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Anisotropic fluids with two-perfect-fluid components

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A two-perfect-fluid model of an anisotropic fluid is presented. The energy-momentum tensor associated with the sum of two perfect fluids, one perfect and one null fluid, and two null fluids is examined. Special attention is devoted to the study of the stress tensor. The special case wherein the two perfect fluids are irrotational is studied. A relation between the Einstein equations for this particular case and the Einstein equations for a massless complex scalar field is found. The general solution of Einstein equations for an anisotropic fluid constructed with two-null-fluid components in the plane-symmetric case is discussed. The energy-momentum tensor of a cloud of strings and the energy-momentum tensor of an anisotropic fluid formed by two null fluids are compared.

I. INTRODUCTION

The energy-momentum tensor associated with a perfect fluid has been widely studied in general relativity (GR) as a source of the gravitational field, mainly to describe models of stars, galaxies, and universes.[1,2] Imperfect fluids have seldom been studied as a source of the gravitational field because of the mathematical difficulties associated with such models, i.e., the field equations cannot be solved exactly even for the most simple cases. In most astrophysical applications perfect-fluid models appear to be adequate.[2]

We believe that the study of imperfect fluids and particularly the study of anisotropic fluids in GR has value if the model possesses some of the following characteristics: (i) The model must be simple enough that it can be solved exactly for some important particular cases. (ii) The physical interpretation of the model can be easily performed and the relation of this model with the perfect-fluid model can be studied in a simple way. (iii) The model can be used to better understand some of the open problems in GR, e.g., to serve as a model for a Kerr metric interior[3] or to serve as a source for a radiating metric,[4] or both.

In this paper we study a model that possesses some of the above-mentioned characteristics. The model shares some common features with the two-fluid model studied in plasma physics.[5] It has the sum of two "currents" (energy-momentum tensors) as the source of the field equations and two "momentum-transfer equations" (energy-momentum conservation equations) as closure relations.

In Sec. II we present a two-perfect-fluid model of an anisotropic fluid. The energy-momentum tensor (EMT) associated with the sum of two perfect fluids, one perfect and one null fluid, and two null fluids is examined. Special attention is devoted to the study of the stress tensor.

In Sec. III we examine the special case wherein the two fluids are irrotational. Particularizing the equation of state of each fluid we find a relation between the Einstein equations for this particular type of anisotropic fluid and the Einstein equations coupled with a massless complex scalar field.

In Sec. IV we solve the field equations for the special case of two null fields with plane symmetry. The solution found is the general one for the special case under consideration.

In Sec. V we discuss some of the possible generalizations and applications of the model. In an appendix we compare the energy-momentum tensor of a cloud of strings with the energy-momentum tensor of an anisotropic fluid formed by two null fluids.

II. THE MODEL

In this section we study some of the algebraic properties of a stress-energy tensor formed from the sum of two tensors, each of which is the EMT of a perfect fluid or a null fluid. We also study the integrability conditions for the Einstein equations coupled with the above-mentioned stress-energy tensors, in other words, we study the integrability conditions for the system of equations[6]

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -T_{\mu\nu}\ , \tag{2.1}$$

where all the symbols in (2.1) have their usual meaning and $T^{\mu\nu}$ is given by one of the following expressions:

$$T^{\mu\nu}(u,v) = t^{\mu\nu}(u) + t^{\mu\nu}(v)\ , \tag{2.2}$$

$$T^{\mu\nu}(u,l) = t^{\mu\nu}(u) + t^{\mu\nu}(l)\ , \tag{2.3}$$

$$T^{\mu\nu}(k,l) = t^{\mu\nu}(k) + t^{\mu\nu}(l)\ . \tag{2.4}$$

The tensors $t^{\mu\nu}(u)$, $t^{\mu\nu}(v)$, $t^{\mu\nu}(l)$, and $t^{\mu\nu}(k)$ are defined as

$$t^{\mu\nu}(u)=(p+w)u^{\mu}u^{\nu}-pg^{\mu\nu}\,, \qquad (2.5)$$

$$t^{\mu\nu}(v)=(q+e)v^{\mu}v^{\nu}-qg^{\mu\nu}\,, \qquad (2.6)$$

$$t^{\mu\nu}(k)=wk^{\mu}k^{\nu}\,, \qquad (2.7)$$

$$t^{\mu\nu}(l)=el^{\mu}l^{\nu}\,, \qquad (2.8)$$

where

$$u^{\mu}u_{\mu}=v^{\mu}v_{\mu}=1,\quad v^{\mu}\neq u^{\mu}\,, \qquad (2.9)$$

$$k^{\mu}k_{\mu}=l^{\mu}l_{\mu}=0,\quad l^{\mu}\neq k^{\mu}\,. \qquad (2.10)$$

The EMT (2.2) is the sum of the energy-momentum tensors of two perfect fluids of pressures p and q and rest energies w and e, respectively. The EMT (2.3) represents the sum of the EMT of a perfect fluid and the EMT of a null fluid of energy w. And the EMT (2.4) describes the sum of the energy-momentum tensors of two null fluids of energies w and e, respectively.

