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Suppression of ultraviolet infinities in gravity-modified field theories

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Ultraviolet infinities in quantum field theories are regularized by including the gravitational interaction and applying nonpolynomial Lagrangian techniques to it in a certain approximation which greatly simplifies calculation and gives the same expressions for the regularized amplitudes as the full tensor gravity in the lowest order in the gravitational coupling constant. The approximation is better than scalar gravity and works for all field theories, including exact and broken Yang-Mills theories and other renormalizable and nonrenormalizable field theories. Special consideration is given to regularization of theories with spontaneously broken symmetries (taking the σ model as an example) and of Yang-Mills fields. Renormalization constants for the general Yang-Mills field and for quantum chromodynamics are explicitly calculated in the lowest order in the gauge coupling constant. Gauge invariance (both Yang-Mills and gravitational) are shown to be preserved.

I. INTRODUCTION

It is by now well known that nonpolynomial Lagrangians have remarkable convergence properties[1] and that, in particular, the nonpolynomial character of the gravitational interaction can be exploited, employing superpropagator methods,[1] to regularize ultraviolet infinities in quantum electrodynamics[2,3] and other field theories; the inverse of the gravitational coupling constant κ provides a natural ultraviolet cutoff. This regularization procedure has at least two advantages over others[4]: Firstly, it is not *ad hoc* but quite natural in that it recognizes the regularizing role of the gravitational interaction which is always there but is usually ignored due to its small strength; secondly, it is universal because gravitation couples to all fields—in fact, it is not difficult to see that its regularizing effect is sufficiently strong to take care of the ultraviolet infinities of even the so-called unrenormalizable field theories.

The complications of tensor gravity, however, make it very difficult to put this scheme of regularization in practice. As a remedy, it was proposed in Ref. 5 to employ exponential couplings of a massless scalar field σ so as to make a Poincaré-invariant Lagrangian conformal invariant and, after appropriate field transformations, exploit the exponential couplings to obtain infinity suppression. This method has, unfortunately, a drawback in that, if one insists, as was done in Ref. 5, on having canonical scale dimensions for various fields, the method does not work for the important case of Yang-Mills fields[6] because the Yang-Mills Lagrangian is conformal invariant[7] by itself. Even for spontaneously broken gauge theories,[8] where the exponential σ couplings can be introduced in the mass terms for the Higgs fields and taken to other terms by field transformations, it is generally not possible to regularize all ultraviolet infinities. One approach to remove this drawback could be, for example, to employ noncanonical scale dimensions for various fields; this, however, does not appear to be a very attractive proposal.

In the present work we explore in some detail an approximation[9] to the usual tensor gravity couplings which has all the simplicity of scalar gravity and works for all field theories. This approximation consists in retaining the $\sqrt{-g}$ factor in the action integral with appropriate exponential parametrization, replacing the metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ elsewhere in the couplings with matter fields by the Minkowski metrics $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$, respectively, and ignoring gravitational self-interactions. This approximation greatly simplifies calculations while it retains all the effectiveness of the regularizing role of full tensor gravity.

In the next section some general features of amplitudes regularized according to the above-mentioned scheme are discussed and illustrated with examples: *viz.*, the $\lambda\Phi^N$ interaction (which is unrenormalizable for $N>4$) and quantum electrodynamics. It is noted that, when a class of gauges for the graviton propagator (containing the Fock–de Donder gauge as a special case) characterized by a parameter ξ is employed, the ξ dependence goes into the renormalization constants and the "physical parts" of amplitudes remain ξ independent. Section III is devoted to the discussion of regularization of theories with spontaneously broken symmetries with the σ model as an example. The main new feature here is the regularization of bubble diagrams. In Sec. IV we discuss regularization of non-Abelian gauge theories. Renormalization constants in lowest nontrivial order are calculated. Due consideration is given to problems associated with gauge invariance. The last section contains some concluding remarks.

II. REGULARIZATION SCHEME WITH (APPROXIMATED) GRAVITY

In this section we shall introduce the approximation to the full gravitational interaction mentioned in the Introduction, discuss important features of amplitudes calculated by applying non-polynomial Lagrangian techniques to this approximated gravitational interaction, and illustrate these with simple examples.

A. Gravity-modified field theories

The fundamental objects in terms of which the gravitational interaction is to be described are the vierbein fields[10] L_μ^a which relate the Riemannian metric tensor $g_{\mu\nu}$ to the Minkowski metric η_{ab} through the relation

$$g_{\mu\nu} = L_\mu^a L_\nu^b \eta_{ab} = L_{\mu a} L_{\nu b} \eta^{ab}, \tag{2.1}$$

where $L_{\mu a} = \eta_{ab} L_\mu^b$. Both the general tensor indices $\mu, \nu, \ldots$ and the Lorentz indices $a, b, \ldots$ take values 0, 1, 2, 3 and the summation convention has been employed. It is convenient to assume that the vierbein field is symmetric:

$$L_{\mu a} = L_{a\mu}; \tag{2.2}$$

the relation (2.1) then fixes $L_{\mu a}$ uniquely for a given $g_{\mu\nu}$. The inverse $L^{\mu a} = L^{a\mu}$ of the vierbein field matrix $L_{\mu a}$ satisfies the relations

$$L^{\mu a} L_{\nu a} = \delta_\nu^\mu, \quad L^{\mu a} L_{\mu b} = \delta_b^a. \tag{2.3}$$

As is well known,[10] when couplings with the gravitational field are introduced in a Poincaré-invariant Lagrangian, the following modifications are to be made:

(i) Ordinary derivatives of fields are to be replaced by "covariant derivatives" employing the coefficients of affine connection.

(ii) The Dirac bilinears, which are tensors with respect to the Lorentz group, are to be converted into tensors with respect to the general coordinate transformations with the help of the vierbein fields.

(iii) All contractions of tensor indices are to be done with the Riemannian metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ instead of the Minkowski metrics η_{ab} and η^{ab}.

(iv) Each term in the Lagrangian is to be multiplied with an appropriate power of $\sqrt{-g}$ where $g \equiv \det(g_{\mu\nu})$ to make it a scalar density.

(v) To the Lagrangian modified as above, one has to add the Einstein Lagrangian for gravity

$$\mathcal{L}_{\text{grav}} = \frac{1}{\kappa^2} R \sqrt{-g} \tag{2.4}$$

and an appropriate gauge-fixing term for the gravitational field.

For the gravitational field we shall employ the exponential parametrization[3]

$$L^{\mu a} = [\exp(\tfrac{1}{2}\kappa\phi)]^{\mu a}, \tag{2.5}$$

which implies

$$g = -\exp(-\kappa \operatorname{Tr}\phi) = -\exp(-2\kappa\chi). \tag{2.6}$$

Here ϕ is the symmetric 4×4 matrix field of gravitons. We shall generally work in the Fock-de Donder gauge[2] in which the graviton propagator is given by

$$\langle 0| T(\phi^{\mu a}(x)\phi^{\nu b}(0))|0\rangle = \tfrac{1}{2}(\eta^{\mu\nu}\eta^{ab} + \eta^{\mu b}\eta^{\nu a} - \eta^{\mu a}\eta^{\nu b}) D(x), \tag{2.7}$$

where $D(x)$ is the propagator for a massless scalar field:

$$D(x) = -\frac{1}{4\pi^2 x^2} \tag{2.8}$$

(our metric has signature $+---$). The effect of employing a more general class of gauges will be considered at the end of this section. We shall be interested only in the regularizing effect of gravity on other interactions and will not consider, for example, processes involving gravitons in the initial and/or final states.

B. The approximation

Our approximation consists in ignoring, for most practical purposes, all gravitational interactions except the $\sqrt{-g}$ factor with the coupling terms in the Lagrangian; the gravitational self-interactions are also to be ignored. As we shall see later, some effects of the discarded terms have to be taken into account to preserve gauge invariance and to regularize the bubble diagrams.

To see how it works, we shall consider a simple example—the self-interaction $\lambda\Phi^N$ $(N \geq 3)$ of a massless scalar field which is unrenormalizable for $N > 4$. The Lagrangian including gravity is

$$\mathcal{L} = \sqrt{-g}\,(\tfrac{1}{2} g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \lambda\Phi^N) + \mathcal{L}_{\text{grav}} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{\text{grav}}, \tag{2.9}$$

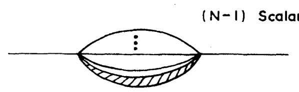

FIG. 1. Scalar self-energy supergraph. The shaded double line represents the graviton superpropagator.

where

$$\mathcal{L}_0=\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi,$$
$$\mathcal{L}_1=-\lambda\sqrt{-g}\,\Phi^N=-\lambda\Phi^N\exp(-\kappa\chi), \tag{2.10}$$
$$\mathcal{L}_2=\tfrac{1}{2}(\sqrt{-g}\,g^{\mu\nu}-\eta^{\mu\nu})\partial_\mu\Phi\partial_\nu\Phi.$$

In our approximation $\mathcal{L}_2$ and terms in $\mathcal{L}_{\text{grav}}$ other than the bilinear ones are to be neglected. We shall consider the supergraph for scalar self-energy in the lowest order in λ (Fig. 1), which gives

$$\Pi(k^2)=A\lambda^2\int d^4x\,e^{ik\cdot x}[D(x)]^{N-1}\exp[-\kappa^2D(x)], \tag{2.11}$$

where A is a numerical constant. Now

$$[D(x)]^{N-1}\exp[-\kappa^2D(x)]=\sum_{n=0}^{\infty}\frac{(-\kappa^2)^n}{n!}[D(x)]^{n+N-1}=\frac{1}{2\pi i}\int_{C_0}dz\,\Gamma(-z)(\kappa^2)^z[D(x)]^{z+N-1}, \tag{2.12}$$

where C_0 is a contour parallel to the imaginary axis with $-1<\text{Re}z<0$. To be able to apply the Gel'fand-Shilov formula[11]

$$[D(x)]^z=\frac{-i\Gamma(2-z)}{16\pi^2\Gamma(z)}\int\frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}\left(\frac{-q^2}{16\pi^2}\right)^{z-2} \tag{2.13}$$
$$(0<\text{Re}z<2)$$

we shift the contour in (2.12) to the left so that $0<\text{Re}(z+N-1)<2$ and obtain

$$\Pi(k^2)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}dz\,\Gamma(-z)\Pi(k^2,z), \tag{2.14}$$

where $-N+2<\alpha<-N+3$ and

$$\Pi(k^2,z)=\frac{-iA\lambda^2}{16\pi^2}(\kappa^2)^z\frac{\Gamma(3-N-z)}{\Gamma(N+z-1)}\left(\frac{-k^2}{16\pi^2}\right)^{z+N-3}. \tag{2.15}$$

On folding the contour in (2.14) to the right on the real axis, we obtain contributions from simple poles at $z=-N+3,\ -N+4,\ldots,-1$ and double poles at $z=0,1,\ldots,$ etc. The simple poles give terms of order $(\kappa^2)^{-N+3},\ldots,\kappa^{-2}$ and the double pole at $z=0$ gives terms in $\ln\kappa$ and zeroth power of κ. The poles at $z=1,2,\ldots$ give terms of order κ and higher. The negative powers of κ and $\ln\kappa$ terms are reminiscent of the ultraviolet divergences in the self-energy graph without gravity; κ^{-1} has appeared as an ultraviolet cutoff.

