PHYSICAL REVIEW D VOLUME 22, NUMBER 4 15 AUGUST 1980

Cosmic censorship and test particles

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(Received 7 January 1980; revised manuscript received 1 April 1980)

In this paper one unambiguous prediction of cosmic censorship is put to the test, namely that it should be impossible to destroy a black hole (i.e. eliminate its horizon) by injecting test particles into it. Several authors have treated this problem and have not found their conclusions in contradiction with the prediction. Here we prove that if a general charged spinning particle (with parameters very much smaller than the respective hole parameters) is injected in an arbitrary manner into an extreme Kerr-Newman black hole, then cosmic censorship is upheld. As a by-product of the analysis a natural proof is given of the Christodoulou-Ruffini conditions on the injection of a spinless particle which yield a reversible black-hole transformation. Finally we consider the injection of particles with parameters that are *not* small compared with those of the hole, for which cosmic censorship is apparently violated. By assuming the validity of cosmic censorship we are led to a few conjectures concerning the extent of the particle's interaction with the hole while approaching it.

I. INTRODUCTION

Possibly the greatest unsolved problem in classical general relativity is Penrose's cosmic censorship hypothesis[1] whereby naked singularities are excluded from the real world. As with the birth of many important ideas in physics cosmic censorship is not yet precisely defined so that many situations are known which contradict a strict interpretation but not the spirit of the hypothesis. However, a clear implication of any reasonable definition is that if we are given a black hole then we should not be able to destroy it, that is remove its horizon, and in particular we should not be able to destroy it by means of injecting test particles. Thus if it could be explicitly shown to be possible to destroy a black hole by firing particles into it then there would seem to be little hope of *any* form of cosmic censorship being true.

It is now well established[2] that all stationary electrovac black holes belong to the three-parameter Kerr-Newman family of exact solutions to Einstein's equations, the parameters being the mass M, charge Q, and specific angular momentum $a=J/M$ with the condition $M^2 \geq a^2+Q^2$. A stationary electrovac spacetime with $M^2<a^2+Q^2$ must then be a naked singularity and not a black hole. Calculations show it is possible to create an extreme black hole (i.e., one for which $M^2=a^2+Q^2$), and indeed calculations on astrophysical accretion[3] lead us to believe that some black holes will be near extreme. Thus the prediction of cosmic censorship is the following: Given an extreme Kerr-Newman black hole we cannot create a spacetime with $M^2<a^2+Q^2$ by injecting test particles into it. If the injected particle contributes energy dM, angular momentum dJ, and charge dQ to the hole, then the final spacetime cannot have $M^2<a^2+Q^2$ if

$$dM-\frac{a}{M^2+a^2}\,dJ-\frac{QM}{M^2+a^2}\,dQ \geq 0\,. \tag{1}$$

So this inequality is the prediction of cosmic censorship to be tested. In deriving this equation we have assumed that $dM \ll M$, $dJ \ll J$, and $dQ \ll Q$. If the particle did not satisfy these conditions we could imagine the following: Take an extreme Kerr hole and drop a charged particle into it. No repulsive force is apparent, and consequently the particle could enter the hole and destroy it. In situations of this kind one must go beyond the test-particle approximation in order to verify cosmic censorship. We shall return to this problem in Sec. IV. However, until then the expression "test particle" should be interpreted as being a particle which satisfies these conditions.

Now the first law of black-hole dynamics states[2]

$$\frac{\kappa}{8\pi}\,d\mathfrak{A}=dM-\Omega\,dJ-\Phi dQ\,, \tag{2}$$

where $\mathfrak{A}$ is the area of the horizon, κ is the surface gravity, Ω is the angular velocity of the hole, and Φ is the electrostatic potential of the horizon:

$$\Omega=\frac{a}{r_+^2+a^2},\quad \Phi=\frac{Qr_+}{r_+^2+a^2}\,,\quad r_+=M+(M^2-a^2-Q^2)^{1/2}\,.$$

Since for an extreme Kerr-Newman black hole $r_+=M$ we can rewrite Eq. (1) as the restriction to the extreme case of

$$dM-\Omega\,dJ-\Phi dQ \geq 0\,, \tag{3}$$

which looking at Eq. (2) is seen to be just the area-theorem inequality restricted to the case of an extreme black hole. Thus it might appear that we could prove the result we require, namely, Eq. (3), by invoking the area theorem $d\mathfrak{A} \geq 0$ and using Eq.

