquad (20)$$

which cancels (19). We have not studied the effect of this field redefinition on fermionic terms of the supersymmetric action. However, it seems clear from the superspace formulation[1] that a fermionic field redefinition of the form $\psi'^i(x) = \psi^i(x) - (1/2\epsilon) D_j V^i(\phi(x))\psi^j(x)$ will also be required. Terms quadratic in V and effects of the field redefinition on the one-loop counterterm (13) must also be considered. It appears that these induce higher-order poles in ϵ which must cancel consistently due to the renormalizable structure. This lemma is easily interpreted in the language of the renormalization group. We are interested[6] only in changes of the metric $g_{ij}(\phi)$ under renormalization which are actually changes of the geometry and therefore must exclude diffeomorphisms of $g_{ij}(\phi)$. A field redefinition is a poor man's diffeomorphism.

Note that this lemma applies to any counterterm of the form $D_i D_j S$ where S is a scalar. When restricted to Kahler manifolds such tensors have both mixed components $\partial_\alpha \partial_{\bar\beta} S$ and unmixed components $D_\alpha \partial_\beta S$. A Kahler tensor has mixed components which can be represented as $\partial_\alpha \partial_{\bar\beta} S$ but unmixed components vanish. Such a tensor cannot be canceled by a field redefinition.

HIGHER-LOOP ORDER

Let us consider tensors T_{ij} which are algebraically constructed from products of curvature tensors with no derivatives. On a Kahler manifold the curvature tensor has an equal number of barred and unbarred indices, so that tensors T_{AB} constructed from them have vanishing unmixed components. Tensors with nonvanishing unmixed components can appear when derivatives of curvature are included and in general they cannot be compensated by field redefinitions. For example, the tensor $D_i R_{klmn} D_j R^{klmn}$ has this property, and is a candidate three-loop counterterm for nonlinear σ models. At present such tensors cannot be excluded by $N=2$ supersymmetry since Zumino's theorem requires a Hermitian metric by hypothesis. However, such tensors could be excluded by a stronger form of the theorem to the effect that $N=2$ supersymmetry is not possible in a manifold which is genuinely non-Hermitian, i.e., not diffeomorphic to a Kahler manifold. We think that this extension is plausible and hope to pursue it elsewhere. Independently of this, however, the tensors in question vanish for locally symmetric spaces and it may be possible to establish higher-loop ultraviolet finiteness for this class of supersymmetric σ model without confronting the problem of non-Hermiticity.

The key problem is still the question of Kahler tensors, that is, tensors of the form

$$T_{AB} = (R \cdots DR \cdots R \cdots)_{AB} \qquad (21)$$

with all indices but two contracted, and with $T_{\alpha\gamma} = 0$ and $\partial_\gamma T_{\alpha\bar\beta} = \partial_\alpha T_{\gamma\bar\beta}$. The discussion at the two-loop level suggests that the vanishing curl condition is very restrictive but does not eliminate all possible tensors. At least those required by the renormalization-group pole equations are allowed. It is also clear that the proof by exhaustion which eliminated all but one two-loop tensor must be replaced by a more general argument. Toward this end it appears useful to discuss the Kahler tensor question in terms of the potential S for which $T_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} S$. If S is not a true scalar or only locally defined on the manifold, as in the case of the Ricci tensor $R_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} \ln\det g_{\gamma\bar\delta}$ or the metric $g_{\alpha\bar\beta} = \partial_\alpha \partial_\beta F$, where F is the Kahler potential, the tensor $T_{\alpha\bar\beta}$ defines a 2-form which is closed but not exact. It may be possible to use cohomology theory to show that the Ricci and metric tensors are the only closed inexact 2-forms algebraically constructed from curvature on a general Kahler manifold. If so, one can restrict one's attention to tensors for which S is a true scalar constructed from curvature as in the case of the two-loop tensor T_{ij} for which $T_{\alpha\gamma} = 0$ and $T_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} R$. All such tensors vanish for locally symmetric spaces. One might also hope that further geometrical argument can establish that the only such tensors which appear are those required by the renormalization-group pole equations. In this case the tensors will vanish on Ricci-flat manifolds.

The discussion of this section is speculative, but we would summarize it by saying that it appears probable that higher-loop counterterms can be excluded by general argument for supersymmetric σ models on locally symmetric manifolds and possibly excluded for Ricci-flat manifolds also.

