Homogeneous space-times of the Gödel type

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The condition for homogeneity of a class of cylindrically symmetric metrics is investigated. This leads to the discovery of a general solution from which a number of metrics (obtained earlier by different authors, including the two forms of the Gödel metric) are recovered by choosing the integration constants suitably. Incidentally, it is found that the same electromagnetic stress-energy tensor admits alternative interpretations as a Maxwell field either without sources or with a continuous distribution of sources.

I. INTRODUCTION

The Gödel[1] metric

$$ds^2 = a^2(dt^2 - dx^2 - dz^2 + \tfrac{1}{2}e^{2x}dy^2 + 2e^x dy\, dt) \qquad (1)$$

represents a homogeneous space-time as is apparent from the fact that it admits the following transformations, which together form a transitive group:

(i) $t' = t + \alpha$,

(ii) $z' = z + \beta$,

(iii) $y' = y + \gamma$,

(iv) $x' = x + \delta, \quad y' = ye^{-\delta}$.

Again, Gödel himself showed that the metric (1) can be transformed to the form

$$ds^2 = dt^2 - dr^2 - dz^2 + 2\sqrt{2}\,\sinh^2 r d\phi dt + (\sinh^4 r - \sinh^2 r)d\phi^2 , \qquad (2)$$

where the coordinate ϕ is an angle coordinate, and while the cylindrical symmetry of the space-time is now obvious the homogeneity is not so apparent. In recent years, quite a number of metrics of the above forms have been studied and apparently it has often been assumed that the situations represented by them are different from the Gödel universe and the question of space-time homogeneity very often remained obscure (e.g., Hoenselaers and Vishveshwara,[2] Reboucas,[3] Banerjee and Banerji[4]). The present authors had reason to suspect, from the behavior of the physical variables such as the density, pressure, vorticity, shear, and expansion, that these solutions are also homogeneous and transformable to the Gödel form. To verify this conjecture, first of all we investigated the condition that a metric of the form

$$ds^2 = dt^2 - dr^2 - dz^2 + 2md\phi dt - ld\phi^2 \qquad (3)$$

(where m and l are functions of r alone) may be homogeneous. In this way we came across a general form for the homogeneous metrics containing three arbitrary constants. Setting suitable values to these constants, we succeed in obtaining both the Gödel forms (1) and (2) as well as the metrics of Reboucas and Hoenselaers and Vishveshwara.

However, the set of homogeneous space-times of the form (3) are not exhausted, and one can obtain solutions where the Einstein equations are satisfied by nonperfect fluids such as electromagnetic fields or, alternately, matter with anisotropic pressure nevertheless geodetic world lines. A study is made of these types of solutions.

II. THE KILLING EQUATIONS AND HOMOGENEITY

The metric (3) obviously admits three translations along the t, z, and ϕ axis. For homogeneity, it must admit one more linearly independent Killing vector ξ^α. From the form of the metric (3) it is obvious that we may take $\xi^2 = 0$ and $\xi^\alpha{}_{,2} = 0$ (we write 0, 1, 2, 3 for t, r, z, and ϕ, respectively, and a comma will indicate a partial derivative while a semicolon indicates a covariant derivative). We shall further assume that the vector $u^\alpha = \delta^\alpha{}_0$ is an invariant vector of the group—as will be the case if, for example, the source of the field is assumed to be a perfect fluid with stress-energy tensor

$$T^\mu{}_\nu = (p+\rho)u^\mu u_\nu - p\,\delta^\mu{}_\nu , \qquad (4)$$

with u^μ representing the velocity vector of matter. Then we have the Lie derivative

$$\xi^\alpha{}_{;\beta}u^\beta - \xi^\beta u^\alpha{}_{;\beta} = 0 , \qquad (5)$$

or

$$\xi^\alpha{}_{,0} = 0 \text{ for all } \alpha. \qquad (6)$$

With these conditions, the nontrivial Killing equations $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$ are