It should be noted that the regularization works even for $N>4$ corresponding to a nonrenormalizable interaction; no new ambiguities appear for this case.

It is important to note the correlation between the ultraviolet singularities of the amplitude in (2.11) with $\kappa=0$ and the poles in the integrand (2.14) for $z=-N+3,\ -N+4,\ldots,-1,0$. This is because, by employing the Sommerfeld-Watson transform in (2.12), these singularities have been translated into those of the distribution $[D(x)]^{z+N-1}$ which are[11] simple poles at $(z+N-1)=+2,+3,\ldots$. This particular feature would persist even if the field χ were massive, because the singular part of the Feynman propagator is independent of mass. The ultraviolet finiteness of amplitudes is realized in the general case in the same manner, as in the above example. We note that the finiteness of $\Pi(k^2)$ above comes about essentially because of the presence of the exponential term (the graviton superpropagator) in (2.11) which is guaranteed[12] to be a decaying exponential by the Euclidity ansatz (which is a part and parcel of the nonpolynomial Lagrangian method). In the general case, every Feynman graph of the original theory is replaced by a supergraph in which there is a graviton superpropagator between every two vertices providing a convergent factor as above. These convergent factors are sufficiently powerful[5] to take care of all ultraviolet divergences including those due to derivative couplings. For closed loops involving more than two vertices, it is, in fact, sufficient to consider only one superpropagator between any two vertices and ignore the other superpropagators. This follows from the fact that a single superpropagator is enough to provide the necessary convergent factors for the loop integral; inclusion of others will not harm this property and will only produce additional terms of $O(\kappa)$ and higher.

As another example, we consider gravity-modified quantum electrodynamics (QED). The Lagrangian is[2,3]

$$\mathcal{L}=\sqrt{-g}\,[\tfrac{1}{2}i(\bar\psi\gamma_a\psi_{;\mu}-\bar\psi_{;\mu}\gamma_a\psi)L^{\mu a}-m\bar\psi\psi+e\bar\psi\gamma_a\psi L^{\mu a}A_\mu-\tfrac{1}{4}F_{\rho\sigma}F_{\mu\nu}g^{\mu\rho}g^{\nu\sigma}]+\mathcal{L}_{\text{grav}}+\mathcal{L}_{\text{gauge}}, \tag{2.16}$$

where

$$\psi_{;\mu} = \partial_\mu \psi - \tfrac{1}{4} i B_{\mu a b}\sigma^{ab}\psi ,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu ,$$
$$B_{\mu ab} = \tfrac{1}{2}(L_a^\nu \partial_\mu L_{\nu b} - L_b^\nu \partial_\mu L_{\nu a}) - \tfrac{1}{2}(L_a^\nu \partial_\nu L_{\mu b} - L_b^\nu \partial_\nu L_{\mu a}) - \tfrac{1}{2} L_{\mu c}(\partial_\lambda L_\nu^c - \partial_\nu L_\lambda^c) L_a^\lambda L_b^\nu ,$$
$$\sigma^{ab} = \frac{i}{4}[\gamma^a, \gamma^b] .$$

Employing the parametrization (2.5), we are left with, in our approximation, the interaction

$$\mathcal{L}_1 = e\bar{\psi}\gamma_a \psi \eta^{a\mu} A_\mu e^{-\kappa\chi} , \tag{2.17}$$

which is the same as $\mathcal{L}_{em}$ of Ref. 5, where nonpolynomial Lagrangian techniques were applied to this interaction to remove ultraviolet infinities from QED. It is interesting to note that, in Ref. 5, the $\ln\kappa$ terms in the self-masses and renormalization constants (calculated in the lowest nontrivial order in e), which replace the usual logarithmic cutoff-dependent terms, were the same as in the theory with full tensor gravity.[2] This, in fact, is a general feature—the $\ln\kappa$ terms in the self-masses and renormalization courtants are the same (at least for graphs with one loop) in our approximation as with full tensor gravity. A formal proof of this is presented in Appendix A where it is also shown that, in the case of (gravity-modified) renormalizable theories, the finite parts of the amplitudes (ignoring terms vanishing with κ) obtained after the usual subtractions, are the same in the theory with full tensor gravity as well as in our approximation as in the conventionally renormalized theory.

A remark about normal ordering is in order. In this section as well as in the following sections we shall take the exponential χ field term [see, for example, Eq. (2.17)] normal ordered. As naive normal ordering is known to interfere seriously with gauge invariance, justification for this is required. Since

$$e^{-\kappa\chi} = \exp\left[\tfrac{1}{2}D(0)\frac{\partial^2}{\partial\chi^2}\right] : e^{-\kappa\chi} : , \tag{2.18}$$

we have, in effect, taken[1] $D(0)=0$. It may be possible to give a formal proof of the consistency of this procedure along the lines of, for example, Ref. 12; we have, however, not attempted this. We shall examine the Yang-Mills gauge covariance of amplitudes at appropriate places. The effect of change of gauge of the graviton propagator on our amplitudes is discussed below.

C. Remarks on gravitational gauge invariance

In all our calculations, we have employed the Fock–de Donder gauge in which the graviton propagator is given by (2.7). We now consider a class of gauges in which the graviton propagator is

$$\langle 0 | T(\phi^{\mu a}(x)\phi^{\nu b}(0)) | 0\rangle = \tfrac{1}{2}(\eta^{\mu\nu}\eta^{ab} + \eta^{\mu b}\eta^{\nu a} - \xi\eta^{\mu a}\eta^{\nu b})D(x) , \tag{2.19}$$

where ξ is a constant parameter. This gives

$$\langle 0 | T(\chi(x)\chi(0)) | 0\rangle = \tfrac{1}{4}\langle 0 | T(\mathrm{Tr}\phi(x)\,\mathrm{Tr}\phi(0)) | 0\rangle = (1-2\xi)D(x) \tag{2.20}$$

and therefore

$$\langle 0 | T(: e^{-\kappa\chi(x)} : : e^{-\kappa\chi(0)} :) | 0\rangle = \exp[-\kappa^2(2\xi - 1)D(x)] . \tag{2.21}$$

The net effect is, therefore, to replace κ^2 by $\kappa^2(2\xi-1)$ in the amplitudes. The physically relevant κ-independent terms will, therefore, remain unaffected. The renormalization constants will, of course be ξ dependent in general.

We conclude this section with some remarks on the role of terms in Lagrangians like $\mathcal{L}_2$ in (2.10). It is formally of order κ and, in our scheme, can be ignored to the extent that, in any supergraph rendered already convergent through the exponential factor in $\mathcal{L}_1$, additional insertions due to $\mathcal{L}_2$ only change terms of order κ or higher. As is well known,[3] however, $\mathcal{L}_2$, acting by itself or with $\mathcal{L}_1$, can produce divergent supergraphs which can, in general, be rendered convergent by including some "supplementary" graph (or supergraphs) in which $\mathcal{L}_0$ has been put into service at appropriately chosen "kinetic energy kinks"—this finally amounts to treating $\mathcal{L}_0 + \mathcal{L}_2$ together. To give an illustration which will be useful later,

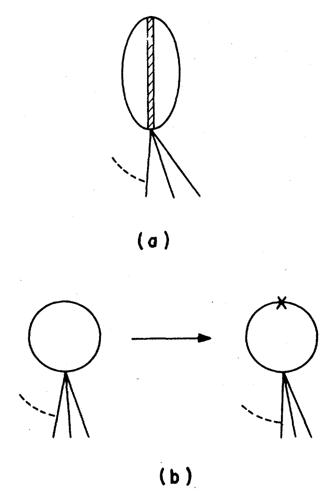

FIG. 2. (a) Supergraph for scalar $(N-2)$-point function. (b) Kinetic energy kink in the bubble diagram for the scalar $(N-2)$-point function.

we replace, for simplicity of argument, $\mathcal{L}_2$ by

$$\mathcal{L}_2' = \tfrac{1}{2}(e^{-\kappa\chi} - 1)(\partial_\mu \phi)^2 . \tag{2.22}$$

To first order in $\mathcal{L}_1$ and $\mathcal{L}_2'$, there arises a contribution to the $(N-2)$-point function [see Fig. 2(a)]

$$-\lambda \int d^4x [e^{\kappa^2 D(x)} - 1](\partial_\mu D)^2 \tag{2.23}$$

in which there is trouble resulting from the second term in the square brackets. This is remedied by adding to (2.23) the contribution from the graph in Fig. 2(b) (arising from $\mathcal{L}_1$ in first order)

$$-\lambda \int d^4x (\partial_\mu D)^2 \tag{2.24}$$

which precisely cancels the troublesome term in (2.23). The cross in the second part of Fig. 2(b) indicates a kinetic energy kink which means that, in the internal propagator of Fig. 2(b), one has to make the replacement

$$\frac{1}{p^2} \to \frac{1}{p^2} p_\mu p^\mu \frac{1}{p^2} \tag{2.25}$$

so that the sum of Figs. 2(a) and 2(b) is equivalent to the operation of $(\mathcal{L}_0 + \mathcal{L}_2')$ at the upper vertex in Fig. 2(a). This example has a lesson in that the "bubble diagrams" like that in Fig. 2(b) can be regularized by putting $\mathcal{L}_2$ into service and including supergraphs like Fig. 2(a); this will, indeed, be done in a later section.

III. REGULARIZATION OF THEORIES WITH SPONTANEOUS SYMMETRY BREAKING: σ MODEL

In this section, we consider infinity suppression in gravity-modified theories with spontaneous symmetry breaking, taking the σ model[14-16] as an example. Some new features appear. The regularization of the bubble diagrams (which arise because the σ-model Lagrangian is not to be normal ordered[15]), for example, cannot be achieved by the mechanism discussed in the preceding section and requires the inclusion of some of the terms which are usually discarded in our approximation; it is then nontrivial to verify that the usual nice features[15,16] of renormalization of the σ model persist, at least to lowest order, in our scheme of regularization.