(2). However, Eq. (2) only makes sense if we have a horizon before and after the perturbation for which to calculate the charge $d\mathcal{A}$. Hence if we used the area theorem we should be assuming the existence of the horizon after injecting the particle. This is the very statement we wish to test. Also in the proof of $d\mathcal{A}\geq 0$ one shows that if $\mathcal{A}$ were to decrease, then within a finite lapse of affine parameter up the generators of the horizon, $\mathcal{A}$ would decrease to zero, at which point generators of the horizon $H=\dot{J}^-(\mathscr{I}^+)$ would intersect each other. This contradicts a theorem of Penrose[4] which states that generators of $\dot{J}^-(\mathscr{I}^+)$ cannot cross to the future of the point at which they enter $\dot{J}^-(\mathscr{I}^+)$ from $J^-(\mathscr{I}^+)$. However, this theorem depends on ruling out the possibility that the generators hit a singularity and hence cease to exist, i.e., it depends on cosmic censorship. Consequently we are barred from using the area theorem to test cosmic censorship, as manifested in Eq. (3), by the twofold *assumption* of cosmic censorship, first in proving $d\mathcal{A}\geq 0$ and second in using Eq. (2).

The approach adopted by previous authors (e.g., Refs. 3, 5–9) is to use an appropriate equation of motion (geodesic, Papapetrou, etc.) and by explicit calculation of turning points or by more general manipulation show that particles with energy E, angular momentum L, and charge e at infinity are repelled before entering the horizon if they violate

$$E-\Omega L-\Phi e\geq 0\,. \tag{4}$$

Before proceeding further we should mention the question of radiation. When a particle falls into a black hole it emits gravitational and, if charged, electromagnetic radiation which will carry energy and angular momentum (though not charge) away from the particle. As a result $dM<E$ and $dJ\neq L$ (E and L defined at infinity) so that one might fear that in testing Eq. (4) one is not really testing the actual prediction of cosmic censorship, namely, Eq. (3). However, calculations indicate (see, e.g., Ref. 10) that the amount of energy radiated from a particle of mass m is less than m^2/M, which is precisely the type of second-order term which was neglected in deriving Eq. (1). Consequently, to first order, Eqs. (3) and (4) may be considered equivalent.

Wald[5] has obtained the general proof of Eq. (4) for spinless particles; however, the complexity of the equations of motion for spinning chargeless particles has prevented detailed analysis of their motion except on the axis of symmetry[5,11] and in the equatorial plane.[6,9] For charged spinning particles only the case of motion in the equatorial plane has been treated.[7] Thus this approach has not produced a general proof of Eq. (4) for an arbitrary method of injecting a charged spinning test particle.

In Sec. II the case of nonspinning particles is considered and particles entering the hole are shown to satisfy Eq. (4) even for a nonextreme black hole. As a by-product of this analysis a transparent proof is given of the Christodoulou-Ruffini conditions[12,13] on the injected particle which yield a reversible black-hole transformation. In Sec. III the basic idea of the proof in Sec. II is generalized to spinning particles, and finally in Sec. IV situations in which cosmic censorship is apparently violated are discussed.

II. NONSPINNING PARTICLES

Consider for the moment an uncharged nonspinning particle with momentum p. If ξ and $\hat{\xi}$ are the time and angular Killing vectors, then they yield two conserved (to first order) quantities $E=p\cdot\xi$ and $L=-p\cdot\hat{\xi}$, the energy and angular momentum of the particle, respectively.