$$\xi^1{}_{,1} = 0 , \qquad (7)$$

$$-\xi^1{}_{,3} + m\xi^0{}_{,1} - l\xi^3{}_{,1} = 0 , \qquad (8)$$

$$m\xi^3{}_{,1} + \xi^0{}_{,1} = 0 , \qquad (9)$$

$$2m\xi^0{}_{,3} - 2l\xi^3{}_{,3} - \xi^1 l_{,1} = 0\,, \tag{10}$$

$$m\xi^3{}_{,3} + \xi^0{}_{,3} + \xi^1 m_{,1} = 0\,. \tag{11}$$

Equations (10) and (11) together give

$$2(m^2 + l)\xi^3{}_{,3} + \xi^1(m^2 + l)_{,1} = 0\,. \tag{12}$$

When we write

$$(m^2 + l) = D^2\,, \tag{13}$$

Eq. (12) reads

$$\xi^3{}_{,3} = -\xi^1 \frac{D_{,1}}{D}\,. \tag{14}$$

Also, from Eqs. (8) and (9)

$$\xi^1{}_{,3} = -D^2 \xi^3{}_{,1}\,. \tag{15}$$

Equations (14) and (15) are consistent only if

$$\xi^1{}_{,33} = \xi^1[D_{,11}D - (D_{,1})^2]\,. \tag{16}$$

In view of Eq. (7), Eq. (16) requires

$$D_{,11}D - (D_{,1})^2 = \text{const} = b^2 \quad (\text{say})\,. \tag{17}$$

Equation (16) now gives

$$\xi^1 = \alpha e^{b\phi} + \beta e^{-b\phi}\,, \tag{18}$$

where, in view of Eqs. (6) and (7), α and β are constants. Equation (14) now gives

$$\xi^3 = -\frac{D_{,1}}{bD}(\alpha e^{b\phi} - \beta e^{-b\phi}) + f(x')\,. \tag{19}$$

Substituting (19) and (18) in (15), we get $f(x')$ = constant. Using (19) in (9), and remembering (17), we get

$$\xi^0{}_{,1} = \frac{mb}{D^2}(\alpha e^{b\phi} - \beta e^{-b\phi})\,. \tag{20}$$

Also, from Eq. (11)

$$\xi^0{}_{,3} = -(\alpha e^{b\phi} + \beta e^{-b\phi})\left(m_{,1} - \frac{mD_{,1}}{D}\right). \tag{21}$$

Equations (20) and (21) are integrable only if

$$\frac{mb^2}{D^2} = -\left(m_{,1} - \frac{mD_{,1}}{D}\right)_{,1}$$

or, if [again using (17)]

$$\frac{m_{,1}}{D} = \text{const} = c \quad (\text{say})\,. \tag{22}$$

Thus an additional Killing vector exists if Eqs. (17) and (22) are satisfied. Equation (17) has the general integral

$$D = (A_1 e^{ar} + A_2 e^{-ar})\,, \tag{23}$$

where the three constants A_1, A_2, and a are subject to the condition

$$4A_1 A_2 a^2 = b^2\,. \tag{24}$$

Equation (17) also admits a singular solution

$$D = r\,, \tag{25}$$

which is obtainable when b^2 is negative and an integration constant is removed by a coordinate transformation. With a real, Eq. (23) would give singularities of D at $\pm\infty$, while with a imaginary, D will periodically vanish. Both these singular behaviors are, however, of no physical significance owing to the demonstrated homogeneity of the space-time. (For examples of such solutions, see Banerjee and Banerji[4] and Ozsvath.[5])