With gravity included, the σ-model Lagrangian (without nucleons) is

$$\mathcal{L} = \sqrt{-g}\,\left[\tfrac{1}{2} g^{\mu\nu}(\partial_\mu \sigma \partial_\nu \sigma + \partial_\mu \vec{\Pi} \cdot \partial_\nu \vec{\Pi}) - \tfrac{1}{2} m_0^2(\sigma^2 + \pi^2) - \tfrac{1}{4}\lambda(\sigma^2+\pi^2)^2 + c\sigma\right] + \mathcal{L}_{\text{grav}} . \tag{3.1}$$

Replacing $g^{\mu\nu}$ in the square brackets by $\eta^{\mu\nu}$, employing the parametrization (2.5), and performing the field shift

$$\sigma = \langle \sigma \rangle_0 + \sigma' \equiv V + \sigma' , \tag{3.2}$$

we have the Lagrangian

$$\mathcal{L}' = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_{\text{grav}} , \tag{3.3}$$

where

$$\begin{aligned}
\mathcal{L}_0 &= \tfrac{1}{2}[(\partial_\mu \vec{\pi})^2 - m_\pi^2 \vec{\pi}^2] + \tfrac{1}{2}[(\partial_\mu \sigma')^2 - m_\sigma^2 \sigma'^2] , \\
\mathcal{L}_1 &= e^{-\kappa\chi}[-\lambda V \sigma'(\sigma'^2 + \vec{\pi}^2) - \tfrac{1}{4}\lambda(\sigma'^2 + \vec{\pi}^2)^2] , \\
\mathcal{L}_2 &= (C - V m_\pi^2)\sigma' , \\
\mathcal{L}_3 &= \tfrac{1}{2}(e^{-\kappa\chi} - 1)[(\partial_\mu \sigma')^2 + (\partial_\mu \vec{\pi})^2] \\
&\quad + (e^{-\kappa\chi} - 1)(C - V m_\pi^2)\sigma' ,
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
m_\sigma^2 &= m_0^2 + 3\lambda V^2 , \\
m_\pi^2 &= m_0^2 + \lambda V^2 .
\end{aligned} \tag{3.5}$$

We now consider regularization of various amplitudes in the lowest nontrivial order in λ. The part $\mathcal{L}_3$ will be required only in the regularization of the bubble diagrams and will be ignored in all other calculations.

A. Vertex correction

We consider the lowest-order corrections to the σ'^4 vertex. Other vertices can be treated in the same fashion. The relevant supergraphs are shown in Fig. 3; they give

$$\begin{aligned}
\Gamma_{\sigma'^4}(k) &= \Gamma^{(a)}_{\sigma'^4}(k) + \Gamma^{(b)}_{\sigma'^4}(k) \\
&= \frac{18\lambda^2}{(2\pi)^8}[3I(k, m_\sigma) + I(k, m_\pi)] ,
\end{aligned} \tag{3.6}$$

where

$$I(k, m) = \int d^4p\, d^4q \frac{1}{p^2 - m^2} \frac{1}{(k-p-q)^2 - m^2} \tilde{\mathfrak{D}}(q) \tag{3.7}$$

and [see Eq. (A3)]

$$\begin{aligned}
\tilde{\mathfrak{D}}(q) &= \int e^{iq\cdot x} \mathfrak{D}(x) \\
&= \frac{1}{2\pi i} \int_{C_0} dz\, \Gamma(-z)(\kappa^2)^z \frac{\Gamma(2-z)}{16\pi^2 i \Gamma(z)} \left(\frac{-q^2}{16\pi^2}\right)^{z-2} ;
\end{aligned} \tag{3.8}$$

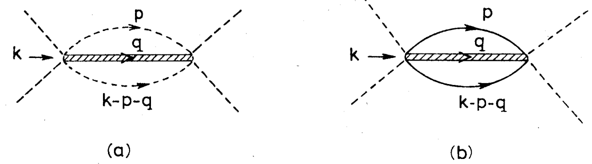

FIG. 3. σ'^4 vertex correction. The dashed lines are σ' lines and the solid single lines are π lines.

we have

$$I(k,m)=\frac{1}{2\pi i}\int_{C_0}dz\,\Gamma(-z)(\kappa^2)^z\frac{\Gamma(2-z)}{16\pi^2 i\Gamma(z)}\times\left(-\frac{1}{16\pi^2}\right)^{z-2}I(k,m;z)\,,\qquad(3.9)$$

where

$$I(k,m;z)=\int d^4p\,d^4q\frac{1}{p^2-m^2}\frac{1}{(k-p-q)^2-m^2}(q^2)^{z-2}$$
$$=-\tfrac{1}{2}\pi^4(-4m^2)^z\frac{\Gamma(z)\Gamma(-z)\Gamma(\frac{1}{2})\Gamma(1-z)}{\Gamma(\frac{3}{2}-z)}\times{}_3F_2(1-z,-z,2-z;\tfrac{3}{2}-z,2;k^2/4m^2)\,.\qquad(3.10)$$

As expected, the integrand in (3.9) has a double pole at $z=0$; evaluating the residue, we get

$$I(k,m)=-8\pi^6 i\left[2\ln\left(\frac{\kappa^2m^2}{4\pi^2}\right)+f(k,m)\right]+O(\kappa^2\ln\kappa)\qquad(3.11)$$

where

$$f(k,m)=\frac{d}{dz}\left\{\frac{\Gamma(\frac{1}{2})\Gamma(2-z)[\Gamma(2-z)]^3}{\Gamma(\frac{3}{2}-z)}\times{}_3F_2(1-z,-z,2-z;\tfrac{3}{2}-z,2;k^2/4m^2)\right\}\Bigg|_{z=0}.\qquad(3.12)$$

Finally, we obtain, to order κ^0,

$$\Gamma_{\sigma'^4}(k)=\frac{-9i\lambda^2}{2\pi^2}\left[\ln\left(\frac{\kappa^2m_0^2}{4\pi^2}\right)+\tfrac{1}{2}f(k,m_0)\right]+O(\lambda^3)\,.\qquad(3.13)$$

Now, defining the vertex renormalization constant $\hat{Z}_1$ by

$$-6i\lambda+\Gamma_{\sigma'^4}(0)=\hat{Z}_1^{\,-1}(-6i\lambda)\,,\qquad(3.14)$$

we obtain

$$\hat{Z}_1=1+\frac{3\lambda^2}{2\pi^2}\left[\ln\left(\frac{2\kappa}{km_0}\right)+O(\kappa^0)\right].\qquad(3.15)$$

The lowest-order correction to λ is

$$\delta\lambda=-\frac{1}{6i}\Gamma_{\sigma'^4}(0)=\frac{3\lambda^2}{2\pi^2}\left[\ln\left(\frac{\kappa m_0}{2\pi}\right)+\tfrac{1}{4}f(0,m_0)\right].\qquad(3.16)$$

B. Regularization of the bubble diagram

To regularize the bubble diagrams (see Fig. 4), we bring into service the Lagrangian $\mathfrak{L}_3$ in (3.4). The amplitude for the diagram 4(b) will be proportional to

FIG. 4. (a) Bubble diagram. The thick line represents either two external meson lines or a single σ' line; the thin line represents a π propagator or a σ' propagator. (b) Bubble diagram with a graviton superpropagator; this represents the diagram (a) plus diagrams with one, two, ... graviton exchanges between A and B arising from the operation of $\mathfrak{L}_3$ at the vertex B.

$$T(m^2)=\frac{1}{(2\pi)^4}\int d^4p\,d^4q\frac{1}{p^2-m^2}\frac{p\cdot(q+p)-m^2}{(q+p)^2-m^2}\tilde{\mathfrak{D}}(q)$$
$$=\frac{1}{(2\pi)^4}\frac{1}{2\pi i}\int_{C_1}dz\frac{\Gamma(-z)\Gamma(2-z)}{16\pi^2 i\Gamma(z)}(\kappa^2)^z\times\left(-\frac{1}{16\pi^2}\right)^{z-2}T(m^2,z)\,,\qquad(3.17)$$

where m is the mass of the particle represented by the thin line, Eq. (3.8) has been used (to achieve convergence we have shifted the contour C_0 to C_1 which is parallel to the imaginary axis with $-2<\operatorname{Re}z<-1$), and

$$T(m^2,z)=\int d^4p\,d^4q\frac{p\cdot(p+q)-m^2}{(p^2-m^2)[(p+q)^2-m^2]}(q^2)^{z-2}$$
$$=(-1)^z\pi^4\frac{[\Gamma(1-z)]^2\Gamma(z)\Gamma(-z-1)}{\Gamma(1-2z)}(m^2)^{z+1}\,.\qquad(3.18)$$

Evaluating the residues of the simple pole at $z=-1$ and double pole at $z=0$ in (3.17), we obtain

$$T(m^2)=-i\pi^2\left[\frac{16\pi^2}{\kappa^2}+m^2\ln\left(\frac{\kappa^2m^2}{16\pi^2}\right)+m^2\frac{d}{dz}\left(\frac{[\Gamma(1-z)]^4\Gamma(2-z)}{\Gamma(1-2z)(z+1)}\right)\Bigg|_{z=0}\right]+O(\kappa^2\ln\kappa)\,.\qquad(3.19)$$

C. Meson self-energies

For σ self-energy, the four diagrams shown in Fig. 5 give

$$\Pi_\sigma(k^2)=\Pi_\sigma^a(k^2)+\cdots+\Pi_\sigma^d(k^2)$$
$$=\frac{3i\lambda}{(2\pi)^4}[T(m_\sigma^2)+T(m_\pi^2)]+\frac{6i\lambda^2V^2}{(2\pi)^8}[3I(k,m_\sigma)+I(k,m_\pi)]\,.\qquad(3.20)$$

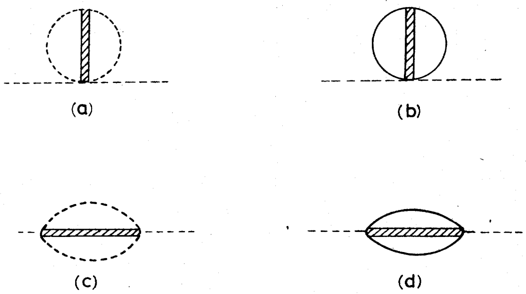

FIG. 5. σ self-energy supergraphs.

