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Higgs-boson decay and the running mass

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We calculate QED and quantum-chromodynamics (QCD) radiative corrections to the fermionic decays of the Higgs boson, $H \to l^+ l^-$ and $H \to q\bar{q}$. A novel feature of the calculation is that renormalization introduces logarithms of the fermion mass into the total decay rate. The order-α QED corrections suppress the decay rate by a few percent. The order-α_s QCD corrections to the hadronic decay rate for a heavy Higgs boson are large, and lowest-order perturbation theory is useful only for quantities in which the logarithms of the quark mass cancel, such as $\Gamma(H \to 2\text{ jets})/\Gamma(H \to \text{hadrons})$. We obtain reliable corrections to the decay rate $H \to q\bar{q}$ by summing the QCD corrections to all orders in the leading-logarithm approximation. The result has a simple interpretation in terms of the "running mass" of the renormalization group: The effective quark–Higgs-boson coupling is proportional to the running mass of the quark evaluated at the Higgs-boson mass.

I. INTRODUCTION

The development of unified models of the weak and electromagnetic interactions, such as the standard $SU(2)\times U(1)$ model,[1] has been one of the great advances in elementary-particle physics. Higgs bosons, introduced for spontaneous symmetry breaking, are necessary ingredients in all such models to guarantee their renormalizability. The possibility of detecting Higgs bosons at present and planned accelerators has been reviewed recently.[2] A careful study of their decay modes must include radiative corrections.

We have calculated the QED and quantum-chromodynamics (QCD) radiative corrections to the decay rate of the Higgs particle into a fermion-antifermion pair in the standard weak (SW) model. A novel feature of this calculation is the appearance of mass singularities (logarithms of the fermion mass) in the total decay rate. Although Kinoshita's theorem[3] guarantees the absence of mass singularities in the *unrenormalized* total decay rate, the renormalization of the fermion–Higgs-boson coupling introduces a logarithm of the fermion–Higgs-boson mass ratio into the radiative corrections. This feature is a consequence of the fermion–Higgs-boson coupling being proportional to the fermion mass.

For leptonic decays $H \to l^+l^-(\gamma)$, we find that QED corrections give a small suppression of the decay rate. For hadronic decays $H \to q\bar{q}(g)$, the QCD corrections, because of the mass singularity, can exceed the lowest-order decay rate. This indicates that the naive application of perturbation theory is not reliable. It is possible, however, to give a reliable prediction for the fraction $\Gamma(H\to 2\text{ jets})/\Gamma(H\to\text{hadrons})$.

Using Kinoshita's theorem and the renormalization properties of the quark–Higgs-boson coupling, one can sum the leading logarithms of the quark–Higgs-boson mass ratio to all orders in QCD perturbation theory. The resulting correction is some power of the ratio of the logarithms of the quark and Higgs-boson masses. This correction has a very simple renormalization-group interpretation. In the context of a mass-independent renormalization prescription, a "running mass" parameter can be defined along with the "running coupling constant." The QCD corrections to the Higgs-boson decay rate in the leading-logarithm approximation can then be obtained by replacing the quark mass in the lowest-order decay rate by the running mass of the quark.

II. THE UNRENORMALIZED DECAY RATE

The lowest-order contribution to the decay of the Higgs boson into fermions comes from the diagram in Fig. 1(a):

$$\Gamma_0 = \frac{C}{8\pi} g^2 M \beta^3, \quad \beta \equiv \left(1 - \frac{4m^2}{M^2}\right)^{1/2}, \tag{1}$$

where M is the Higgs-boson mass and m is the fermion mass. g is the fermion–Higgs-boson coupling constant, which in the SW model is proportional to the fermion mass $g = m(G_F\sqrt{2})^{1/2}$. C is a color factor: $C=1$ for leptons, $C=3$ for quarks.

Kinoshita's theorem[3] states that, in each order of perturbation theory, the *unrenormalized* total decay rate of an unstable particle will contain no mass singularities (logarithms of the masses of the decay products). The word "unrenormalized" requires a little explanation. The matrix element for the unrenormalized decay rate to a given order is obtained by summing the contributions from all Feynman diagrams of that order, *excluding* diagrams which contain wave-function renormalization counterterms on internal lines and/or vertex