The integrability conditions for the system of equations (2.1) is

$$T^{\mu\nu}{}_{;\nu}=0\,, \qquad (2.11)$$

where the semicolon denotes a covariant derivative. Note that Eq. (2.11) is not enough to determine all the unknowns in the system of Eq. (2.1) when $T_{\mu\nu}$ is given by any of the expressions (2.2)–(2.4). In particular, for the Einstein equations coupled to the EMT given by (2.2) we have as unknowns $g_{\mu\nu}$, p, w, q, e, u^{μ}, and v^{μ}, i.e., 20 unknowns [u^{μ} and v^{μ} have only three independent components due to (2.9)]. And we have 14 equations, the 10 Einstein equations (2.1) and the four Bianchi identities (2.11). So we need to add new conditions to have a well-defined problem. For the system of Eqs. (2.1) and (2.2) we shall add the supplementary conditons

$$t^{\mu\nu}(u)_{;\nu}=0\,, \qquad (2.12a)$$

$$t^{\mu\nu}(v)_{;\nu}=0\,, \qquad (2.12b)$$

$$F_1(p,w,q,e)=0 \quad\text{or}\quad h_1(p,w)=0\,, \qquad (2.12c)$$

$$F_2(p,w,f,e)=0 \quad\text{or}\quad h_2(q,e)=0\,. \qquad (2.12d)$$

Conditions (2.12a) and (2.12b) are the most simple conditions sufficient to satisfy (2.11). Note that they are a type of minimal coupling condition for the two fluids, a more general condition will be discussed in the last section of the present paper. The conditions (2.12c) and (2.12d) are state equations for the two fluids, in particular we shall assume that each fluid has its own equation of state of the form $h_1(p,w)=0$. Note that the system of Eqs. (2.1), (2.2), and (2.12) gives us a well-defined mathematical problem with the same number of equations and unknowns.

For the systems of Eqs. (2.1), (2.2) and (2.1), (2.3), we choose the following set of supplementary conditions, respectively:

$$t^{\mu\nu}(l)_{;\nu}=0\,, \qquad (2.13a)$$

$$t^{\mu\nu}(u)_{;\nu}=0\,, \qquad (2.13b)$$

$$F_3(p,w,e)=0 \quad\text{or}\quad h_3(p,w)=0\,, \qquad (2.13c)$$

$$t^{\mu\nu}(k)_{;\nu}=0\,, \qquad (2.14a)$$

$$t^{\mu\nu}(l)_{;\nu}=0\,. \qquad (2.14b)$$

Note that in each case we have a well-defined mathematical problem. Now we shall analyze the physical meaning of each of the energy-momentum tensors (2.2)–(2.4). To do so we need to cast them in the general form of the EMT for a single fluid,[7] i.e.,

$$T^{\mu\nu}=\rho U^{\mu}U^{\nu}+S^{\mu\nu} \qquad (2.15)$$

with

$$S^{\mu\nu}U_{\nu}=0\,, \qquad (2.16)$$

$$U^{\mu}U_{\mu}=1\,, \qquad (2.17)$$

$$\rho>0\,. \qquad (2.18)$$

$\rho U^{\mu}U^{\nu}$ represents the EMT kinetical part. ρ is the rest energy density and $S^{\mu\nu}$ is the stress tensor.

Let us start by analyzing the EMT (2.2). First, we notice that the "quadratic form"

$$(p+w)u^{\mu}u^{\nu}+(q+e)v^{\mu}v^{\nu} \qquad (2.19)$$

is invariant under the transformation

$$u^{\mu}\to u^{*\mu}=\cos\alpha u^{\mu}+\left(\frac{q+e}{p+w}\right)^{1/2}\sin\alpha v^{\mu}\,, \qquad (2.20a)$$

$$v^{\mu}\to v^{*\mu}=-\left(\frac{p+w}{q+e}\right)^{1/2}\sin\alpha u^{\mu}+\cos\alpha v^{\mu}\,. \qquad (2.20b)$$

Thus

$$T^{\mu\nu}(u,v)=T^{\mu\nu}(u^*,v^*)\,. \qquad (2.21)$$

Now we shall "rotate" the vectors u^{μ} and v^{μ} in such a way that one becomes timelike and the other spacelike. This condition is implemented by requiring that

$$u^{*\mu}v^*_{\nu}=0\,. \qquad (2.22)$$

From (2.22) and (2.20) we get

$$\tan(2\alpha)=\frac{[(p+w)(q+e)]^{1/2}}{p+w-q-e}\,2v^{\mu}u_{\mu}\,. \qquad (2.23)$$

Note that Eqs. (2.9), (2.20), and (2.23) and the fact that u^{μ} and v^{μ} are future oriented imply that $u^{*\mu}u^*_{\mu}>0$ and $v^*_{\mu}v^{*\mu}<0$, i.e., u^*_{μ} is a timelike vector and v^*_{μ} is a spacelike vector.