This gives

$$\delta m_\sigma^2 = \Pi_\sigma(0)$$
$$= \delta m^2 + 3(\delta\lambda)V^2 + O(\kappa^2 \ln\kappa)\,, \qquad (3.21)$$

where

$$\delta m^2 = \frac{6i\lambda}{(2\pi)^4} T(m_0^2) \qquad (3.22)$$

is the self-mass in the symmetric theory. The wave-function renormalization constant Z_σ is, to this order,

$$Z_\sigma = 1 + \left.\frac{\partial \Pi_\sigma(k^2)}{\partial k^2}\right|_{k^2=0}$$
$$= 1 + \frac{3\lambda^2 V^2}{16\pi^2} \frac{\partial}{\partial k^2}[3f(k, m_\sigma^2) + f(k, m_\pi^2)]\bigg|_{k^2=0}$$
$$+ O(\kappa^2 \ln\kappa)\,. \qquad (3.23)$$

For the π self-energy, the three diagrams shown in Fig. 6 give

$$\Pi_\pi(k^2) = \Pi_\pi^{(a)}(k^2) + \Pi_\pi^{(b)}(k^2) + \Pi_\pi^{(c)}(k^2)$$
$$\simeq \frac{i\lambda}{(2\pi)^4}[5T(m_\pi^2) + T(m_\sigma^2)] + \frac{4i\lambda^2 V^2}{(2\pi)^8} I(k, m_0)\,, \qquad (3.24)$$

where a contribution of $O(\lambda^3)$ has been ignored in $\Pi_\pi^{(c)}$. This gives

$$\delta m_\pi^2 = \delta m^2 + (\delta\lambda)V^2 + O(\kappa^2 \ln\kappa) \qquad (3.25)$$

and

$$Z_\pi = 1 + \frac{\lambda^2 V^2}{8\pi^2} \frac{\partial}{\partial k^2} f(k, m)\bigg|_{k^2=0} + O(\kappa^2 \ln\kappa)\,. \qquad (3.26)$$

Note that the self-masses are consistent with (3.5). The absence of inverse powers of κ and $\ln\kappa$ terms in Z_σ and Z_π corresponds to the fact

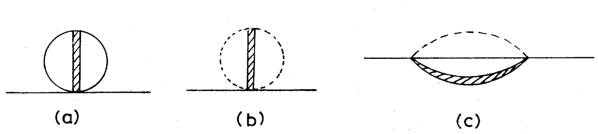

FIG. 6. π self-energy supergraphs.

FIG. 7. Supergraphs for corrections to $\langle\sigma\rangle_0$ in the lowest order.

that, to one-loop order, they are finite in the conventional theory.

D. Goldstone theorem

The vacuum expectation value $V = \langle\sigma\rangle_0$ is determined from the equation $\langle\sigma'\rangle = 0$ which gives

$$C - Vm_\pi^2 + S(V) = 0\,, \qquad (3.27)$$

where $S(V)$ is the sum of one-loop and higher-order diagrams in which a σ' line vanishes into the vacuum. The lowest-order diagrams contributing to $S(V)$ are shown in Fig. 7; these give

$$S(V) = -\frac{3i\lambda}{(2\pi)^4}[T(m_\pi^2) + T(m_\sigma^2)]\,. \qquad (3.28)$$

Keeping only terms of order λ^2 in (3.28) we obtain

$$S(V) = -\delta m_\pi^2 + O(\kappa^2 \ln\kappa)\,. \qquad (3.29)$$

Equation (3.27) now gives

$$C - V(m_\pi^2 + \delta m_\pi^2) + O(\kappa^2 \ln\kappa) = 0\,. \qquad (3.30)$$

To order κ^0, we therefore have the usual situation: When $C = 0$ either $V = 0$ (no spontaneous breaking) or $m_\pi^2 + \delta m_\pi^2 = 0$ thereby verifying the Goldstone theorem in the lowest nontrivial order.

E. Partially conserved axial-vector current (PCAC)

Defining the axial-vector current $A_\mu^\alpha(x)$, the pion propagator $\Delta^\pi(k^2)$, and the pion–axial-vector vertex $\Gamma_\mu^5(k)$ through the equations

$$A_\mu^\alpha = \pi^\alpha \partial_\mu \sigma - \sigma \partial_\mu \pi^\alpha\,, \quad \alpha = 1, 2, 3 \qquad (3.31)$$

$$\delta^{\alpha\beta}\Delta^\pi(k^2) = \int e^{ik\cdot x}\langle 0|\, T(\pi^\alpha(x)\pi^\beta(0))|\, 0\rangle\,, \qquad (3.32)$$

$$\int e^{ik\cdot x}\langle 0|\, T(A_\mu^\alpha(x)\pi^\beta(0))|\, 0\rangle d^4x = \Gamma_\mu^5(k)\Delta^\pi(k^2)\delta^{\alpha\beta}\,, \qquad (3.33)$$

we obtain, after the usual manipulations,[15] the PCAC relation

$$k_\mu \Gamma^{5\mu}(k) = -V[\Delta^\pi(k^2)^{-1} - \Delta^\pi(0)^{-1}] + O(\kappa)\,. \qquad (3.34)$$

We shall verify (3.34) to the lowest nontrivial order in perturbation theory; the relevant super-

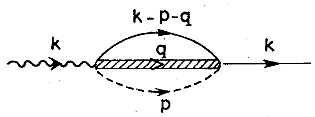

FIG. 8. Lower-order supergraph for the pion–axial-vector vertex. The wavy line represents the axial-vector current.

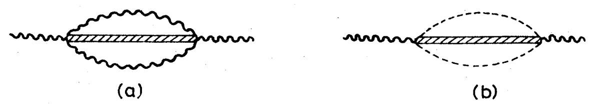

FIG. 9. Lowest-order supergraphs for gauge particle self-energies. Wavy lines represent the gauge particles and dashed lines the ghosts.

graph is shown in Fig. 8. To this order, we have

$$\Gamma^{5\mu}(k)=iVk^{\mu}+\frac{2\lambda V}{(2\pi)^8}\int d^4p\,d^4q\,\frac{(2p-k+q)^{\mu}}{(p^2-m_\sigma^{\,2})[(k-p-q)^2-m_\pi^{\,2}]}\times\tilde{\mathfrak{D}}(q)\,. \tag{3.35}$$

Evaluating the integrals, one finally obtains

$$k_\mu\Gamma^{5\mu}(k)=iVk^2-i[\Pi_\pi(k^2)-\Pi_\pi(0)]+O(\kappa^2\ln\kappa)\,. \tag{3.36}$$

In fact, the relation (3.36) can be established without evaluating[17] the integrals in (3.35). In the relevant order, (3.36) is equivalent to (3.34).

IV. REGULARIZATION OF NON-ABELIAN GAUGE THEORIES

We now proceed to consider the regularization of non-Abelian gauge theories. These require special consideration because maintenance of gauge invariance is a nontrivial problem in such schemes of regularization.[3,5] Indeed, it will be seen that the "point-separation method" for the definition of the current operator has to be employed to maintain gauge invariance appropriately.

A. Pure Yang-Mills fields

We shall first consider pure Yang-Mills fields, i.e., without any other matter fields. With gravity included, the Lagrangian is (with the usual notation[18])

$$\mathfrak{L}=\sqrt{-g}\left(-\tfrac{1}{4}g^{\mu\lambda}g^{\nu\rho}F^a_{\mu\nu}F^a_{\lambda\rho}-\frac{1}{2\alpha}g^{\mu\nu}g^{\lambda\rho}A^a_{\nu;\mu}A^a_{\rho;\lambda}+g^{\mu\nu}\partial_\mu C^{\dagger a}\partial_\nu C^a+gf_{abc}g^{\mu\nu}\partial_\mu C^{\dagger a}A^b_\nu C^c\right)+\mathfrak{L}_{\rm grav}\,, \tag{4.1}$$

where

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf_{abc}A^b_\mu A^c_\nu$$

and the semicolons represent covariant derivatives with respect to general coordinate transformation; the gauge group G is assumed, as usual, to be a compact simple Lie group.

Replacing $g_{\mu\nu}$ by $\eta_{\mu\nu}$ in the parentheses in (4.1) and employing the exponential parametrization (2.5), we have

$$\mathfrak{L}=\mathfrak{L}_{\rm bilinear}+\mathfrak{L}_1+\mathfrak{L}'\,, \tag{4.2}$$

where

$$\mathfrak{L}_1=e^{-\kappa\chi}(-gf_{abc}\partial^\mu A^{a\nu}A^b_\mu A^c_\nu-\tfrac{1}{4}g^2f_{abc}f_{ab'c'}A^b_\mu A^c_\nu A^{b'\mu}A^{c'\nu}+gf_{abc}\partial^\mu C^{\dagger a}A^b_\mu C^c) \tag{4.3}$$

and

$$\mathfrak{L}'=(e^{-\kappa\chi}-1)\left[-\tfrac{1}{4}(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu)^2-\frac{1}{2\alpha}(\partial_\mu A^{\mu a})^2+\partial^\mu C^{\dagger a}\partial_\mu C^a\right]. \tag{4.4}$$

As before, $e^{-\kappa\chi}$ is understood to be the normal-ordered expression $:e^{-\kappa\chi}:$. The propagators are

$$D^{ab}_{\mu\nu}(x)\equiv\langle 0|\,T(A^a_\mu(x)A^b_\nu(0))\,|0\rangle=-\delta_{ab}\left[\eta_{\mu\nu}-(1-\alpha)\frac{\partial_\mu\partial_\nu}{\partial^2}\right]D(x)\,,$$
$$\langle 0|\,T(C^a(x)C^{\dagger b}(0))\,|0\rangle=\delta_{ab}D(x)\,. \tag{4.5}$$

We shall now consider the primitive divergents of the Yang-Mills theory, calculate the renormalization constants to order $\ln\kappa$ (in the lowest nontrivial order in the gauge coupling constant g), and verify that they are consistent with Ward identities.[19,20] For the time being we shall ignore $\mathfrak{L}'$.