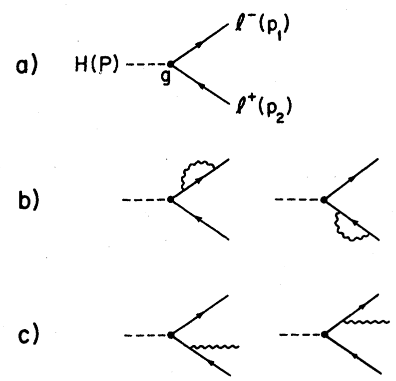

FIG. 1. Feynman diagrams for Higgs-boson decay $H \to l^+ l^- (\gamma)$: (a) lowest order, (b) virtual-photon corrections, (c) real-photon corrections. Diagrams with counterterms have been omitted for simplicity.

counterterms. Diagrams with mass counterterms are included, and the theorem assumes that these mass counterterms are calculated on the mass shell. Diagrams with self-energy parts on external lines contribute the usual multiplicative factor $\sqrt{Z}$, where Z is the appropriate wave-function renormalization constant. To show that our results satisfy Kinoshita's theorem, we will first calculate the order-α QED corrections to the unrenormalized Higgs-boson decay rate into leptons, so that we can verify the cancellation of the mass singularities.

The virtual corrections of order $\alpha = e^2/4\pi$ come from the interference between the diagrams of Fig. 1(b) and the lowest-order diagram, Fig. 1(a). We calculated these diagrams in the unitary gauge, renormalizing the leptons on the mass shell. Ultraviolet and infrared divergences were regulated by calculating in $4-2\epsilon$ dimensions.[4] The mass counterterm δm and the wave-function renormalization constant Z_2 for the leptons are extracted from the lepton self-energy parts in the first two diagrams of Fig. 1(b):

$$\delta m = m\frac{\alpha}{4\pi}\left(\frac{4\pi}{M^2}\right)^{\epsilon}\left[3\Gamma(\epsilon)_{\rm UV} - 6\ln\frac{m}{M} + 4\right], \tag{2}$$

$$Z_2 = 1 + \frac{\alpha}{4\pi}\left(\frac{4\pi}{M^2}\right)^{\epsilon}\left[-\Gamma(\epsilon)_{\rm UV} - 2\Gamma(\epsilon)_{\rm IR} + 6\ln\frac{m}{M} - 4\right]. \tag{3}$$

We have distinguished the divergences of ultraviolet origin from those of infrared origin with a subscript. The third diagram of Fig. 1(b) provides a vertex correction $gG(P, p_1, p_2)$ to the lowest-order vertex, g. We need only the real part of this vertex correction, evaluated with all external particles on their mass shells:

$$\mathrm{Re}G = \frac{\alpha}{4\pi}\left(\frac{4\pi}{M^2}\right)^{\epsilon}\left\{4\Gamma(\epsilon)_{\rm UV} + L\frac{1+\beta^2}{\beta}\Gamma(\epsilon)_{\rm IR} + 6 - 8\ln\frac{m}{M} + \left(\frac{2}{\beta} - 2\beta\right)L + \frac{1+\beta^2}{\beta}\left[\tfrac{1}{2}L^2 - 2L\ln\beta + 2\,\mathrm{Sp}\left(\frac{1-\beta}{1+\beta}\right) + \tfrac{2}{3}\pi^2\right]\right\}, \tag{4}$$

where $L = \ln(1+\beta)/(1-\beta)$ and $\mathrm{Sp}(x) = -\int_0^x (dt/t)\ln(1-t)$ is the Spence function.

The unrenormalized decay rate into a lepton-antilepton pair to order α is obtained from Eq. (1) by replacing g^2 by

$$|(\sqrt{Z_2})^2(g + gG)|^2 \simeq g^2[1 + 2\,\mathrm{Re}G + 2(Z_2 - 1)]\,. \tag{5}$$

Using (3) and (4), the order-α virtual correction becomes

$$\Gamma_1^V = \Gamma_0\frac{\alpha}{2\pi}\left(\frac{4\pi}{M^2}\right)^{\epsilon}\left\{3\Gamma(\epsilon)_{\rm UV} + \left(\frac{1+\beta^2}{\beta}L - 2\right)\Gamma(\epsilon)_{\rm IR} + 2 - 2\ln\frac{m}{M} + \left(\frac{2}{\beta} - 2\beta\right)L + \frac{1+\beta^2}{\beta}\left[\tfrac{1}{2}L^2 - 2L\ln\beta + 2\,\mathrm{Sp}\left(\frac{1-\beta}{1+\beta}\right) + \tfrac{2}{3}\pi^2\right]\right\}\,. \tag{6}$$