Now, defining

$$U^\mu \equiv u^{*\mu}/(u^{*\alpha}u^*_\alpha)^{1/2}\,, \tag{2.24a}$$

$$\chi^\mu \equiv v^{*\mu}/(-v^{*\alpha}v_\alpha)^{1/2}\,, \tag{2.24b}$$

$$\rho \equiv T^{\mu\nu}U_\mu U_\nu = (p+w)u^{*\alpha}u^*_\alpha - (p+q)\,, \tag{2.25a}$$

$$\sigma \equiv T^{\mu\nu}\chi_\mu\chi_\nu = p+q-(q+e)v^{*\alpha}v^*_\alpha\,, \tag{2.26a}$$

$$\pi \equiv p+q\,, \tag{2.27}$$

we can cast (2.21) as follows:

$$T^{\mu\nu} = (\rho+\pi)U^\mu U^\nu + (\sigma-\pi)\chi^\mu\chi^\nu - \pi g^{\mu\nu} \tag{2.28a}$$

$$= \rho U^\mu U^\nu + S^{\mu\nu}\,, \tag{2.28b}$$

where

$$S^{\mu\nu} = (\sigma-\pi)\chi^\mu\chi^\nu - \pi(g^{\mu\nu} - U^\mu U^\nu)\,. \tag{2.29}$$

Note that

$$U^\mu U_\mu = -\chi^\mu\chi_\mu = 1\,, \tag{2.30a}$$

$$\chi^\mu U_\mu = 0\,, \tag{2.30b}$$

$$S^{\mu\nu}U_\nu = 0\,, \tag{2.31}$$

$$S^{\mu\nu}\chi_\nu = -\sigma\chi^\mu\,. \tag{2.32}$$

A direct computation shows that

$$\rho = +\tfrac{1}{2}(w-p+e-q) + \tfrac{1}{2}\{(p+w+q+e)^2 + 4(p+w)(q+e)[(u^\mu v_\mu)^2 - 1]\}^{1/2}\,, \tag{2.25b}$$

$$\sigma = -\tfrac{1}{2}(w-p+e-q) + \tfrac{1}{2}[(p+w-q-e)^2 + 4(u_\mu v^\mu)^2(p+w)(q+e)]^{1/2}\,. \tag{2.26b}$$

Thus ρ and σ are positive quantities.

Now let us write the matrix whose elements are $S^{\mu\nu}$ given by Eq. (2.29) in the tangent space ($g_{\mu\nu} = \eta_{\mu\nu}$), when $U^\mu = \delta^\mu_0$ and $\chi^\mu = \delta^\mu_1$,

$$\|S^{\mu\nu}\| = \begin{bmatrix} 0 & & & \\ & \sigma & & \\ & & \pi & \\ & & & \pi \end{bmatrix}\,. \tag{2.33}$$

So the EMT (2.28) describes an anisotropic fluid, i.e., we have a pressure σ along the δ^μ_1 direction and a pressure π on the perpendicular plane to δ^μ_1. Note that from (2.26) and (2.27) we have $\sigma > \pi$.

The EMT (2.3) can also be cast in the form (2.28). But now the transformation that leaves (2.3) invariant is

$$u^{*\mu} = \cos\alpha\, u^\mu + \left(\frac{e}{w+p}\right)^{1/2}\sin\alpha\, l^\mu\,, \tag{2.34a}$$

$$l^{*\mu} = -\left(\frac{w+p}{e}\right)^{1/2}\sin\alpha\, u^\mu + \cos\alpha\, l^\mu\,. \tag{2.34b}$$

The condition

$$u^{*\mu} l^*_\mu = 0 \tag{2.35}$$

gives us

$$\tan(2\alpha) = \left(\frac{e}{w+p}\right)^{1/2} 2u^\alpha l_\alpha\,. \tag{2.36}$$

From (2.36), (2.34), and (2.35) we get that $u^{*\mu}$ and $l^{*\mu}$ are a timelike and a spacelike vector, respectively. Now the quantities that appear in (2.28) have the following values, U^μ is defined as in (2.24a), and

$$\chi^\mu = l^{*\mu}/(-l^*_\alpha l^{*\alpha})^{1/2}\,, \tag{2.37}$$

$$\rho = \tfrac{1}{2}\{w-p+[(w+p)^2 + 4e(p+w)(u^\alpha l_\alpha)^2]^{1/2}\} > 0\,, \tag{2.38}$$

$$\sigma = \tfrac{1}{2}\{p-w+[(w+p)^2 + 4e(p+w)(u^\alpha l_\alpha)^2]^{1/2}\} > 0\,, \tag{2.39}$$

$$\pi = p\,. \tag{2.40}$$

Finally, we have that EMT (2.4) is invariant under the transformation

$$k^{*\mu} = \cos\alpha\, k^\mu + (e/w)^{1/2}\sin\alpha\, l^\mu\,, \tag{2.41a}$$

$$l^{*\mu} = -(w/e)^{1/2}\sin\alpha\, k^\mu + \cos\alpha\, l^\mu\,. \tag{2.41b}$$

The condition

$$k^{*\mu} l^*_\mu = 0 \tag{2.42}$$

gives us

$$\tan^2\alpha = 1\,. \tag{2.43}$$

The null vectors l^μ and k^μ are assumed to be future oriented, i.e., each one represents the velocity of a *bona fide* null fluid. Thus, from (2.41) and (2.43) we have that $k^{*\mu} k^*_\mu > 0$ and $l^{*\mu} l^*_\mu < 0$.