1. Self-energy of the gauge particles

A direct momentum-space calculation gives a non-gauge-invariant result with a longitudinal part in the self-energy function $\Pi^{\mu\nu}$ proportional to κ^{-2}. (This is analogous to the situation regarding photon self-energy in QED.[2,5,21]) Following Ref. 3, we shall present a coordinate-space calculation employing the "calculus of derivatives" and the normal ordering for $\mathfrak{L}_1$. The relevant lowest-order supergraphs are shown in Fig. 9. After a straightforward calculation, we have

$$\Pi^{\mu\nu}(x)=\Pi^{\mu\nu(a)}(x)+\Pi^{\mu\nu(b)}(x)\,, \tag{4.6}$$

where

$$\Pi^{\mu\nu(a)}(x) = -ig^2C_2(G)\Big\{-\eta^{\mu\nu}\partial^2(D^2\mathfrak{D}) + \partial^\mu\partial^\nu(D^2\mathfrak{D}) + 2\partial^\mu(\partial^\nu DD\mathfrak{D}) - 2\eta^{\mu\nu}\partial_\lambda(\partial^\lambda DD\mathfrak{D}) - 2(\partial^\mu\partial^\nu D)D\mathfrak{D} + 3\partial^\mu D\partial^\nu D\mathfrak{D}$$
$$-\eta^{\mu\nu}\partial^2 DD\mathfrak{D} + \alpha'\partial^\lambda\partial^\rho\left[\eta^{\mu\nu}\left(\frac{\partial_\lambda\partial_\rho D}{\partial^2}\right)D\mathfrak{D} + \eta_{\lambda\rho}\left(\frac{\partial^\mu\partial^\nu D}{\partial^2}\right)D\mathfrak{D} - \eta^\mu_\rho\left(\frac{\partial^\nu\partial_\lambda D}{\partial^2}\right)D\mathfrak{D} - \eta^\mu_\lambda\left(\frac{\partial_\rho\partial^\nu D}{\partial^2}\right)D\mathfrak{D}\right]$$
$$-2\alpha'\partial^\lambda\left[\eta^\mu_\lambda\partial^\rho D\left(\frac{\partial^\nu\partial_\rho D}{\partial^2}\right)\mathfrak{D} - \eta^{\mu\nu}\partial^\rho D\left(\frac{\partial_\lambda\partial_\rho D}{\partial^2}\right)\mathfrak{D}\right]$$
$$+\alpha'\left[\eta^{\lambda\rho}\partial^\mu\partial^\nu D\left(\frac{\partial_\lambda\partial_\rho D}{\partial^2}\right)\mathfrak{D} - \eta^{\lambda\mu}\partial^\rho\partial^\nu D\left(\frac{\partial_\lambda\partial_\rho D}{\partial^2}\right)\mathfrak{D} - \eta^{\lambda\nu}\partial^\rho\partial^\mu D\left(\frac{\partial_\lambda\partial_\rho D}{\partial^2}\right)\mathfrak{D} + \eta^{\mu\nu}\partial^\lambda\partial^\rho D\left(\frac{\partial_\lambda\partial_\rho D}{\partial^2}\right)\mathfrak{D}\right]$$
$$-\alpha'^2\partial^\lambda\partial^\rho\left[\left(\frac{\partial_\lambda\partial_\rho D}{\partial^2}\right)\left(\frac{\partial^\mu\partial^\nu D}{\partial^2}\right)\mathfrak{D} - \left(\frac{\partial^\mu\partial_\lambda D}{\partial^2}\right)\left(\frac{\partial^\nu\partial_\rho D}{\partial^2}\right)\mathfrak{D}\right]\Big\} \quad (4.7)$$

and

$$\Pi^{\mu\nu(b)}(x) = ig^2C_2(G)\partial^\mu D\partial^\nu D\mathfrak{D}(x)\,. \quad (4.8)$$

Here we have put $\alpha' = 1-\alpha$, $\mathfrak{D}(x)$ is defined in Appendix A, and $C_2(G)$ is defined by

$$\sum_{c,d} f_{acd}f_{bcd} = C_2(G)\delta_{ab}\,.$$

Employing (A3), the formulas of the calculus of derivatives given in Appendix B and proceeding as in the previous sections, we finally obtain

$$\Pi^{\mu\nu}(k) \equiv \int e^{-ik\cdot x}\Pi^{\mu\nu}(x)d^4x$$
$$= \frac{-g^2C_2(G)}{16\pi^2}\Bigg\{(k^\mu k^\nu - k^2\eta^{\mu\nu})\left(\frac{5}{3}+\frac{\alpha'}{2}\right)\ln\left(\frac{-\kappa^2k^2}{16\pi^2}\right)$$
$$+\frac{d}{dz}\left[\frac{[\Gamma(1-z)]^2}{\Gamma(z+2)}\left(1+\frac{8-\alpha'^2z}{4(z+2)} - \frac{(2-3\alpha')}{(z+2)(z+3)}\right)(k^\mu k^\nu - \eta^{\mu\nu}k^2)\right.$$
$$\left.+\frac{1}{2}\frac{(3\alpha'-2)z}{(z+1)(z+2)(z+3)}k^2\eta^{\mu\nu}\right]\Bigg|_{z=0}\Bigg\} + O(\kappa^2\ln\kappa)\,. \quad (4.9)$$

We note that, whereas the $\ln\kappa$ term is gauge invariant, the κ^0 term is not. This shows that, contrary to the conjecture made in Refs. 3 and 22, normal ordering plus calculus of derivatives do not, in general, ensure gauge invariance. We shall do what is needed to restore gauge invariance later in this section. We note, however, that $\Pi^{\mu\nu}(0)=0$ (to order κ^0) so that masslessness of the gauge quantum is ensured even in the present calculation.

To $O(g^2)$, we have the unrenormalized gauge particle propagator as

$$D^{ab}_{\text{un}\,\mu\nu}(k) = D^{ab}_{\mu\nu}(k) - iD^{ac}_{\mu\lambda}(k)\Pi^{\lambda\rho}(k)D^{cb}_{\rho\nu}(k)\,. \quad (4.10)$$

Defining the renormalization constant Z_3 by

$$D^{\text{tr}\,ab}_{\text{un}\,\mu\nu}(k)\big|_{k^2=-\mu^2} = \frac{i}{\mu^2}Z_3\left(\eta_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2}\right)\delta_{ab} \quad (4.11)$$

(the superscript tr indicates the transverse part), we obtain

$$Z_3 = 1+\frac{g^2}{16\pi^2}C_2(G)\left[\left(\tfrac{13}{3}-\alpha\right)\ln\left(\frac{4\pi}{\kappa\mu}\right)+O(\kappa^0)\right]. \quad (4.12)$$

2. Ghost self-energy

The relevant supergraph is shown in Fig. 10. This gives

$$\Pi_G(x^2) = -ig^2C_2(G)\partial^\mu\left[\left(\eta_{\mu\nu}-\alpha'\frac{\partial_\mu\partial_\nu}{\partial^2}\right)D\partial^\nu D\mathfrak{D}\right], \quad (4.13)$$

which, following the above procedure, finally gives

$$\Pi_G(k^2) = \frac{-g^2C_2(G)}{32\pi^2}\left(\tfrac{3}{2}-\tfrac{1}{2}\alpha\right)k^2\left[\ln\left(\frac{-\kappa^2k^2}{16\pi^2}\right)+O(\kappa^0)\right]. \quad (4.14)$$

Defining $\tilde{Z}_3$ by

$$D^{Gab}_{\text{un}}(-\mu^2) = \frac{i\delta_{ab}}{k^2-\Pi_G(k^2)}\bigg|_{k^2=-\mu^2} = \frac{-i}{\mu^2}\tilde{Z}_3\delta_{ab}, \quad (4.15)$$

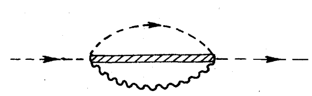

FIG. 10. Lowest-order supergraph for ghost self-energy.

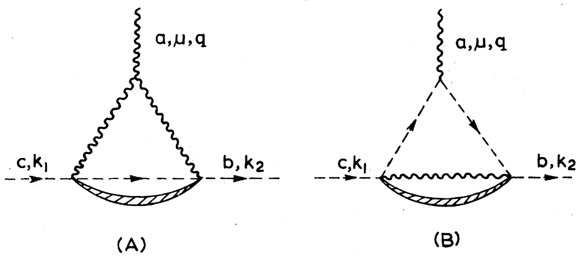

FIG. 11. Supergraphs for lowest-order corrections to the ghost-ghost-vector vertex.

we obtain

$$\tilde{Z}_3 = 1 + \frac{g^2}{16\pi^2} C_2(G)\left[\left(\tfrac{3}{2} - \tfrac{1}{2}\alpha\right) \ln\left(\frac{4\pi}{\kappa\mu}\right) + O(\kappa^0)\right]. \tag{4.16}$$

3. The ghost-ghost-vector vertex

The relevant supergraphs are shown in Fig. 11. As discussed in Sec. II, it is sufficient to include only one superpropagator in each graph to secure ultraviolet convergence. Proceeding as above, we obtain, after a fairly lengthy calculation,[17]

$$\tilde{\Lambda}^{\mu abc}(k_1, k_2, q)|_{k_1^2=k_2^2=q^2=-\mu^2} = \frac{-\alpha g^3}{16\pi^2} C_2(G) f_{abc} k_2^\mu \left[\ln\left(\frac{\kappa\mu}{4\pi}\right) + O(\kappa^0)\right]. \tag{4.17}$$

Defining $\tilde{Z}_1$ by

$$\begin{aligned}\tilde{\Gamma}^{\mu abc}(k_1, k_2, q)|_{k_1^2=k_2^2=q^2=-\mu^2} &= g k_2^\mu f_{abc} + \tilde{\Lambda}^{\mu abc}(k_1, k_2, q)|_{k_1^2=k_2^2=q^2=-\mu^2} \\ &= (\tilde{Z}_1)^{-1} g k_2^\mu f_{abc},\end{aligned} \tag{4.18}$$

we obtain

$$\tilde{Z}_1 = 1 - \frac{\alpha g^2 C_2(G)}{16\pi^2}\left[\ln\left(\frac{4\pi}{\kappa\mu}\right) + O(\kappa^0)\right]. \tag{4.19}$$

4. Vector-vector-vector vertex

The relevant supergraphs are shown in Fig. 12. Denoting the correction to this vertex by $\Lambda^{abc}_{\alpha\beta\gamma}(q_1, q_2, q_3)$ and defining Z_1 by

$$\Lambda^{abc}_{\alpha\beta\gamma}(q_1, q_2, q_3)|_{q_1^2=q_2^2=q_3^2=-\mu^2} = (Z_1^{-1} - 1)(-g) f_{abc}[\eta_{\alpha\beta}(q_1 - q_2)_\gamma + \eta_{\beta\gamma}(q_2 - q_3)_\alpha + \eta_{\gamma\alpha}(q_3 - q_1)_\beta],$$

we obtain

$$Z_1 = 1 + \frac{g^2 C_2(G)}{16\pi^2}\left(\tfrac{17}{6} - \tfrac{3}{2}\alpha\right) \ln\left(\frac{4\pi}{\kappa\mu}\right) + O(\kappa^0). \tag{4.20}$$