For a charged particle $E=\pi\cdot\xi$ and $L=-\pi\cdot\hat{\xi}$, where $\pi=$ generalized momentum $=p+eA$ and A is the vector potential of the hole with the gauge freedom $\underline{A}\to\underline{A}+\underline{d}f(r,\theta)$ ($\underline{d}$ is the exterior derivative).

We now prove Eq. (3) for the injection of a spinless charged particle into a general Kerr-Newman black hole. Now the product of any future timelike vector with any future null vector is strictly positive, so if we contract the momentum of the particle with the tangent vector χ to the null generators of H as the particle passes through H we get a strictly positive quantity, i.e.,

$$\langle\underline{p},\chi\rangle>0\,.$$

Then since $\underline{p}=\underline{\pi}-e\underline{A}-e\underline{d}f(\underline{r},\theta)$ and $\chi=\xi+\Omega\hat{\xi}$ we obtain

$$\langle\underline{\pi}-e\underline{A}-e\underline{d}f(r,\theta),\,\xi+\Omega\hat{\xi}\rangle>0$$

or

$$dM-\Omega dJ-\langle\underline{A},\chi\rangle\,dQ-\langle\underline{d}f(r,\theta),\,\xi+\Omega\hat{\xi}\rangle>0\,.$$

A simple calculation shows $\langle\underline{A},\chi\rangle=\Phi$ [see Eq. (2)], and noting $\langle\underline{d}\theta,\chi\rangle=\langle\underline{d}r,\chi\rangle=0$ we obtain

$$dM-\Omega dJ-\Phi dQ>0\,, \tag{5}$$

which is the required result. If the particle does not enter the hole we cannot contract its momentum with χ so we do not obtain the above restriction on the properties of the particle, in accord with the idea of particles not obeying Eq. (5) being repelled before reaching H.

Note that for a general stationary axisymmetric black hole (e.g., hole with a ring of matter around it) general theory shows[2] that we may still write

$\chi = \xi + \Omega\hat{\xi}$ and that $\Phi = \langle \underline{A}, \chi\rangle$, thus Eq. (5) holds in this case also.

Equation (2) tells us that Eq. (5) is equivalent to $d\mathfrak{A} > 0$ for massive particles. The connection between $\mathfrak{A}$ and entropy leads one to call a process for which $d\mathfrak{A} = 0$ reversible. Christodoulou and Ruffini[12,13] have shown that the injection of a particle is a reversible process if and only if the particle is massless and $p^r = p^\theta = 0$ at H. This result may be deduced very simply by the method of this paper without the detailed algebra of the original proof. Since we have proved $d\mathfrak{A} > 0$ for massive particles we see that if $d\mathfrak{A} = 0$ is possible at all then the particle must be massless, and since there are no massless charged particles in nature Eq. (2) requires that for a reversible process

$$0 = d\mathfrak{A} = \frac{8\pi}{\kappa}(dM - \Omega dJ) = \frac{8\pi}{\kappa}\, p\cdot\chi\,,$$

and indeed since p is now null we can make it orthogonal to χ, which we could not do before with massive particles. The condition $d\mathfrak{A} = 0$ now becomes

$$p\cdot\chi = 0\,,$$

and this implies that p points along χ. But $\chi = \xi + \Omega\hat{\xi}$ so $\chi^r = \chi^\theta = 0$ and thus

$$p^r = p^\theta = 0\,.$$

III. SPINNING PARTICLES

In this section the argument of the preceding section is generalized to the case of chargeless and then charged spinning particles with the restriction that the black hole is now extreme.