CONCLUSIONS

We have demonstrated that there is a powerful connection between the ultraviolet properties of supersymmetric σ models and Kahler geometry. This connection is not exhausted by present results but can probably be pushed further to derive stronger results on higher-loop finiteness. In four-dimensional field theories it is well known that there is improvement of the ultraviolet divergence structure in global supersymmetry and supergravity. What is discussed here is the analog in two-dimensional nonlinear theories. It is surprising that there is no improvement due to supersymmetry in one-loop order, but remarkable consequences in higher order.

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APPENDIX

In this appendix we will show in detail that the tensor T_{ij} of (17) is the only linear combination of the first six tensors in the two-loop counterterm (16) which is a Kahler tensor. We start with the manifest Kahler tensor

$$\begin{aligned} T_{\alpha\bar{\beta}} &= \partial_\alpha \partial_{\bar{\beta}} R \\ &= 2\partial_\alpha \partial_{\bar{\beta}} g^{\gamma\bar{\delta}} \partial_\gamma \partial_{\bar{\delta}} \ln \det g\,. \end{aligned} \tag{A1}$$

We move the ∂_α and $\partial_{\bar{\beta}}$ derivatives to the right using the relations

$$\begin{aligned} &R_{\bar{\alpha}\beta\gamma}{}^{\delta} = \partial_{\bar{\alpha}} \Gamma^{\delta}{}_{\beta\gamma}, \quad \Gamma^{\alpha}{}_{\beta\gamma} = g^{\alpha\bar{\delta}} \partial_\beta g_{\gamma\bar{\beta}}\,, \\ &\partial_\beta g^{\gamma\bar{\delta}} = -g^{\gamma\bar{\epsilon}} \Gamma^{\bar{\delta}}{}_{\bar{\beta}\bar{\epsilon}}, \quad \partial_\alpha g^{\gamma\bar{\delta}} = -g^{\epsilon\bar{\delta}} \Gamma_{\alpha\epsilon} \end{aligned} \tag{A2}$$

which are correct for general Kahler manifolds.[11] After using $R_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \ln \det g$, we find

$$T_{\alpha\bar{\beta}} = D^A D_A R_{\alpha\bar{\beta}} + 2R_{\alpha\bar{\gamma}\delta\bar{\beta}} R^{\bar{\gamma}\delta} + 2R_{\alpha\bar{\gamma}} R^{\bar{\gamma}}{}_{\bar{\beta}} \tag{A3}$$

which is the restriction of (17) to a Kahler manifold.

To show that T_{ij} of (17) is the only curl-free linear combination, we study the five independent tensors

$$\begin{aligned} T^{(1)}{}_{\alpha\bar{\beta}} &= R_{\alpha\bar{\gamma}\delta\bar{\eta}} R_{\bar{\beta}}{}^{\bar{\gamma}\delta\bar{\eta}}\,, \\ T^{(2)}{}_{\alpha\bar{\beta}} &= R_{\alpha\bar{\beta}} R\,, \\ T^{(3)}{}_{\alpha\bar{\beta}} &= g_{\alpha\bar{\beta}} R^2\,, \\ T^{(4)}{}_{\alpha\bar{\beta}} &= R_{\alpha\bar{\gamma}\delta\bar{\beta}} R^{\bar{\gamma}\delta}\,, \\ T^{(5)}{}_{\alpha\bar{\beta}} &= R_{\alpha\bar{\gamma}} R^{\bar{\gamma}}{}_{\bar{\beta}} \end{aligned} \tag{A4}$$

obtained by restriction of the tensors of (16) to a Kahler manifold. Since only $T^{(1)}$ contributes for Ricci-flat Kahler manifolds ($R_{\alpha\bar{\beta}} = 0$) we can study this tensor alone, independent of linear combinations with others. Taking the trace of the curl and using the Bianchi identity, we find

$$g^{\epsilon\bar{\beta}}[\partial_\epsilon T^{(1)}{}_{\alpha\bar{\beta}} - \partial_\alpha T^{(1)}{}_{\epsilon\bar{\beta}}] = R_{\alpha\bar{\gamma}\delta\bar{\eta}} D^{\bar{\gamma}} R^{\delta\bar{\eta}} - \tfrac{1}{2}\partial_\alpha (R^{\epsilon\bar{\gamma}\delta\bar{\beta}} R_{\epsilon\bar{\gamma}\delta\bar{\beta}})\,. \tag{A5}$$

The second term is nonvanishing for general Ricci-flat Kahler manifolds, so that $T^{(1)}$ is not a Kahler tensor.