Since Kinoshita's theorem refers to the *total* decay rate, we must also include the order-α corrections from the emission of real photons. The relevant diagrams are shown in Fig. 1(c). We calculated separately the hard- and soft-photon contributions, defining soft photons by an infinitesimal energy cutoff λ. Dimensional regularization was used to calculate the soft-photon contribution. Adding together the two contributions, the λ dependence cancels leaving

$$\Gamma_1^R = \Gamma_0\frac{\alpha}{2\pi}\left(\frac{4\pi}{M^2}\right)^{\epsilon}\left\{-\left(\frac{1+\beta^2}{\beta}L - 2\right)\Gamma(\epsilon)_{\rm IR} + 8 - \frac{3}{4}\frac{1+\beta^2}{\beta^2} + 8\ln\frac{m}{M} - 8\ln\beta + \left(\frac{3}{\beta} + \frac{3}{8}\frac{(1-\beta^2)^2}{\beta^3}\right)L \right.$$
$$\left. + \frac{1+\beta^2}{\beta}\left[-\tfrac{1}{2}L^2 + 4L\ln\frac{1+\beta}{2} - 2L\ln\beta - 2\ln\frac{1+\beta}{2}\ln\frac{1-\beta}{2} + 6\,\mathrm{Sp}\left(\frac{1-\beta}{1+\beta}\right) - 4\,\mathrm{Sp}\left(\frac{1-\beta}{2}\right) - \tfrac{2}{3}\pi^2\right]\right\}. \tag{7}$$

The total unrenormalized decay rate to order α is obtained by adding (6) and (7) to the lowest-order decay rate:

$$\Gamma_u = \frac{1}{8\pi} g^2 M\beta^3 + \Gamma_1^V + \Gamma_1^R . \qquad (8)$$

The infrared divergences cancel between Γ_1^V and Γ_1^R as usual, but the ultraviolet divergences do not. Taking the limit $m/M \to 0$, we find that Eq. (8) reduces to

$$\Gamma_u = \frac{1}{8\pi} g^2 M \left\{ 1 + \frac{\alpha}{2\pi}\left(\frac{4\pi}{M^2}\right)^{\epsilon} \left[3\Gamma(\epsilon)_{\rm UV} + \tfrac{17}{2}\right] \right\} . \qquad (9)$$

This contains no $\ln m$ terms, demonstrating that the mass singularities have canceled in accordance with Kinoshita's theorem.

III. RENORMALIZATION OF FERMION-HIGGS-BOSON COUPLING

In a spontaneously broken gauge theory in which the fermion acquires its mass from the Higgs mechanism, the mass of the fermion and its coupling to the Higgs boson cannot be renormalized independently because they arise from the same term in the Lagrangian. In the SW model, the relevant terms in the unrenormalized Lagrangian are[5]

$$\mathcal{L} = \bar\psi_0 i\partial\!\!\!/ \psi_0 - g_0 \bar\psi_0 \psi_0 (v + h) - e_0 \bar\psi_0 \gamma_\mu \psi_0 A_0^\mu , \qquad (10)$$

where ψ_0 and A_0^μ are the unrenormalized lepton and photon fields, v is the vacuum expectation value of the scalar field, and h is the Higgs field. To lowest order in g, the Higgs field needs no wave-function renormalization. The unrenormalized electron mass can be identified as $m_0 = g_0 v$. We now write this Lagrangian in terms of the renormalized fields ψ and A^μ, defined by $\psi_0 = \sqrt{Z_2}\psi$ and $A_0^\mu = \sqrt{Z_3} A^\mu$, expressing it as the sum of a renormalized Lagrangian $\mathcal{L}_R$ and a counterterm Lagrangian $\mathcal{L}_c$:

$$\begin{aligned} \mathcal{L}_R &= \bar\psi (i\partial\!\!\!/ - m)\psi - g\bar\psi\psi h - e\bar\psi\gamma_\mu\psi A^\mu , \\ \mathcal{L}_c &= -(1 - Z_2)[\bar\psi(i\partial\!\!\!/ - m)\psi] + Z_2 \delta m \bar\psi\psi \\ &\quad - (Z_g - 1) g\bar\psi\psi h - (Z_1 - 1) e\bar\psi\gamma_\mu\psi A^\mu . \end{aligned} \qquad (11)$$

The renormalized mass is given by $m = gv$ and the relations between renormalized and unrenormalized parameters are

$$\begin{aligned} m_0 &= m - \delta m , \\ g_0 &= Z_g g Z_2^{-1} , \\ e_0 &= Z_1 e \cdot Z_2^{-1} Z_3^{-1/2} . \end{aligned} \qquad (12)$$

Renormalizing the fermion on its mass shell determines the counterterm δm and the renormalization constant Z_2 and consequently also the counterterm for g via the relation $m - \delta m = m Z_g Z_2^{-1}$.