The EMT (2.4) reduces to

$$T^{\mu\nu}(k,l) = \rho U^\mu U^\nu + \sigma\chi^\mu\chi^\nu\,, \tag{2.44}$$

where

$$\rho = \sigma = \sqrt{we}\; l^\mu k_\mu > 0\,, \tag{2.45}$$

$$U^\mu = k^{*\mu}/(k^{*\alpha}k^*_\alpha)^{1/2}\,, \tag{2.46a}$$

$$\chi^\mu = l^{*\mu}/(-l^{*\alpha}l^*_\alpha)^{1/2}\,. \tag{2.46b}$$

Now we shall discuss a property common to the fluids described by the EMT (2.3) and (2.4). First, we notice that since the null vectors k^μ and l^μ are not restricted in any way they can be replaced by λk^ν and γl^ν, where λ and γ are arbitrary scalar functions different from zero. Then w and e become w/λ^2 and e/γ^2, respectively. Under these "gauge" transformations the EMT (2.3) and (2.4) are invariant, as well as the equations of motion (2.13a), (2.13b), (2.14) and the relations (2.36), (2.38), (2.39), and (2.45). If we transform $k^{*\mu}$ and

$l^{*\mu}$ in the same way as k^μ and l^μ, hence as well as the transformations (2.34) and (2.41), the relations (2.37), (2.44), and (2.46) are invariant; in other words, the complete theory is invariant under the above-mentioned gauge transformations.

In the case of a fluid described by the EMT (2.3), we can use this gauge invariance to "normalize" l^μ by requiring $u^\mu l_\mu =$ constant. And for the fluid described by (2.4) we can partially normalize l^μ and k^μ by requiring $k^\mu l_\mu =$ constant, i.e., it is always possible to find a λ and γ to achieve these types of normalizations.

To end this section we want to add that instead of using transformations like (2.20) to cast each of the energy-momentum tensors (2.2)–(2.4) in the form (2.15), we could have used the equivalent method of solving the eigenvalue problem for each of the above-mentioned tensors. It happens that in this case the method that we have followed is simpler.

III. ANISOTROPIC FLUIDS WITH IRROTATIONAL FLUID COMPONENTS

In this section we study the model of the anisotropic fluid presented in Sec. II in the particular case that each of the perfect-fluid components used to describe the model are irrotational, i.e., their rotation tensors are zero,

$$\omega_{\mu\nu}(u) \equiv h^\alpha{}_\mu(u) h^\beta{}_\nu(u) u_{[\alpha;\beta]} = 0 , \tag{3.1a}$$

$$\omega_{\mu\nu}(v) \equiv h^\alpha{}_\mu(v) h^\beta{}_\nu(v) v_{[\alpha;\beta]} = 0 , \tag{3.1b}$$

where $h^\alpha{}_\mu(u)$ and $h^\beta{}_\mu(v)$ are defined by

$$h^\alpha{}_\mu(u) = \delta^\alpha{}_\mu - u^\alpha u_\mu , \tag{3.2a}$$

$$h^\beta{}_\mu(v) = \delta^\beta{}_\mu - v^\alpha v_\mu . \tag{3.2b}$$

First, let us study the integrability condition (2.12a). From (2.12a) and (2.5) we get

$$u^\nu w_{,\nu} = -(p+w) u^\nu{}_{;\nu} , \tag{3.3}$$

$$h^\nu{}_\mu(u) p_{,\nu} = (p+w) u^\nu u_{\mu;\nu} . \tag{3.4}$$

Now assuming an equation of state for the fluid with velocity u^μ of the form

$$p = p(w) , \tag{3.5}$$

(3.3) can be written as[8]

$$[e^{-\Sigma}(p+w)u^\mu]_{;\mu} = 0 , \tag{3.6}$$

where

$$\Sigma(w) = \int \frac{dp}{p+w} . \tag{3.7}$$

From (3.1) we get

$$u_\mu = \phi_{,\mu} / (\phi_{,\alpha}\phi^{,\alpha})^{1/2} , \tag{3.8}$$

$$v_\mu = \psi_{,\mu} / (\psi_{,\alpha}\psi^{,\alpha})^{1/2} , \tag{3.9}$$

where ϕ and ψ are arbitrary functions subject to the conditions $\phi^{,\alpha}\phi_{,\alpha} > 0$ and $\psi_{,\alpha}\psi^{,\alpha} > 0$.