The most straightforward way to show that there are no curl-free linear combinations among the remaining four tensors in (A4) is to compute these tensors for one explicit Kahler metric. With bad luck one might choose a particular metric where these are allowed linear combinations but this does not turn out to be the case. A suitable metric is defined by the Kahler potential $F(u,\bar{u}) = e^{u\cdot\bar{u}}$ with $u\cdot\bar{u} = \sum_\alpha u^\alpha \bar{u}^\alpha$, $\alpha,\beta = 1,\ldots,N$ and $u_\alpha = \delta_{\alpha\beta} u^\beta$, $\bar{u}_\beta = \delta_{\beta\gamma}\bar{u}^\gamma$. For this manifold

$$\begin{aligned} &g_{\alpha\bar{\beta}} = e^{u\cdot\bar{u}}(\delta_{\alpha\beta} + \bar{u}_\alpha u_\beta)\,, \\ &g^{\alpha\bar{\beta}} = e^{-u\cdot\bar{u}}\left(\delta^{\alpha\beta} - \frac{u^\alpha \bar{u}^\beta}{1+u\cdot\bar{u}}\right), \\ &R_{\alpha\bar{\beta}} = \left(N + \frac{1}{1+u\cdot\bar{u}}\right)\delta_{\alpha\beta} - \frac{\bar{u}_\alpha u_\beta}{(1+u\cdot\bar{u})^2}\,, \\ &\Gamma^{\alpha}{}_{\beta\gamma} = \delta^{\alpha}{}_{\beta}\bar{u}_\gamma + \delta^{\alpha}{}_{\gamma}\bar{u}_\beta - \frac{u^\alpha \bar{u}_\beta \bar{u}_\gamma}{1+u\cdot\bar{u}}\,, \\ &R^{\alpha}{}_{\beta\gamma\bar{\delta}} = \delta^{\alpha}{}_{\beta}\delta_{\gamma\delta} + \delta^{\alpha}{}_{\gamma}\delta_{\beta\delta} \\ &\qquad - \frac{u^\alpha}{1+u\cdot\bar{u}}(\delta_{\beta\delta}\bar{u}_\gamma + \delta_{\gamma\delta}\bar{u}_\beta) + \frac{u^\alpha \bar{u}_\beta \bar{u}_\gamma u_\delta}{(1+u\cdot\bar{u})^2}\,. \end{aligned} \tag{A6}$$

Each tensor of (A4) has the generic form

$$T^{(i)}{}_{\alpha\bar{\beta}} = \delta_{\alpha\beta} f^{(i)}(x) - \bar{u}_\alpha u_\beta g^{(i)}(x)\,, \tag{A7}$$

where $x = u\cdot\bar{u}$, and curls take the form

$$\begin{aligned} &\partial_\gamma T^{(i)}{}_{\alpha\bar{\beta}} - \partial_\alpha T^{(i)}{}_{\gamma\bar{\beta}} = (\delta_{\alpha\beta}\bar{u}_\gamma - \delta_{\beta\gamma}\bar{u}_\alpha) C^{(i)}(x)\,, \\ &C^{(i)}(x) = \frac{d}{dx} f^{(i)}(x) + g^{(i)}(x)\,. \end{aligned} \tag{A8}$$

Using (A6) we explicitly compute the functions $f^{(i)}(x)$ and $g^{(i)}(x)$ for $i = 2, 3, 4, 5$ and then compute $C^{(i)}(x)$ which appears in the curl. We find that each $C^{(i)}(x)$ can be written as $C^{(i)}(x) = e^{-x}P^{(i)}(y)$ where the $P^{(i)}(y)$ are polynomials in the variable $y = (1+x)^{-1}$, three polynomials being of 5th order and one of 7th order. One readily tests that these polynomials are linearly independent which shows that no linear combination of $T^{(2)}$, $T^{(3)}$, $T^{(4)}$, or $T^{(5)}$ is a Kahler tensor.

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[17]If an even-dimensional Riemannian manifold is really a Kahler manifold in disguise, then there is a preferred class of real coordinate charts $\phi^1, \ldots, \phi^n, \phi^{n+1}, \ldots, \phi^{2n}$, such that when complex coordinates are defined by $z^\alpha = \phi^\alpha + i\phi^{n+\alpha}$ and $\bar{z}^\alpha = \phi^\alpha - i\phi^{n+\alpha}$ for $\alpha = 1, \ldots, n$, the transformed metric g_{AB} is Hermitian and satisfies (3). We say that such coordinate charts are adapted to the Kahler structure.

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