The renormalization constants calculated above agree with the conventional ones[20] with the ultraviolet cutoff Λ replaced by κ^{-1} and satisfy, to order g^2, the Ward identity

$$\frac{Z_3}{Z_1} = \frac{\tilde{Z}_3}{\tilde{Z}_1}. \tag{4.21}$$

B. Gauge invariance

To make up for the lack of gauge invariance, as reflected in the longitudinal part of the gauge particle self-energy (in order κ^0), we shall follow a procedure analogous to the one adopted in Ref. 5, namely, to employ gauge-covariant point separation of field operator products and to incorporate appropriately some effects of the Lagrangian $\mathcal{L}'$ in (4.4). Gauge-covariant Yang-Mills field tensors have been constructed in Ref. 23 which should replace $F^a_{\mu\nu}$ above. These are conveniently expressed in terms of the gauge-covariant phase factors[24]

$$\underline{Y}(x) = P\left[\exp\left(\frac{ig}{2}\int_{-\infty}^{x} \underline{A}_\mu(\xi)\, d\xi^\mu\right)\right], \tag{4.22}$$

where P is the ξ-ordering operator, $\underline{A}_\mu \equiv A^a_\mu t_a$, and t_a are matrices for the generators of G in the adjoint representation normalized as $\mathrm{Tr}(t_a t_b) = \delta_{ab}$. The desired field tensor matrix $\underline{F}'_{\mu\nu}$ (which

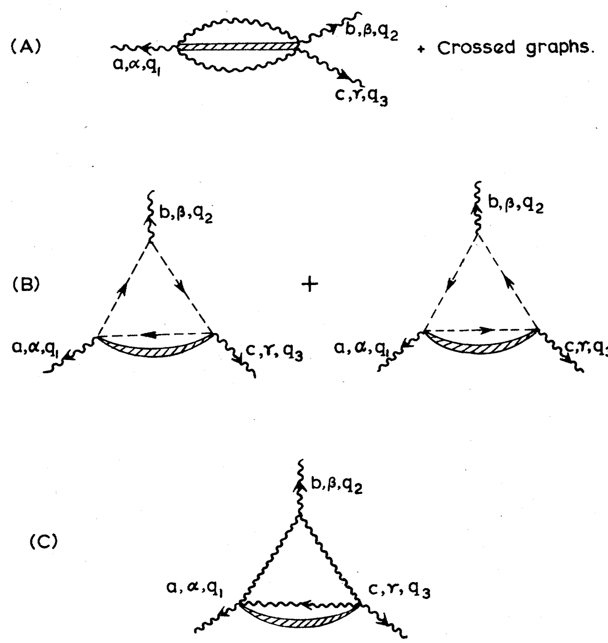

FIG. 12. Supergraphs for lowest-order corrections to the vector-vector-vector vertex.

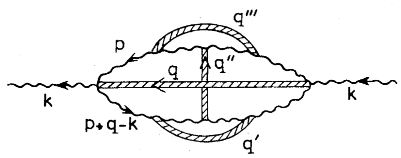

FIG. 13. A "gauge-covariant" supergraph for gauge particle self-energy.

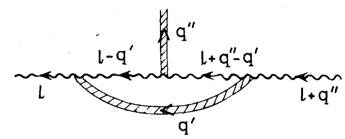

FIG. 14. Vector-vector-graviton supervertex.

replaces $\underline{F}_{\mu\nu}\equiv F^a_{\mu\nu}t_a$ above) is defined as

$$\underline{F}'_{\mu\nu}(x)=\tfrac{1}{2}Y^{-1}(x,x-\epsilon)D_\mu(x-\epsilon)\{Y(x,x-\epsilon)\underline{A}_\nu(x)\}-\tfrac{1}{2}Y^{-1}(x,x-\epsilon)D_\nu(x-\epsilon)\{Y(x,x-\epsilon)\underline{A}_\mu(x)\}+\text{H.c.}, \quad (4.23)$$

where

$$D_\mu(x)=\partial_\mu-\frac{ig}{2}\underline{A}_\mu(x)$$

and

$$Y(x,x-\epsilon)=\underline{Y}^{-1}(x-\epsilon)\underline{Y}(x)=I-\frac{g}{2}\epsilon^\lambda\underline{A}_\lambda(x-\epsilon/2)+O(\epsilon^2). \quad (4.24)$$

Taking into account the extra ϵ-dependent terms and going to the limit $\epsilon\to 0$ in the end, it has been shown[23] that in the radiation gauge, $\Pi_{\mu\nu}$ has a gauge-invariant structure in the one-loop approximation; one expects the same in a general Lorentz-covariant gauge.

To convince oneself it is useful to recall that with the modified Yang-Mills Lagrangian constructed as above, one could follow Mandelstam's procedure[25] employing path-dependent gauge-covariant objects maintaining manifest gauge covariance at every stage. More precisely, writing

$$\Pi_{\mu\nu}(k,\epsilon)=(k_\mu k_\nu-k^2\eta_{\mu\nu})C(k^2,\epsilon)+\eta_{\mu\nu}D(k^2,\epsilon), \quad (4.25)$$

one has

$$\lim_{\epsilon^\nu\to 0}D(k^2,\epsilon)=0. \quad (4.26)$$

This situation is analogous to the one in QED where a similar gauge-covariant point separation ensures[26] vanishing of $D(k^2)$ in the photon self-energy.

When this modified Yang-Mills theory is coupled to gravity, the functions $C(k^2,\epsilon)$ and $D(k^2,\epsilon)$ (now free of UV infinities) are also functions of κ and it is again nontrivial to verify that to order κ^0, the property (4.26) is preserved. With lowest-order supergraphs as in Fig. 9, we will now have

$$D(k^2,\epsilon)=\frac{1}{2\pi i}\int_{C_0}dz\,\Gamma(-z)D(k^2,\epsilon;z)=D(k^2,\epsilon;0)+\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}dz\,\Gamma(-z)D(k^2,\epsilon;z),\quad 0<\beta<1 \quad (4.27)$$

where the first term on the right-hand side is nothing but the quantity $D(k^2,\epsilon)$ calculated in the theory without gravity and satisfying (4.26). Now, if the quantity $D(k^2,\epsilon;z)$ in the second term were finite for $\text{Re}\,z>0$, we will have, on folding the contour on the real axis, terms of order $\kappa^2\ln\kappa$ and higher, thereby ensuring (4.26) to order κ^0. This, however, is not so; the contour in (4.27), in fact, has to be shifted to the left of the imaginary axis to secure ultraviolet convergence of the momentum integrals in $D(k^2,\epsilon;z)$. Following Ref. 5, we will now show that by suitably incorporating effects of $\mathcal{L}'$ in (4.4), we can secure convergence of $D(k^2,\epsilon;z)$ for $\text{Re}\,z>0$, thereby eliminating the need of shifting the contour to the left and ensuring (4.26) to order κ^0.

To incorporate these effects, it is sufficient[3,5] to replace the supergraph in Figs. 9(a) by the one in Fig. 13; a similar replacement should be made for the supergraph in Fig. 9(b). The supergraph in Fig. 13 represents the sum of diagrams with and without the indicated modifications due to $\mathcal{L}'$ so that each of the superpropagator lines with momenta q', q'', and q''' represents the sum of zero, one, two, ... graviton exchanges (and not merely one, two, ... graviton exchanges as the structure of $\mathcal{L}'$ might suggest). In the language of Ref. 3, "kinking" and "cradling" is supposed to have been done.

To discuss the convergence of the supergraph in Fig. 13, we follow the traditional approach[27] and examine the large-momentum behavior of the integrand with respect to various subintegrations. First, we consider the "supervertex" in Fig. 14; the corresponding amplitude may be written

$$M(l,q'')=\int_{\beta-i\infty}^{\beta+i\infty}dz\,(\kappa^2)^z h(z)\int d^4q'\,l\cdot(l-q')\frac{1}{(l-q')^2}(l-q')\cdot(l+q''-q')\frac{1}{(l+q''-q')^2}\times(l+q''-q')\cdot(l+q'')(q'^2)^{z-2}, \quad (4.28)$$

where $-1<\beta<0$ and the Feynman gauge for the vector propagator has been used for simplicity (the convergence arguments are clearly unaffected by a different choice of gauge). The function $h(z)$ includes various z-dependent factors and constants. The q' integration is clearly convergent. Moreover, setting $|q''|=s^2$ and $|l|=t^2$, we have, for $-1<\beta<-\frac{3}{4}$,

$$\lim_{s\to\infty} M(l,s,\hat{q}'') \lesssim s^{-1}(\ln s)^{n_1}\,, \tag{4.29}$$

$$\lim_{t\to\infty} M(t,\hat{l},q'') \lesssim t(\ln t)^{n_2}\,, \tag{4.30}$$

where n_1 and n_2 are non-negative integers. Finally, to find the behavior of $M(l,q'')$ when both l and q'' tend to infinity, we put $q''=\omega^2\hat{q}''$, $1=a\omega^2\hat{l}$ where a is a fixed positive number, then, for $-1<\beta<-\frac{3}{4}$,

$$\lim_{\omega\to\infty} M(\omega,a\hat{l},\hat{q}'') \lesssim \omega(\ln\omega)^{n_3}\,. \tag{4.31}$$

Now, the amplitude corresponding to Fig. 13 may be written

$$\begin{aligned} D(k^2)^{(a)} \sim \int_{\beta-i\infty}^{\beta+i\infty} dz\, h(z)(\kappa^2)^z \int_{\gamma-i\infty}^{\gamma+i\infty} dz'\,h(z')(\kappa^2)^{z'} \int d^4p\, d^4q''\, d^4q\, [a_1 p + b_1(p+q-k)]^\lambda \frac{1}{p^2} M(p,-q'')\frac{1}{(p+q'')^2} \\ \times \frac{1}{(p+q-k)^2} M(p+q-q''-k,q'')\frac{1}{(p-q''+q-k)^2} \\ \times [a_2(p+q'')+b_2(p-q''+q-k)]_\lambda (q^2)^{z-2}(q''^2)^{z-2}\,, \\ 0<\beta<1\,,\quad -1<\gamma<0 \end{aligned} \tag{4.32}$$

where a_1, a_2, b_1, b_2 are fixed numbers. By power counting we can see that q'', p, and q integrations are all convergent provided we take $0<\beta<\frac{3}{4}$ and $-1<\gamma<-\frac{3}{4}$. No shifting of any contour past an integral value of z is required. This ensures (4.26) for $D(k^2)^{(a)}$; a similar argument works for $D(k^2)^{(b)}$. We expect that similar procedures will ensure gauge invariance in other amplitudes; however, we have not attempted a formal general proof of gauge invariance in the theory.