From (3.4), (3.7), and (3.8) we have

$$[\Sigma(w) - \ln(\phi_{,\alpha}\phi^{,\alpha})^{1/2}]_{,\mu} = \phi_{,\mu} F(\phi_{,\alpha}, \phi_{;\alpha\beta}, \Sigma_{,\lambda}) , \tag{3.10}$$

where F is a function of the indicated variables.[9]

Equation (3.10) tells us that

$$\Sigma(w) - \ln(\phi_{,\alpha}\phi^{,\alpha})^{1/2}$$

is a function only of ϕ. And from the fact that (3.8) is invariant under a transformation of ϕ of the form $\phi \to f(\phi)$, it follows that there is no loss of generality in choosing

$$\Sigma = \tfrac{1}{2} \ln(\phi_{,\alpha}\phi^{,\alpha}) . \tag{3.11}$$

From (3.11), (3.8), and (3.6) we get

$$[e^{-2\Sigma(w)}(p+w)\phi_{,\mu} g^{\mu\nu}\sqrt{-g}\,]_{,\nu} = 0 . \tag{3.12}$$

One of the most important particular cases of state equations of the form (3.5) is[10]

$$p = (1-\gamma)w - p_0 , \quad 0 \leq \gamma < 1 , \tag{3.13}$$

where γ and p_0 are constants.

From (3.7), (3.11), and (3.13) we obtain

$$w = \frac{1}{2-\gamma}\left[(\phi_{,\alpha}\phi^{,\alpha})^{(1-\gamma/2)/(1-\gamma)} + p_0\right] \tag{3.14}$$

and from (3.12), (3.13), (3.11), and (3.14) we have

$$[(\phi_{,\alpha}\phi^{,\alpha})^{\gamma/2(1-\gamma)}\phi_{,\mu} g^{\mu\nu}\sqrt{-g}\,]_{,\nu} = 0 . \tag{3.15}$$

Assuming the following equations of state for the fluid with velocity v^μ,

$$q = (1-\beta)e - q_0 , \tag{3.16}$$

we can cast e and the integrability condition (2.12b) in a similar form to (3.14) and (3.15), i.e.,

$$e = \frac{1}{2-\beta}\left[(\psi_{,\alpha}\psi^{,\alpha})^{(1-\beta/2)/(1-\beta)} + q_0\right] , \tag{3.17}$$

$$[(\psi_{,\alpha}\psi^{,\alpha})^{\beta/2(1-\beta)}\psi_{,\mu} g^{\mu\nu}\sqrt{-g}\,]_{,\nu} = 0 . \tag{3.18}$$

In this case the EMT (2.2) reduces to

$$T_{\mu\nu}(u,v) = (\phi_{,\alpha}\phi^{,\alpha})^{\gamma/2(1-\gamma)}\phi_{,\mu}\phi_{,\nu} + (\psi_{,\alpha}\psi^{,\alpha})^{\beta/2(1-\beta)}\psi_{,\mu}\psi_{,\nu} - \left[\frac{1-\gamma}{2-\gamma}(\phi_{,\alpha}\phi^{,\alpha})^{(1-\gamma/2)/(1-\gamma)} + \frac{1-\beta}{2-\beta}(\psi_{,\alpha}\psi^{,\alpha})^{(1-\beta/2)/(1-\beta)}\right] g_{\mu\nu} - \lambda g_{\mu\nu} , \tag{3.19a}$$

where

$$\lambda \equiv \frac{p_0}{\gamma-2} + \frac{q_0}{\beta-2}. \tag{3.19b}$$

Note that the inclusion of the constants p_0 and q_0 in (3.13) and (3.16) has the same effect as adding a cosmological constant to the left-hand side of Eq. (2.1). It is also interesting to point out that the EMT (3.19), as well as the motion equations (3.18) and (3.15), can be derived in the usual form from the action constructed with the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left[-\lambda + \frac{1-\gamma}{2-\gamma}(\phi^{,\alpha}\phi_{,\alpha})^{(1-\gamma/2)/(1-\gamma)} + \frac{1-\beta}{2-\beta}(\psi^{,\alpha}\psi_{,\alpha})^{(1-\beta/2)/(1-\beta)}\right] \tag{3.20}$$

$$= \sqrt{-g}\,(p+q). \tag{3.21}$$

The relation of ϕ and ψ with the one-fluid variables can be found using the relation

$$\tan 2\alpha = 2\phi^{,\mu}\psi_{,\mu}\frac{(\phi^{,\alpha}\phi_{,\alpha})^{(\gamma/2)/(1-\gamma)}(\psi^{,\nu}\psi_{,\nu})^{(\beta/2)/(1-\beta)}}{(\phi^{,\alpha}\phi_{,\alpha})^{(1-\gamma/2)/(1-\gamma)} - (\psi^{,\nu}\psi_{,\nu})^{(1-\beta/2)/(1-\beta)}}. \tag{3.22}$$

Let us further particularize the state equation of the perfect-fluid components by assuming

$$\gamma = \beta = p_0 = q_0 = 0, \tag{3.23}$$

i.e.,

$$p = w, \quad q = e. \tag{3.24}$$

Thus, each fluid component obeys the "stiff" equation of state, pressure equal to rest energy density.