III. DIFFERENT FORMS OF THE HOMOGENEOUS METRIC

With the metric (3), the nonvanishing components of the Ricci tensor are (cf. Van Stockum[6])

$$-R_1^1 = \frac{D_{,11}}{D} - \frac{(m_{,1})^2}{2D^2}\,, \tag{26}$$

$$-R_2^2 = 0\,, \tag{27}$$

$$-R_3^3 = \frac{1}{2D}\frac{d}{dr}\left(\frac{l_{,1} + mm_{,1}}{D}\right), \tag{28}$$

$$+R_0^3 = \frac{1}{2D}\frac{d}{dr}\left(\frac{m_{,1}}{D}\right), \tag{29}$$

$$+R_3^0 = \frac{1}{2D}\frac{d}{dr}\left(\frac{ml_{,1} - lm_{,1}}{D}\right), \tag{30}$$

$$-R_0^0 = \frac{1}{2D}\frac{d}{dr}\left(\frac{mm_{,1}}{D}\right). \tag{31}$$

For the perfect fluid distribution, $R_1^1 = R_2^2$, and hence from (26) and (27)

$$(m_{,1})^2 = 2DD_{,11}\,. \tag{32}$$

Equation (32) is consistent with (22) and (23) if

$$c = \sqrt{2}\,a \tag{33}$$

and consequently

$$m = \sqrt{2}\,(A_1 e^{ar} - A_2 e^{-ar} + B)\,, \tag{34}$$

where B is another arbitrary constant. With the singular solution (25), (32) would require m to vanish and the metric would be transformable to the static form. It is easy to verify that with (13), (23), and (34), the equations (26)–(31) are satisfied if $p = \rho$ = constant and also the rotation is rigid.

Now these are the conditions characteristic of the Gödel universe, and when one recalls the theorem that the Gödel universe is the only homogeneous universe having these properties, it is clear that all the solutions obtainable with different values of A_1, A_2, and B must be transformable into the Gödel metric. Indeed, it is fairly easy to find out the transformation formulas from one form to another by calculating the explicit form of the

Killing vector for different values of the constants. In particular we observe that with $A_2 = B = 0$ and the constant a absorbed within r by a scale transformation, the metric assumes the form (1), while with $A_1 = -A_2 = \frac{1}{4}$, $a = 2$, and $B = -\frac{1}{2}$, the metric goes over to the form (2).

The Hoenselaers-Vishveshwara metric

$$ds^2 = -(dr^2+dz^2)+\tfrac{1}{2}A^2(\cosh K^{1/2}x - 3)(\cosh K^{1/2}x - 1)d\phi^2 + A(\cosh K^{1/2}x - 1)^2 d\phi dt + (1+\tfrac{1}{2}\sinh^2 K^{1/2}x)dt^2 \quad (35)$$

(where A and $K^{1/2}$ are constants related by $AK^{1/2} = 1$) may appear not to be of our type (3) as g_{00} is not constant. However, with the transformation

$$\phi' = \phi + t/A\,,$$

the metric assumes the form (dropping primes)

$$ds^2 = -(dr^2+dz^2)+\tfrac{1}{2}A^2(c-3)(c-1)d\phi^2 + 2A(c-1)d\phi dt + dt^2 \quad (36)$$

(where c stands for $\cosh K^{1/2}x$), so that

$$D = \frac{1}{\sqrt{2}} A \sinh K^{1/2}x\,,$$

$$m_1 = AK^{1/2}\sinh K^{1/2}x\,,$$

which are obtained from (23) and (34) by taking

$$A_1 = -A_2 = A/2\sqrt{2}\,, \quad B = 0\,,$$

$$a = K^{1/2}\,,$$

which is also in agreement with (33).

The Reboucas metric

$$ds^2 = dt^2 - dr^2 - dz^2 + \frac{4\Omega}{a}\cosh ar d\phi dt + \left[\left(\frac{\Omega^2+\alpha^2}{\Omega^2-\alpha^2}\right)\cosh^2 ar + 1\right]d\phi^2\,, \quad (37)$$

where Ω, a, α are constants and $a^2 = 2(\Omega^2 - \alpha^2)$, has

$$D = \sinh ar$$

and

$$m_1 = 2\Omega\sinh ar\,,$$

which are obtainable from (22) and (23) with

$$A_1 = -A_2 = \tfrac{1}{2}, \quad c = 2\Omega\,.$$

However, (33) is satisfied only if $\alpha = 0$. This is indeed as it should be, for (33) is true only for a perfect fluid distribution, while the Reboucas metric with $\alpha \neq 0$ corresponds to a nonvanishing electromagnetic field along with the fluid distribution.