C. Renormalization constants in quantum chromodynamics

In this section, we present the calculation of renormalization constants in quantum chromodynamics,[28] the theory of strong interactions based on unbroken SU(3) (color) gauge-invariant interactions of fermionic quarks and octet fluons. The Lagrangian of Eq. (4.1) (when now $a,b,\ldots=1,2,\ldots,8$) is to be supplemented by the quark terms (for simplicity, we will consider only one quark flavor) giving

$$\begin{aligned}\mathcal{L}_{\text{total}} = \sqrt{-g}\,[\tfrac{1}{2}i(\bar{\psi}\gamma_\alpha\psi_{;\mu} - \bar{\psi}_{;\mu}\gamma_\alpha\psi)L^{\mu\alpha} - m\bar{\psi}\psi \\ + g\bar{\psi}\gamma_\alpha A_\mu^a T_a\psi L^{\mu\alpha}] + \mathcal{L}(4.1)\,,\end{aligned} \tag{4.33}$$

where T_a are the matrices representing the SU(3) generators in the quark representation. In our approximation, $\psi_{;\mu}$ will be replaced by $\partial_\mu\psi$ and $L^{\mu\alpha}$ by $\eta^{\mu\alpha}$.

The renormalization constants $\tilde{Z}_3$ and $\tilde{Z}_1$ are the same as those given in Eqs. (4.16) and (4.19) with $C_2(G)=3$. The relevant details for other renormalization constants are given below.

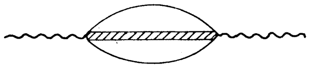

FIG. 15. Fermion-loop supergraph for gluon self-energy.

1. Gluon self-energy

The gluon (gauge particle) self-energy has an additional contribution from the fermion-loop diagram shown in Fig. 15, which, except for the factor $T(R)$, is the same as the expression for photon self-energy in Ref. 5, giving, for the functions $C(k^2)$ and $D(k^2)$, the additional terms

$$\begin{aligned} C^F(k^2) &= \frac{g^2 T(R)}{12\pi^2}\left[\ln\left(\frac{4\pi^2}{\kappa^2 m^2}\right) + O(\kappa^0)\right], \\ D^F(k^2) &= \frac{2}{\kappa^2} g^2 T(R) + O(\ln\kappa)\,. \end{aligned} \tag{4.34}$$

Here $T(R)=\mathrm{Tr}(T_a{}^2)$; for triplets $T(R)=\frac{1}{2}$. Again, there is a longitudinal part in the self-energy, which can be removed by the procedure discussed in Sec. IV B. (The gauge-covariant "point-separated" fermionic current operator in non-Abelian gauge theories is discussed in Ref. 29.) Including the fermionic contribution $-C^F(0)$ to Z_3,

$$\begin{aligned} Z_3 = 1 + \frac{g^2}{16\pi^2} C_2(G)\left[(\tfrac{13}{3}-\alpha)\ln\left(\frac{4\pi}{\kappa\mu}\right) + O(\kappa^0)\right] \\ - \frac{g^2}{6\pi^2} T(R)\left[\ln\left(\frac{2\pi}{m\kappa}\right) + O(\kappa^0)\right]. \end{aligned} \tag{4.35}$$

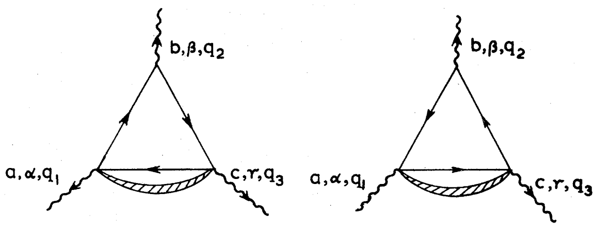

FIG. 16. Fermion-loop supergraphs for the vector-vector-vector vertex.

2. Vector-vector-vector vertex

The additional fermion-loop supergraphs are shown in Fig. 16. By manipulations with Dirac matrices similar to those in the proof of the Ward identity in QED, one easily finds that the contribution of these diagrams to Z_1 is equal to the contribution Z_3^F of Fig. 15 to Z_3 [namely, $-C^F(0)$] giving, finally,

$$Z_1 = 1 + \frac{g^2}{16\pi^2} C_2(G)(\tfrac{17}{6} - \tfrac{3}{2}\alpha)\left[\ln\left(\frac{4\pi}{\kappa\mu}\right) + O(\kappa^0)\right] - \frac{g^2}{6\pi^2} T(R)\left[\ln\left(\frac{2\pi}{m\kappa}\right) + O(\kappa^0)\right]. \tag{4.36}$$

3. Fermion self-energy

The relevant supergraph is shown in Fig. 17. This gives, for the fermion self-energy function,

$$\Sigma(p)_{bc} = \delta_{bc}[\gamma\cdot p A(p^2) + mB(p^2)], \tag{4.37}$$

with

$$A(p^2) = \frac{g^2}{16\pi^2} C_2(R)\alpha\left[\ln\left(\frac{\kappa^2 m^2}{16\pi^2}\right) + O(\kappa^0)\right],$$
$$B(p^2) = -\frac{g^2}{16\pi^2} C_2(R)(3+\alpha)\left[\ln\left(\frac{\kappa^2 m^2}{16\pi^2}\right) + O(\kappa^0)\right], \tag{4.38}$$

where $C_2(R)$, the quadratic Casimir invariant in the representation of the fermions, is defined by

$$\sum_a (T_a{}^2)_{bc} = C_2(R)\delta_{bc}; \tag{4.39}$$

in the triplet representation of SU(3), $C_2(R) = \frac{4}{3}$. From (4.38) we obtain the fermion self-mass

$$\delta m = m[A(m^2) + B(m^2)] = 3m\frac{g^2 C_2(R)}{8\pi^2}\left[\ln\left(\frac{4\pi}{\kappa m}\right) + O(\kappa^0)\right]. \tag{4.40}$$

The fermion wave-function renormalization constant is given by

$$Z_2 = \{1 - A(m^2) - 2m^2[A'(m^2) + B'(m^2)]\}^{-1} = 1 - \frac{\alpha g^2}{8\pi^2} C_2(R)\left[\ln\left(\frac{4\pi}{\kappa m}\right) + O(\kappa^0)\right]. \tag{4.41}$$

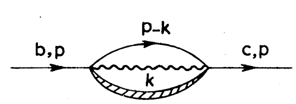

FIG. 17. Fermion self-energy supergraph.

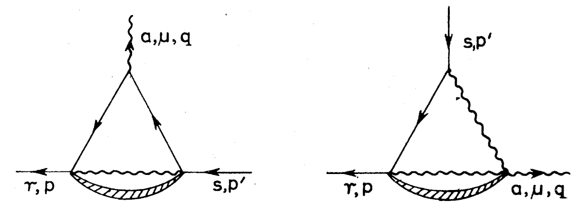

FIG. 18. Supergraphs for fermion-fermion-vector vertex.

4. Fermion-fermion-vector vertex

The relevant supergraphs are shown in Fig. 18; their respective contributions to the vertex correction are

$$\Lambda_{FFV}^{\mu a r s(A)}(p,p') = \frac{ig^3\alpha}{16\pi^2}(T_a)_{rs}\gamma^\mu[C_2(R) - \tfrac{1}{2}C_2(G)] \times\left[\ln\left(\frac{\kappa^2 m^2}{16\pi^2}\right) + O(\kappa^0)\right], \tag{4.42}$$

$$\Lambda_{FFV}^{\mu a r s(B)}(p,p') = -\frac{3i(1+\alpha)g^3}{64\pi^2} C_2(G)(T_a)_{rs}\gamma^\mu \times\left[\ln\left(\frac{\kappa^2 m^2}{16\pi^2}\right) + O(\kappa^0)\right]. \tag{4.43}$$

Defining Z_1' by

$$\Gamma_{FFV\,\text{un}}^{\mu a r s}\big|_{\gamma\cdot p = \gamma\cdot p' = m} = ig(T_a)_{rs}\gamma^\mu + [\Lambda_{FFV}^{\mu a r s(A)}(p,p') + \Lambda_{FFV}^{\mu a r s(B)}(p,p')]_{\gamma\cdot p=\gamma\cdot p'=m} = (Z_1')^{-1} ig(T_a)_{rs}\gamma^\mu, \tag{4.44}$$

we obtain

$$Z_1' = 1 - \frac{g^2}{16\pi^2}\left[[2\alpha C_2(R) + \tfrac{1}{2}(\alpha+3)C_2(G)]\ln\left(\frac{4\pi}{\kappa m}\right) + O(\kappa^0)\right]. \tag{4.45}$$

The expression for the various renormalization constants in the lowest order in g given above correspond to the usual ones[28] with the cutoff Λ replaced by κ^{-1}. To order $\ln\kappa$, the following Ward identities are satisfied:

$$\frac{Z_3}{Z_1} = \frac{\tilde{Z}_3}{\tilde{Z}_1} = \frac{Z_2}{Z_1'}. \tag{4.46}$$

D. Theories with spontaneously broken gauge symmetries

Spontaneously broken gauge theories (SBGT's) combine the features of both—the unbroken gauge theories and the theories with spontaneous symmetry breaking—which we have already considered separately. No new difficulties are expected to arise in the regularization of these theories in our program.

It should be clear from the general discussion in Sec. II that, in our regularization scheme, all amplitudes will be finite in the unitary gauge as well as in any other gauge. Corresponding to higher divergences of the usual unitary gauge amplitudes, our amplitudes will contain higher powers of κ^{-1}; however, renormalized amplitudes (to order κ^0) would be the same as in any other gauge. We believe that, in our framework, it should be possible to evolve a consistent renormalization program in the unitary gauge although we have not attempted that.

SBGT's appear to provide the basis for unified field theories of weak, electromagnetic, and strong interactions. The ultraviolet cutoff provided by gravity has some interesting implications for such unified theories. Using renormalization-group arguments, it appears possible to understand the different strengths of these interactions if one assumes that at energies of the order $\kappa^{-1} \simeq 10^{19}$ GeV the coupling constants for these interactions are the same.[30,31] Successful implementation of this idea puts some constraints on the possible models.[31,32]

V. CONCLUDING REMARKS

The main strong points of regularization by gravity are its naturalness and universality. Its main weak point is its mathematical complexity which has been shown in the present work to be greatly reduced—without sacrificing any of its good features—by employing a suitable approximation.