From (3.23), (3.15), and (3.18) we get

$$(\phi_{,\mu}g^{\mu\nu}\sqrt{-g})_{,\nu} = 0, \tag{3.25a}$$

$$(\psi_{,\mu}g^{\mu\nu}\sqrt{-g})_{,\nu} = 0. \tag{3.25b}$$

And from (3.23), (3.19), and (3.20) we have

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} + \psi_{,\mu}\psi_{,\nu} - \tfrac{1}{2}(\phi_{,\alpha}\phi^{,\alpha} + \psi_{,\beta}\psi^{,\beta})g_{\mu\nu}, \tag{3.26}$$

$$\mathcal{L} = \tfrac{1}{2}\sqrt{-g}\,(\phi^{,\alpha}\phi_{,\alpha} + \psi^{,\alpha}\psi_{,\alpha}). \tag{3.27}$$

Note that (3.25), (3.26), and (3.27) are, respectively, the field equations, the metric EMT, and the Lagrangian density for a massless, complex scalar field. Now let us define a complex scalar field Λ as follows:

$$\Lambda \equiv \phi + i\psi, \quad \bar{\Lambda} \equiv \phi - i\psi. \tag{3.28}$$

It is a matter of direct computation to cast all the anisotropic fluid variables in terms of Λ,

$$\tan 2\alpha = \frac{\mathrm{Im}\Lambda_{,\alpha}\Lambda^{,\alpha}}{\mathrm{Re}\Lambda_{,\beta}\Lambda^{,\beta}}, \tag{3.29}$$

$$e^{2i\alpha} = \frac{\Lambda^{,\alpha}\Lambda_{,\alpha}}{|\Lambda^{,\beta}\Lambda_{,\beta}|}, \tag{3.30}$$

$$\sqrt{2}\,e^{i\alpha} = \left[1 + \frac{\mathrm{Re}\Lambda_{,\alpha}\Lambda^{,\alpha}}{|\Lambda_{,\beta}\Lambda^{,\beta}|}\right]^{1/2} + i\left[1 - \frac{\mathrm{Re}\Lambda_{,\alpha}\Lambda^{,\alpha}}{|\Lambda_{,\beta}\Lambda^{,\beta}|}\right]^{1/2}, \tag{3.31}$$

$$\rho = \sigma = \tfrac{1}{2}|\Lambda^{,\alpha}\Lambda_{,\alpha}|, \tag{3.32}$$

$$\pi = \mathcal{L}/\sqrt{-g} = \tfrac{1}{2}\bar{\Lambda}^{,\alpha}\Lambda_{,\alpha}, \tag{3.33}$$

$$U_\mu = \mathrm{Re}(e^{i\alpha}\bar{\Lambda}_{,\mu})/[\mathrm{Re}(e^{i\alpha}\bar{\Lambda}_{,\nu})\,\mathrm{Re}(e^{i\alpha}\bar{\Lambda}^{,\nu})]^{1/2}, \tag{3.34}$$

$$\chi_\mu = -\mathrm{Im}(e^{i\alpha}\bar{\Lambda}_{,\mu})/[-\mathrm{Im}(e^{i\alpha}\bar{\Lambda}_{,\nu})\,\mathrm{Im}(e^{i\alpha}\bar{\Lambda}^{,\nu})]^{1/2}. \tag{3.35}$$

Equation (3.32) tells us that the stiffness condition for each fluid, Eq. (3.24), is propagated along the direction of anisotropy. And Eq. (3.34) says that U_μ cannot be cast in a form similar to (3.8); thus the condition of irrotationality of each component is not propagated to the anisotropic fluid model.

It is a rather surprising result that the theory of Tabensky and Taub,[8] relating the Einstein equations coupled to an irrotational perfect fluid with stiff equation of state to the Einstein equations coupled with a massless scalar field,[11] can be generalized to include *complex* scalar fields.

IV. PLANE-SYMMETRIC ANISOTROPIC FLUID WITH TWO-NULL-FLUID COMPONENTS

In this section we study the solution to Einstein equations coupled to the EMT (2.4) when the space-time has plane symmetry. The most general form of the metric for plane symmetry is[12]

$$ds^2 = e^{\omega}du\,dv - e^{\mu}(dx^2 + dy^2), \tag{4.1}$$

where ω and μ are functions of u and v. Two different null vectors for plane symmetry are

$$k^\mu = \delta^\mu_{(+)}, \tag{4.2}$$

$$l^\mu = \delta^\mu_{(-)}, \tag{4.3}$$

where by (+) and (−) we denote the coordinates u and v, respectively.