The singular solution (25) along with (22) leads to the metric

$$ds^2 = dt^2 - dr^2 - dz^2 + 2ar^2 d\phi\, dt - (r^2 - a^2r^4)d\phi^2\,. \quad (38)$$

It has been studied previously by Som and Raychaudhuri[7] and represents a distribution of charged dust with the charge density equal to double the mass density. However, that the space-time is homogeneous was not shown there.

In view of the results of Som and Raychaudhuri as well as Reboucas, it seems worthwhile to enquire whether all homogeneous space-times which satisfy (22) and (23) [or (25)] but not (33) can be interpreted as due to an electromagnetic field superposed on a perfect fluid distribution. The following investigation shows that this is indeed the case. Equations (26)–(31) along with (22) and (23) give

$$a^2 - \tfrac{1}{2}c^2 = 8\pi[-\tfrac{1}{2}(\rho - p) + \mathcal{T}^1_1]\,, \quad (39)$$

$$0 = 8\pi[-\tfrac{1}{2}(\rho - p) + \mathcal{T}^2_2]\,, \quad (40)$$

$$a^2 - \tfrac{1}{2}c^2 = 8\pi[-\tfrac{1}{2}(\rho - p) + \mathcal{T}^3_3]\,, \quad (41)$$

$$0 = \mathcal{T}^3_0\,, \quad (42)$$

$$m(a^2 - c^2) = -8\pi[(\rho + p)m + \mathcal{T}^0_3]\,, \quad (43)$$

$$\tfrac{1}{2}c^2 = 8\pi[\tfrac{1}{2}(\rho + 3p) + \mathcal{T}^0_0]\,, \quad (44)$$

where we have assumed that the velocity vector $u^\mu = \delta^\mu{}_0$ and $\mathcal{T}^i_k$ is to represent the electromagnetic stress-energy tensor. In case we take the singular solution (25) instead of (23), Eqs. (39)–(44) would remain valid with $a = 0$. From Eqs. (39)–(44) we get

$$8\pi\rho = \tfrac{1}{2}(c^2 - a^2)\,, \quad (45)$$

$$8\pi p = \tfrac{1}{2}a^2\,, \quad (46)$$

$$\mathcal{T}^1_1 = -\mathcal{T}^2_2 = \mathcal{T}^3_3 = -\mathcal{T}^0_0 = \frac{1}{16\pi}(a^2 - \tfrac{1}{2}c^2)\,, \quad (47)$$

$$\mathcal{T}^0_3 = -\frac{1}{8\pi} ma^2\,. \quad (48)$$

Physical constraints on the values of ρ and p would impose some restrictions on c and a, e.g., if $\rho > p$, then $c^2 > 2a^2$, which also happens to be the condition for energy density $\mathcal{T}^0_0$ to be positive With (42), (47), and (48), the Rainich algebraic conditions are satisfied; therefore, one can find an electromagnetic field tensor subject to the arbitrariness of "duality rotations." In this case, this freedom allows a nonunique interpretation of the sources of the electromagnetic fluid. Thus (47) and (48) may be considered to be due to the electromagnetic field tensor with the nonvanishing contravariant components

$$F^{13} = -F^{31} = \pm\frac{1}{2D}(c^2 - 2a^2)^{1/2}\,, \qquad F^{10} = -mF^{13}\,. \quad (49)$$

This would mean a charge density

$$4\pi\sigma = \tfrac{1}{2}c(c^2 - 2a^2)^{1/2}.$$

However, one may as well take the nonvanishing contravariant components to be

$$F^{13} = -F^{31} = \pm\frac{1}{2D}(c^2 - 2a^2)^{1/2}\cos\alpha,$$
$$F^{10} = -F^{01} = \mp\frac{m}{2D}(c^2 - 2a^2)^{1/2}\cos\alpha, \quad (50)$$
$$F^{20} = \pm\tfrac{1}{2}(c^2 - 2a^2)^{1/2}\sin\alpha,$$

which satisfy the source-free Maxwell equations if $\alpha = cz$.