For spinless chargeless particles we showed that $dM - \Omega dJ$ can be written as the contraction of a future timelike vector (namely p) and the tangent to the null generator of H. We now do the same for chargeless spinning particles. For such particles the conserved (to first order) energy and angular momentum are given by[6]

$$E = p\cdot\xi + \tfrac{1}{2}\xi_{a;b}S^{ab} \quad\text{and}\quad L = -(p\cdot\hat{\xi} + \tfrac{1}{2}\hat{\xi}_{a;b}S^{ab})\,.$$

Now

$$\begin{aligned} dM - \Omega dJ &= E - \Omega L \\ &= p\cdot\chi + \tfrac{1}{2}\chi_{a;b}S^{ab}\,. \end{aligned} \tag{6}$$

However, on H we have[2] $\chi_{[a}\chi_{b;c]} = 0$ so there exists a V such that

$$\chi_{a;b} = 2\chi_{[a}V_{b]}\,. \tag{7}$$

Using Eq. (7) we may rewrite Eq. (6) as

$$dM - \Omega dJ = \lambda\cdot\chi\,, \tag{8}$$

where we define $\lambda^a \equiv p^a + S^{ab}V_b$.

If λ is timelike then we obtain $dM - \Omega dJ > 0$ as required. Before looking more closely at λ, note that there is a degree of freedom in the definition [Eq. (7)] of V, namely, $V \to V + g\chi$, where g is some real constant. Thus while $\lambda\cdot\chi$ is invariant under such a change, λ is not; so what we must prove is that there exists a single V satisfying Eq. (7) such that λ is timelike, for then $\lambda\cdot\chi > 0$ for any V. Therefore we need to show that regardless of the point at which the particle crosses H there exists a V for which $\lambda\cdot\lambda > 0$.

Now[11] $p_a S^{ab} = p\cdot S = 0$ and $S^{ab} = \epsilon^{abcd}S_c\hat{p}_d$ where a caret denotes the unit vector. Using the expansion of $\epsilon^{abcd}\epsilon_{aefg}$ we obtain

$$\lambda\cdot\lambda = p^2 - |S^2|[(\hat{p}\cdot V)^2 - (\hat{S}\cdot V)^2 - V^2]\,. \tag{9}$$

Since $p\cdot S = 0$ we may construct two null vectors

$$\delta \equiv \frac{\hat{p} + \hat{S}}{\sqrt{2}} \quad\text{and}\quad \gamma \equiv \frac{\hat{p} - \hat{S}}{\sqrt{2}} \quad\text{with } \delta\cdot\gamma = 1\,,$$

and using these Eq. (9) may be written as

$$\lambda\cdot\lambda = m^2 - |S^2|[2(\delta\cdot V)(\gamma\cdot V) - V^2]\,. \tag{10}$$

Since we are considering an extreme Kerr-Newman black hole $\kappa = 0$, so $\nabla_\chi \chi = \kappa\chi = 0$ and consequently $0 = \chi^b\chi_{a;b} = (\chi\cdot V)\chi_a$, that is,

$$\chi\cdot V = 0\,. \tag{11}$$

If $V \to V + g\chi$ then $V^2 \to V^2 + 2g(\chi\cdot V) = V^2$, so V^2 does not depend on the choice of V, and since $\chi\cdot V = 0$, V must be spacelike or null. In the Appendix we calculate V^2 and find $0 \geq V^2 \geq -1/M^2$ and that V is only null at the poles.

Since at the poles V is null and $V\cdot\chi = 0$, V points along χ, which implies $\chi_{a;b} = 2\chi_{[a}V_{b]} = 0$. Substituting this into Eq. (6) we obtain

$$dM - \Omega dJ = p\cdot\chi > 0$$

if the particle crosses H at the poles.

Off the axis V is spacelike so we can add on multiples of χ to make V orthogonal to δ or γ. For V chosen in this way Eq. (10) becomes

$$\lambda\cdot\lambda = m^2 + |S^2|V^2 \geq m^2 - \frac{|S^2|}{M^2}\,. \tag{12}$$

However, a particle with spin $|S^2|^{1/2}$ and mass m has a minimum size $|S^2|^{1/2}/m$, and for it to be considered a test particle this must be much less than the size of the hole, so the test particle condition is

$$\frac{|S^2|}{m^2} \ll M^2\,. \tag{13}$$

Using Eq. (13) in Eq. (12) we obtain $\lambda\cdot\lambda > 0$ for the V such that $V\cdot\delta = 0$ or $V\cdot\gamma = 0$ so $\lambda\cdot\chi > 0$ for all V. Consequently

$$dM - \Omega dJ > 0$$

away from the poles as well as at them. Thus the required result is obtained for the general injection of a chargeless spinning particle.