The Einstein equations (2.1) coupled to the energy-momentum tensor (2.4) constructed with the null vectors (4.2) and (4.3) for the metric (4.1) reduce to

$$\mu_{++}+\tfrac{1}{2}\mu_+{}^2-\mu_+\omega_+=-\tfrac{1}{4}e\exp(2\omega)\,, \tag{4.4}$$

$$\mu_{--}+\tfrac{1}{2}\mu_-^2-\mu_-\omega_-=-\tfrac{1}{4}w\exp(2\omega)\,, \tag{4.5}$$

$$\mu_{-+}+\tfrac{1}{2}\mu_+\mu_-+\omega_{+-}=0\,, \tag{4.6}$$

$$(e^{\mu})_{+-}=0\,, \tag{4.7}$$

where we have introduced the notation $\mu_+=\partial\mu/\partial u$, $\mu_-=\partial\mu/\partial v$, etc.

The integrability conditions (2.14) give us

$$w_+ + w(2\omega_+ + \mu_+)=0\,, \tag{4.8}$$

$$e_- + e(2\omega_- + \mu_-)=0\,. \tag{4.9}$$

The general solutions to Eqs. (4.7), (4.8), and (4.9) are, respectively,

$$e^{\mu}=t\equiv f(u)+h(v)\,, \tag{4.10}$$

$$w=4A(v)\exp[-(2\omega+\mu)]\,, \tag{4.11}$$

$$e=4B(u)\exp[-(2\omega+\mu)]\,, \tag{4.12}$$

where f, h, A, and B are arbitrary functions of the indicated arguments. From (4.4), (4.5), and (4.10)–(4.12) we get

$$\omega_+=(f_{++}+B)/f_+ - f_+/2t\,, \tag{4.13}$$

$$\omega_-=(h_{--}+A)/h_- - h_-/2t\,. \tag{4.14}$$

Note that Eq. (4.6) is fulfilled by either (4.13) or (4.14). From (4.13) and (4.14) we obtain

$$\omega=\ln(f_+h_-)-\tfrac{1}{2}\ln t+b(u)+a(v)+\omega_0\,, \tag{4.15}$$

where ω_0 is an integration constant and

$$b(u)\equiv\int\frac{B(u)}{f_+}du\,, \tag{4.16a}$$

$$a(v)=\int\frac{A(v)}{h_-}dv\,. \tag{4.16b}$$

Thus the metric (4.1) in the present case takes the form

$$ds^2=\frac{e^{a+b+\omega_0}}{\sqrt{t}}(f_+du)(h_-dv)-t(dx^2+dy^2)\,. \tag{4.17}$$

e and w can be cast as

$$e=4\frac{B(u)}{(f_+h_-)^2}\exp[|-2(b+a+\omega_0)|]\,, \tag{4.18}$$

$$w=4\frac{A(v)}{(f_+h_-)^2}\exp[|-2(b+a+\omega_0)|]\,. \tag{4.19}$$

Now the variables that describe the anisotropic fluid can be written as

$$\rho=\sigma=2\left(\frac{AB}{t}\right)^{1/2}e^{-(a+b+\omega_0)}\Big/h_-f_+\,, \tag{4.20}$$

$$U^{\mu}=e^{-\omega/2}\left(\frac{e}{w}\right)^{-1/4}\left[\delta^{\mu}_{(+)}+\left(\frac{e}{w}\right)^{1/2}\delta^{\mu}_{(-)}\right]\,, \tag{4.21}$$

$$\chi^{\mu}=e^{-\omega/2}\left(\frac{w}{e}\right)^{-1/4}\left[\left(\frac{w}{e}\right)^{1/2}\delta^{\mu}_{(+)}-\delta^{\mu}_{(-)}\right]\,. \tag{4.22}$$

Note that taking $\omega_0=i\pi$, $A=B=0$, and performing the change of variables $t=f+h$ (as before) and $z=h-f$, the metric (4.17) reduces to Taub's metric.[12] Also note that the metric (4.17) has a singularity at $t=0$ of the same type as Taub's metric. The energy density ρ is also singular at $t=0$.

V. DISCUSSION

In Sec. IV, as an example, we presented a particular case of anisotropic fluid with a particular symmetry that can be solved exactly. We have been able to find different exact solutions to the presented model of anisotropic fluid with plane, cylindrical, and spherical symmetry. The solution presented here is one of the simplest, the others will be published elsewhere.

The problem of finding a Kerr metric interior can be attacked in the following way: One starts with a metric that can be matched to Kerr's metric and later computes the EMT associated with this metric.[13] Some of the energy-momentum tensors computed in this way are similar to the EMT (2.28). This problem is under active consideration by the author.