In fact, Reboucas has interpreted his solution as representing a source-free electromagnetic field, while Som and Raychaudhuri have considered their field to be due to a continuous charge distribution. In either case however, the Lorentz force vanishes, for with (49), the electric field $F_{\mu\nu}u^\nu$, as seen by the charge distribution, vanishes while in (50) there is no charge current whatsoever.

Incidentally, this failure of the metric tensor to determine uniquely the sources of the electromagnetic field seems to us to be a strong argument against the philosophy of the "already unified theory."

We may note that while the different metrics satisfying (22), (23), and (33) differ merely by a scale factor, solutions which do not satisfy (33) depend essentially on two parameters, c and a, and differ essentially from one another as is evident from the varying ratios of the pressure, matter density, and charge density. However, as all the perfect fluid solutions are transformable to the Gödel form, the different solutions with the same values of c/a may be shown to be equivalent.

IV. CONCLUDING REMARKS

It is interesting to consider the question of closed timelike lines in these solutions. Their presence in the Gödel universe is most easily demonstrated by considering the metric (2). Here $g_{\phi\phi} \to 0$ as $r \to 0$, however, $g_{\phi\phi}/g_{rr} \to r^2$ as $r \to 0$ and hence by a transformation $X = r\cos\phi$, $Y = r\sin\phi$ we may obtain analyticity of the metric tensor at $r = 0$ (Maitra[8]). This shows that ϕ is to be regarded as an angular coordinate. Hence, it turns out that the line $t = z = 0$, $r = \text{constant} > \ln(1+\sqrt{2})$ is a circular closed timelike line. Judged by the same criterion, y in (1) and ϕ in (37) are not angular coordinates and hence one cannot say that in (37) "the circles defined by $r = \text{constant}$, $t = z = 0$ are closed timelike curves (for all r).[3]" In fact these curves are in general neither circles nor closed.

However, one may transform to cylindrical polar coordinates. In such coordinates

$$D = \frac{1}{2a}(e^{ar} - e^{-ar})$$

and

$$m = \frac{c}{2a^2}(e^{ar/2} - e^{-ar/2}).$$

Thus, it is clear that with $c^2 > a^2$ (a condition ensured by the reality of electromagnetic field and/or positivity of p and ρ) for sufficiently large r, $l = D^2 - m^2$ would be negative and hence we would have closed timelike lines. However, for small enough r, l is always positive and thus one would have a region $r < r_{\text{crit}}$, where no closed timelike line exists.

It is possible that the solutions which we have interpreted as due to a fluid combined with an electromagnetic field may admit alternative interpretations. In fact, while we have considered the fluid velocity vector to be $\delta^\mu{}_0$, Ozsvath, by retaining the cosmological term and considering a different velocity vector (which, however, is also a Killing vector), has interpreted the metrics as due to a dust combined with an electromagnetic field. There seems further possibilities in view of the work of Tupper,[9] who has demonstrated that numerous electrovac metrics may be considered to be due to distributions of matter with viscosity. However, we have not tried to investigate these possibilities.

The fluid solutions (23) and (34) are characterized by $p = \rho$ [we are not considering the original Gödel version of $p = 0$ and a nonvanishing cosmological constant (see Raychaudhuri[10])]. This makes it impossible to cut off a portion of the solution and fit it consistently with an outside empty space. In fact, this difficulty led Hoenselaers and Vishveshwara to introduce a surface stress-energy tensor with a corresponding "kink" in the metric tensor. The difficulty, however, need not occur when electromagnetic fields are also present—there, $p = 0$ if $a = 0$ and one gets the solution of Som and Raychaudhuri. In fact, they noted the possibility of a fit with an outside empty space having electromagnetic fields only.

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