An important test for any acceptable regularization scheme is that it should respect various global and local symmetries in the theory. In our formalism, whereas there is no problem regarding Poincaré invariance and global internal symmetries, including spontaneously broken symmetries and chiral symmetries (where the dimensional regularization scheme has problems[33,34], local gauge invariance poses special problems. While we have found a way to solve these problems, it is clearly desirable to have a simpler alternative.

When fermions are present in a gauge theory, the axial-vector Ward identity has anomalous terms.[35] A regularization scheme should reproduce these anomalies correctly. Our scheme satisfies this requirement.[17] Instead of presenting detailed calculations[17] verifying this, we would like to present a simple argument why it should be so. The point is that, once one has ensured that well-defined, gauge-covariant objects are being employed (after point separation, etc.), even equations of motion give the anomalies correctly.[35,36] The possibility of a perturbation-theoretic calculation betraying such authentic equations of motion arises only when the calculational procedure employs ill-defined objects (e.g., divergent integrals) or breaks some symmetries. With our amplitudes always finite and gauge invariance, etc., correctly incorporated, both possibilities are eliminated.

APPENDIX A: FULL TENSOR GRAVITY VERSUS THE APPROXIMATED GRAVITY

With full tensor gravity, the following superpropagator[2,3,13] appears in calculations:

$$\begin{aligned}\mathfrak{D}^{\mu a,\nu b}(x) &\equiv \left\langle 0 \left| T\left(\frac{L^{\mu a}(x)}{\det L(x)}\,\frac{L^{\nu b}(0)}{\det L(0)}\right)\right| 0\right\rangle \\ &= \sum_{n=0}^{\infty}[\eta^{\mu a}\eta^{\nu b}\mathfrak{D}^{(0)}(n)+\tfrac{1}{2}(\eta^{\mu\nu}\eta^{ab}+\eta^{\mu b}\eta^{\nu a}-\eta^{\mu a}\eta^{\nu b})\mathfrak{D}^{(1)}(n)]\frac{1}{n!}[-\kappa^2 D(x)]^n \\ &= \frac{1}{2\pi i}\int_{C_0} dz\,\Gamma(-z)[\eta^{\mu a}\eta^{\nu b}\mathfrak{D}^{(0)}(z)+\tfrac{1}{2}(\eta^{\mu\nu}\eta^{ab}+\eta^{\mu b}\eta^{\nu a}-\eta^{\mu a}\eta^{\nu b}\mathfrak{D}^{(1)}(z)][\kappa^2 D(x)]^z,\end{aligned} \tag{A1}$$

where $\det L \equiv \det(L^{\mu\nu})$. It is clear that

$$\mathfrak{D}^{(0)}(0)=1\,,\quad \mathfrak{D}^{(1)}(0)=0\,, \tag{A2}$$

whatever the parametrization for $L^{\mu a}(x)$. Detailed expressions for these functions may be found in Ref. 13 for the exponential parametrization (2.5) and in Ref. 2 for the rational parametrization $L^{\mu a}=\eta^{\mu a}+\frac{1}{2}\kappa\phi^{\mu a}$; we shall, however, require only the property (A2) in our discussion.

In our approximation, the above superpropagator is replaced by $\eta^{\mu a}\eta^{\nu b}\mathfrak{D}(x)$, where

$$\begin{aligned}\mathfrak{D}(x) &\equiv \langle 0|\,T(:e^{-\kappa\chi(x)}:\,:e^{-\kappa\chi(0)}:)|\,0\rangle \\ &= \exp[-\kappa^2 D(x)] \\ &= \frac{1}{2\pi i}\int_{C_0} dz\,\Gamma(-z)[\kappa^2 D(x)]^z\,.\end{aligned} \tag{A3}$$

This clearly corresponds to replacing $\mathfrak{D}^{(0)}(z)$ and $\mathfrak{D}^{(1)}(z)$ in (A1) by their values at $z=0$.

We now consider the regularization of a general amplitude through the graviton superpropagator. We shall restrict ourselves to graphs with one loop only. As stated in Sec. II, it is sufficient, for loops with more than two vertices, to include only one superpropagator. The amplitude $A(p)$ corresponding to such a one-loop graph with one superpropagator may be written as

$$A(p)=\int d^4q\, d^4k\, f_{\mu\nu ab}(p,q,k)\tilde{\mathfrak{D}}^{\mu a,\nu b}(q)\,, \tag{A4}$$

where p symbolically represents all external momenta, q is the superpropagator momentum, and k is an appropriately chosen loop momentum. Using (A1) and (2.13), we have

$$A(p)=\frac{1}{2\pi i}\int_{C_n} dz\,\Gamma(-z)(\kappa^2)^z K^{\mu\nu ab}(z)F_{\mu\nu ab}(p,z)\,, \tag{A5}$$

where $K^{\mu\nu ab}(z)$ stands for the quantity in square brackets containing $\mathfrak{D}^{(0)}(z)$ and $\mathfrak{D}^{(1)}(z)$ in (A1) and the contour C_0 of (A1) has been shifted appropriately to the left to guarantee convergence; moreover,

$$F_{\mu\nu ab}(p,z)=-i\frac{\Gamma(2-z)(4\pi)^{2-2z}}{\Gamma(z)}\times\int d^4q\, d^4k\, f_{\mu\nu ab}(p,q,k)(-q^2)^{z-2}\,. \tag{A6}$$

In the unregularized theory, one would have

$$A(p)_{\text{unreg}}=\eta^{\mu a}\eta^{\nu b}F_{\mu\nu ab}(p,0)\,, \tag{A7}$$

which is infinite if the amplitude has ultraviolet divergences. (We ignore infrared divergences to simplify the argument.) As explained in Sec. II, these ultraviolet infinities are reflected in the singularities of $F_{\mu\nu ab}(p,z)$ in z which are simple poles at $z=-n,\ -n+1,\dots,-1,0$. Here n denotes the degree of ultraviolet divergence in the original amplitude.

To evaluate $A(p)$, one folds the contour C_n in (A5) towards the right so as to enclose the part of the real axis with $\mathrm{Re}\,z\geq -n$ and evaluate contributions from the simple poles at $z=-n,\ -n+1,\dots,-1$ and the double pole at $z=0$. The $\ln\kappa$ term in $A(p)$ comes from the double pole at $z=0$; the equality of this term in the theory with full tensor gravity with that in our approximation is guaranteed by Eq. (A2).

We note that, when the amplitude (A7) of the unregularized theory has no ultraviolet divergences, the function $F_{\mu\nu ab}(p,z)$ will not have any singularities at $z=0,\ -1,\ -2,\dots$, and in Eq. (A5), the contribution of the simple pole at $z=0$ will now give precisely the amplitude (A7).

Now, in a renormalizable theory, when the usual subtractions are made in $A(p)$ (assumed divergent, as above), the integrand in (A5) will contain, instead of $F_{\mu\nu ab}(p,z)$, a function $\tilde{F}_{\mu\nu ab}(p,z)$ which will not have singularities at $z=0,\ -1,\ -2,\dots$ (for the same reasons as in the preceding paragraph); the contribution from the simple pole at $z=0$ now gives the usual finite part

$$\eta^{\mu a}\eta^{\nu b}\tilde{F}_{\mu\nu ab}(p,0) \tag{A8}$$

both in the full tensor gravity theory as well as in our approximation. Indeed, it was verified in second-order calculations in Ref. 5 that, after the usual subtractions, the κ^0 terms in the electron and photon self-energy functions were the same as their usual finite parts in conventional QED.

APPENDIX B: CALCULUS OF DERIVATIVES

Salam and co-workers[3,22] have proposed the following formulas which one gets after naive manipulations and which conserve the number of derivatives for products of distributions containing derivatives (the idea is to avoid encountering worse singularities by actually carrying out the differentiations in expressions like $\partial_\mu\partial_\nu[D(x)]^z$ and then taking products of the resulting objects with other similar distributions):

$$D^{z_1}\partial_\mu D^{z_2}=\frac{z_2}{z_1+z_2}\partial_\mu D^{z_1+z_2}\,, \tag{B1}$$

$$(\partial_\mu D^{z_1})\partial_\nu D^{z_2}=\frac{z_1z_2}{(z_1+z_2)(z_1+z_2+1)}\left(\partial_\mu\partial_\nu+\frac{1}{2(z_1+z_2-1)}\eta_{\mu\nu}\partial^2\right)D^{z_1+z_2}\,, \tag{B2}$$

$$D^{z_1}\partial_\mu\partial_\nu D^{z_2}=\frac{z_2}{(z_1+z_2)(z_1+z_2+1)}\left((1+z_2)\partial_\mu\partial_\nu-\frac{z_1}{2(z_1+z_2-1)}\eta_{\mu\nu}\partial^2\right)D^{z_1+z_2}\,. \tag{B3}$$

These are supposed to be valid for all z_1 and z_2 except when they are obviously meaningless. They have coined the term "calculus of derivatives" for this procedure. The equalities above should be understood in terms of the equalities of the Fourier transforms of the two sides. Some formal justification for these may be found in Ref. 12.

In our gauge theory calculations the above three formulas are sufficient to cope with the derivatives if one works in the Feynman gauge ($\alpha = 1$). In a general gauge we employed, besides the above formulas, the following additional ones:

$$D^z \frac{\partial_\mu \partial_\nu}{\partial^2} D = \frac{1}{2}\left[\frac{z}{z+1}\eta_{\mu\nu} - \frac{2(z-1)}{z+1}\frac{\partial_\mu\partial_\nu}{\partial^2}\right] D^{z+1}, \tag{B4}$$

$$\partial^\nu \left[(\partial^\rho D)\left(\frac{\partial^\mu \partial_\rho D}{\partial^2}\right) D^z \right] = -\frac{1}{2(z+2)}\partial^\mu \partial^\nu D^{z+2}, \tag{B5}$$

$$(\partial^\rho \partial^\nu D)\left(\frac{\partial^\mu \partial_\rho D}{\partial^2}\right) D^z = -\frac{1}{2(z+2)(z+3)}\left[\partial^\mu \partial^\nu + \frac{z+4}{2(z+1)}\eta^{\mu\nu}\partial^2\right] D^{z+2}, \tag{B6}$$

$$\partial^\lambda \partial^\rho \left[\left(\frac{\partial_\lambda \partial_\rho}{\partial^2} D\right)\left(\frac{\partial^\mu \partial^\nu D}{\partial^2}\right) D^z - \left(\frac{\partial^\mu \partial_\lambda D}{\partial^2}\right)\left(\frac{\partial^\nu \partial_\rho D}{\partial^2}\right) D^z \right] = \frac{z}{4(z+2)}(\partial^\mu \partial^\nu - \eta^{\mu\nu}\partial^2) D^{z+2}. \tag{B7}$$

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