Possible generalizations of the model are of two kinds. First, instead of taking a sum like (2.2) we can take

$$T^{\mu\nu}(u,v,n,m)=at^{\mu\nu}(u)+bt^{\mu\nu}(v)ct^{\mu\nu}(n)+dt^{\mu\nu}(m)\,, \tag{5.1}$$

where each of the energy-momentum tensors $t^{\mu\nu}(u),\ldots,t^{\mu\nu}(m)$ satisfy an equation such as (2.12). In principle, we can diagonalize the EMT (5.1) and we shall end up with three different pressures in the three different spatial directions. This generalized model will be useful in describing the general case of an anisotropic fluid. A second kind of generalization results if one changes the equations such as (2.12) by

$$t^{\mu\nu}(u)_{;\nu}=f^{\mu}(u,v)\,, \tag{5.2}$$

$$t^{\mu\nu}(v)_{;\nu}=-f^{\mu}(u,v)\,. \tag{5.3}$$

In other words, we now consider a collision term $f^{\mu}(u,v)$ different from zero. This collision term in principle can be computed using kinetic theory. Of course, one can put together both kinds of generalizations in a single model.

APPENDIX

In this appendix we compare the EMT associated with a cloud of strings with the EMT of an anisotropic fluid with two-null-fluid components.

The EMT for a cloud of strings can be written as[14]

$$T^{\mu\nu}=\rho_s\Sigma^{\mu\lambda}\Sigma_{\lambda}{}^{\nu}/(-\gamma)\,, \tag{A1}$$

where ρ_s is the string-cloud, gauge-invariant proper density[15] and

$$\Sigma^{\mu\nu}=a^{\mu}b^{\nu}-a^{\nu}b^{\mu}\ , \tag{A2}$$

$$\gamma=a^{\mu}a_{\mu}b^{\nu}b_{\nu}-(a_{\mu}b^{\mu})^2\ , \tag{A3}$$

$$a^{\alpha}a_{\alpha}>0,\quad b^{\alpha}b_{\alpha}<0\ . \tag{A4}$$

Thus

$$T^{\mu\nu}=\rho_s\frac{a_{\lambda}b^{\lambda}(a^{\mu}b^{\nu}+b^{\mu}a^{\nu})-b^{\lambda}b_{\lambda}a^{\mu}a^{\nu}-a^{\lambda}a_{\lambda}b^{\mu}b^{\nu}}{-a_{\alpha}a^{\alpha}b_{\beta}b^{\beta}+(a_{\alpha}b^{\alpha})^2}\ . \tag{A5}$$

Taking a string gauge such that[14]

$$a^{\mu}b_{\mu}=0\ , \tag{A6}$$

Eq. (A4) reduces to

$$T^{\mu\nu}=\rho_s\left(\frac{a^{\mu}a^{\nu}}{a_{\alpha}a^{\alpha}}+\frac{b^{\mu}b^{\nu}}{b_{\beta}b^{\beta}}\right)\ . \tag{A7}$$

Defining

$$U^{\mu}=a^{\mu}/(a_{\alpha}a^{\alpha})^{1/2}\ , \tag{A8}$$

$$\chi^{\mu}=b^{\mu}/(-b_{\beta}b^{\beta})^{1/2}\ , \tag{A9}$$

we can cast (A7) as

$$T^{\mu\nu}=\rho_s(U^{\mu}U^{\nu}-\chi^{\mu}\chi^{\nu})\ . \tag{A10}$$

Note that

$$U^{\mu}U_{\mu}=-\chi^{\mu}\chi_{\mu}=1\ , \tag{A11}$$

$$U^{\mu}\chi_{\mu}=0\ . \tag{A12}$$

It is interesting to compare (A10) with (2.44). We notice that the only difference is the sign of the anisotropic "pressure," i.e., in (A10) we have tension rather than pressure. Now we shall consider the possibility of generating (A10) from a different linear combination of EMT than (2.4). The minus sign in (A10) suggests that we consider an EMT such as

$$\tau^{\mu\nu}=t^{\mu\nu}(k)-t^{\mu\nu}(l)\ . \tag{A13}$$

If one tries to put (A13) in the form (A10), one finds that it is not possible to do so. A modification of the method used in Sec. II to the hyperbolic case can be used to prove the previous statement.

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[4]F. A. E. Pirani, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), pp. 201–212.

[5]See, for instance, N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill, New York, 1973), pp. 88–89.

[6]We use units such that $c=8\pi G=1$. The Greek indices run from 0 to 3. The signature of the metrics $g_{\mu\nu}$ is taken as −2.

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[8]R. Tabensky and A. H. Taub, Commun. Math. Phys. 29, 61 (1973).

[9]The explicit form of F is given in Ref. 8.

[10]This equation of state has been widely studied in the context of general relativity in the case $p_0=0$. See, for instance, Refs. 1 and 2.

[11]The massless scalar field case has been widely studied. See, for instance, P. S. Letelier and R. Tabensky, Nuovo Cimento 28B, 407 (1975); J. Math. Phys. 16, 8 (1975); P. S. Letelier, *ibid.* 16, 1488 (1975); 20, 2078 (1979); D. Ray, *ibid.* 17, 1171 (1976); J. Wainwright *et al.*, Gen. Relativ. Gravit. 10, 259 (1979).

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[15]In this appendix we shall not consider the string's end points. Note that the ρ_s used in the present paper differs by a factor of $\sqrt{-\gamma}$ from that in Ref. 14.