Note that when the particle crosses at the poles we do not impose the size condition whereas in general we do. Comparing this with previous calculations we see that when Wald[5] sends the particle down the axis of symmetry he does not have to impose Eq. (13), whereas in the equatorial plane Tod *et al.*[6] are forced to if the hole is not to be destroyed. The same feature is present in the above proof. Let $(\gamma, \delta, \mu, \bar{\mu})$ be a null tetrad defined along the world line of the particle; then we may write Eq. (10) as

$$\lambda^2 = m^2 - 2|S^2||(\mu \cdot V)|^2 .$$

But $\mu \cdot V$ is independent of $|S^2|$ and can only be made to vanish at the poles, so that if we do not demand Eq. (13) away from the poles then we can choose $|S^2|$ large enough to make λ spacelike for any V. We are thus unable to deduce $\lambda \cdot \chi > 0$.

For charged spinning particles[7]

$$E = \pi \cdot \xi + \tfrac{1}{2}\xi_{a;b} S^{ab}, \quad L = -(\pi \cdot \hat{\xi} + \tfrac{1}{2}\xi_{a;b} S^{ab}) ,$$

where as before $\pi = p + eA$ is the generalized momentum. We have seen that $\lambda \cdot \chi > 0$ regardless of the point at which the particle crosses H, so

$$(p^a + S^{ab} V_b)\chi_a > 0$$

and thus

$$(\pi^a + S^{ab} V_b)\chi_a - A \cdot \chi dQ > 0$$

and thus

$$dM - \Omega dJ - \Phi dQ > 0 . \tag{14}$$

Therefore we have shown that the injection of a charged spinning test particle in an arbitrary manner cannot destroy an extreme Kerr-Newman black hole.

IV. CONJECTURES

In this section we discuss situations in which cosmic censorship is apparently violated, and in attempting to resolve these problems we are led to make a number of conjectures concerning the interaction between the black hole and the injected particles.

In a recent paper Hiscock[14] claims to have violated the area theorem by firing a ring of uncharged particles into a Schwarzschild hole. Since cosmic censorship is manifestly the weakest assumption in the proof of the area theorem, he deduces that cosmic censorship is violated. In more detail, the area theorem applied in this case dictates that a particle entering the hole must induce changes dM and dJ such that

$$dM \geq \frac{(dJ)^2}{8M^3} . \tag{15}$$

However, the geodesic equation allows particles violating Eq. (15) to enter.

The problem stems from the fact that in Secs. II and III the particle could be thought of as propagating in a fixed background geometry and that second-order terms, corresponding to particle-hole interactions, could be neglected. But here there is no first-order term in L, and consequently we *must* look at the interaction of the particle's angular momentum with the hole. We can also no longer ignore the second-order effect of gravitational radiation, and before looking at the particle-hole interaction we must first consider this radiation.

An uncharged particle is normally considered as moving on a geodesic ($\nabla_p p = 0$), and consequently $\nabla_p(p \cdot \xi) = \xi \cdot \nabla_p p = 0$, which leads to the identification of $p \cdot \xi$ with the particle's energy E. However, if the particle is fired toward a black hole it will emit a small amount of gravitational radiation and will deviate slightly from geodesic motion ($\nabla_p p \neq 0$) so that $p \cdot \xi$ will no longer be a constant of the motion. Nevertheless we wish in what follows to maintain the identification of $p \cdot \xi$ as energy, i.e., $E = p \cdot \xi$. (L is exactly conserved, since ring + hole forms an axisymmetric system which therefore cannot radiate angular momentum.) Although we cannot furnish a proof of this, it can be made quite plausible by the following two arguments. First suppose the particle emits gravitational waves of sufficiently high frequency to obey geometrical optics; then the wave can be thought of as being made up of gravitons with definite momenta following definite world lines, and as usual for the emission of particles the energy is still $p \cdot \xi$ despite the fact that it changes. Second, imagine that we fire a particle toward the black hole from the asymptotically flat region where its energy and momentum are E_1 and p_1; it then passes near the hole with the emission of some gravitational radiation, most of which will be low frequency. Then it escapes to the asymptotically flat region where its energy and momentum are E_2 and p_2. The waves have carried away energy so that $E_1 \neq E_2$; however, because of asymptotic flatness, E_1 and E_2 are manifestly $p_1 \cdot \xi$ and $p_2 \cdot \xi$. The second argument justifies the identity $E = p \cdot \xi$ initially and finally for all frequencies, whereas the first justifies it at all times but only for high-frequency waves.

We can now return to Hiscock's problem. As the ring of particles spirals into the black hole, it emits gravitational waves; also, it interacts with the hole so that the initially static Schwarzschild horizon generators start to rotate until, at the point of capture, Ω has increased from zero to a value Ω_c. After entering the horizon, the ring cannot radiate anything more to $\mathscr{I}^+$; hence $dM_{\text{particle}} = E|_H$ and from the preceding paragraph $E = p\cdot\xi$ so that $dM_{\text{particle}} = (p\cdot\xi)|_H$. Clearly dM, but not dJ, will have a small contribution from radiation entering the hole, but since the waves carry no angular momentum this contribution is positive (see Ref. 15), hence $dM = dM_{\text{particle}} + dM_{\text{radiation}} > dM_{\text{particle}}$. Contracting p with χ at the point of capture gives

$$(p\cdot\xi)|_H + \Omega_c(p\cdot\hat{\xi})|_H > 0$$

or

$$dM_{\text{particle}} > \Omega_c\, dJ \tag{16}$$

or

$$dM > \Omega_c dJ\,.$$

If the area theorem is to be upheld by Eq. (16), we deduce

$$\Omega_c > \frac{L}{8M^3}\,. \tag{17}$$

As would be expected from an interaction picture, the lower bound of Ω_c is *proportional* to L (the angular momentum of the ring).

Will[16] has performed a perturbation calculation for this problem and finds $\Omega_c = L/4M^3$ which satisfies Eq. (17), and the area theorem is upheld. In fact, putting $\Omega_c = L/4M^3$ in Eq. (16) yields $d\mathcal{A} > 4\pi L^2/M^2$. The neglect of particle-hole interactions in Hiscock's analysis has already been pointed out by Abramowicz *et al.*[17] However, they did not show how this led to the area theorem being satisfied.

Essentially the same trouble arises if we inject a ring of uncharged particles with orbital angular momentum into an extreme Reissner-Nordström hole, for it would then appear that we could destroy its horizon. The horizon-nondestructibility condition is derived by defining

$$f(M,J,Q) = M^2 - \frac{J^2}{M^2} - Q^2\,,$$

which is non-negative when a horizon is present. If there is a horizon after the injection, then

$$df \geq 0\,. \tag{18}$$

In Secs. II and III, second-order terms in Eq. (18) could be neglected, yielding Eq. (1), but in this case we obtain

$$dM \geq \frac{(dJ)^2}{2M^3}\,. \tag{19}$$

If, as was the case for a Schwarzschild hole, Eq. (19) is to be a consequence of $p\cdot\chi > 0$, we are led to conjecture that

$$\Omega_c > \frac{L}{2M^3}\,. \tag{20}$$

Precisely the same reasoning applied to the case of firing a charged particle radially into an extreme Kerr hole leads us to conjecture that as the particle approaches, it induces a potential (see Ref. 8) on the hole with a value at the point of capture of

$$\Phi_c > \frac{e}{4M}\,. \tag{21}$$

We therefore suggest that if perturbation calculations were performed for extreme Kerr and Reissner-Nordström black holes in order to find Ω_c and Φ_c, then they should satisfy Eq. (20) and Eq. (21), respectively, if cosmic censorship is to be upheld.

ACKNOWLEDGMENTS

It is a pleasure to thank Dr. Paul Tod for many helpful conversations and for constant encouragement during the development of this work. The author would also like to thank Dr. Robert Wald for suggesting that the argument might be generalized to spinning particles and for his constructive criticisms. The financial support of a Science Research Council (S.R.C.) Research Studentship is also gratefully acknowledged.

APPENDIX

In this appendix we calculate V^2 using spin coefficients and show $0 \geq V^2 \geq -1/M^2$, V being null at the poles. Let $(l', n', m, \bar{m})$ be the Kinnersley frame[19] for which all the spin coefficients have been written down:

$$l'^a = \frac{1}{\Delta}(r^2+a^2, \Delta, 0, a), \quad n'^a = \frac{1}{2\rho^2}(r^2+a^2, -\Delta, 0, a)\,,$$

$$m^a = \frac{1}{\sqrt{2}\,\bar{\rho}}(ia\sin\theta, 0, 1, i\csc\theta)\,,$$

where $\bar{\rho} = r + ia\cos\theta$, $\bar{\rho}^* = r - ia\cos\theta$, $\rho^2 = \bar{\rho}\bar{\rho}^*$, $\Delta = r^2 - 2Mr + a^2 + Q^2$. Various inner products occurring in the following calculation blow up on the horizon, and thus it is more convenient to use the $(l, n, m, \bar{m})$ frame defined by $l = \Delta l'$, $n = \Delta^{-1} n'$.

On H, l points along χ so that $V\cdot\chi = 0$ implies $V\cdot l = 0$, so

$$V^2 = 2(l\cdot V)(n\cdot V) - 2(m\cdot V)(\bar{m}\cdot V) = -2(m\cdot V)(\bar{m}\cdot V)\,.$$

We must therefore calculate $m\cdot V$. Now $\chi_{a;b} = 2\chi_{[a}V_{b]}$ so noting $\chi\cdot l = \chi\cdot m = \chi\cdot\bar{m} = 0$ we get

$$n^a m^b \chi_{a;b} = (m\cdot V)(\chi\cdot n)\,. \qquad \text{(A1)}$$

To find $n^a m^b \chi_{a;b}$ expand χ: $\chi_a = (\chi\cdot n)l_a + (\chi\cdot l)n_a - (\chi\cdot m)\bar{m}_a - (\chi\cdot \bar{m})m_a$, so that a short calculation gives

$$n^a m^b \chi_{a;b} = (\chi\cdot n)n^a m^b l_{a;b} + \nabla_m(\chi\cdot n)\,, \qquad \text{(A2)}$$

and a further simple calculation shows

$$\nabla_m(\chi\cdot n) = \frac{a^2 \sin 2\theta (M^2 - r^2)}{2\sqrt{2}\,\bar{\rho}\rho^4(M^2+a^2)} = 0 \quad \text{on } H\,.$$

Thus since $n^a m^b l_{a;b}$ is known explicitly in terms of spin coefficients to be $\alpha^* + \beta$, combining (A1) and (A2) yields

$$m\cdot V = \alpha^* + \beta\,.$$

However, since the transformation $l = \Delta l'$, $n = \Delta^{-1} n'$ does not change $\alpha^* + \beta$ we can simply look at the spin coefficients in the $(l', n', m, \bar{m})$ Kinnersley frame and see[19]

$$\alpha^* + \beta = -\frac{ia\sin\theta}{\sqrt{2}\,(\bar{\rho})^2}\,.$$

Hence

$$V^2 = -2\,|\alpha^* + \beta|^2 = -\frac{a^2\sin^2\theta}{(M^2 + a^2\cos^2\theta)^2} \geq -\frac{1}{M^2}$$

and is zero at the poles.

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