The renormalized decay rate Γ is calculated by including the Feynman diagrams with vertex counterterms and wave-function renormalization counterterms. The net effect of the extra diagrams is to make the following substitutions in Γ_u: $g \to (Z_g g) Z_2^{-1} = g_0$, $e \to (Z_1 e) Z_2^{-1} Z_3^{-1/2} = e_0$. To order α, the only renormalization necessary is that of g. So the renormalized decay rate is obtained by replacing g^2 in Eq. (8) by

$$g_0{}^2 = g^2 \left(\frac{m - \delta m}{m}\right)^2 \simeq g^2 \left(1 - 2\frac{\delta m}{m}\right). \qquad (13)$$

Using δm from Eq. (2), the total renormalized decay rate to order α is

$$\begin{aligned} \Gamma = \Gamma_0 \Bigg(1 + \frac{\alpha}{2\pi} \Bigg\{ & 6 - \frac{3}{4}\frac{1+\beta^2}{\beta^2} + 12 \ln\frac{m}{M} - 8\ln\beta + \left(\frac{5}{\beta} - 2\beta + \frac{3}{8}\frac{(1-\beta^2)^2}{\beta^3}\right) L \\ & + \frac{1+\beta^2}{\beta}\left[4L \ln\frac{1+\beta}{2\beta} - 2\ln\frac{1+\beta}{2}\ln\frac{1-\beta}{2} + 8\,\mathrm{Sp}\left(\frac{1-\beta}{1+\beta}\right) - 4\,\mathrm{Sp}\left(\frac{1-\beta}{2}\right)\right]\Bigg\}\Bigg) . \end{aligned} \qquad (14)$$

The ultraviolet divergences have canceled as expected. Taking the limit $m/M \to 0$, we find that the total decay rate now contains a $\ln m$ term:

$$\Gamma \to \frac{1}{8\pi} g^2 M \left[1 + \frac{\alpha}{2\pi}\left(\frac{9}{2} - 6\ln\frac{M}{m}\right)\right]. \qquad (15)$$

Thus a mass singularity has been introduced by the renormalization procedure. Note, however, that the limit $m \to 0$ is still smooth because of the overall factor of m^2 contained in g^2.

The effect of this mass singularity in the QED radiative corrections is not very dramatic. Even if the Higgs boson is as heavy as the W boson (~85 GeV), the order-α corrections decrease the decay rate into electrons by only 8.4%. If the Higgs boson is much heavier than the weak vector bosons, then weak corrections are comparable in magnitude to the electromagnetic corrections and the complete one-loop weak corrections would be needed.

IV. QCD CORRECTIONS AND JETS

The order-α_s QCD corrections to the decay rate of the Higgs boson into a quark-antiquark pair can be obtained from Eq. (14) by replacing α by

$\frac{4}{3}\alpha_s(M)$, where $\frac{4}{3}$ is a color factor and $\alpha_s(M)$ is the QCD coupling constant at the mass scale of the Higgs boson:

$$\alpha_s(M) \simeq \frac{1}{b}\frac{1}{\ln(M/\Lambda_{\rm QCD})}, \quad b = \frac{33-2N_f}{6\pi}. \qquad (16)$$

N_f is the number of quark flavors with threshold below the Higgs-boson mass, and $\Lambda_{\rm QCD}$ is determined experimentally to be approximately 0.5 GeV. In the limit $m/M \to 0$, the order-α_s QCD correction is

$$\Gamma_1 = \Gamma_0 \frac{2\alpha_s(M)}{3\pi}\left(\frac{9}{2} - 6\ln\frac{M}{m}\right). \qquad (17)$$

Assuming 6 flavors of quarks and letting $M \to \infty$, we see that Γ_1 asymptotically approaches $-\frac{8}{7}\Gamma_0$. Hence, in spite of the inverse-logarithmic decrease of α_s for a very heavy Higgs boson, the order-α_s corrections can exceed the lowest-order decay rate. Even for a moderately heavy Higgs, the effects can be large, as shown in Fig. 2. This implies that higher-order terms in the perturbation expansion may be important, so we can trust neither Eq. (17) nor the lowest-order result, Eq. (1).

We can get reliable predictions from the order-α_s QCD corrections if we consider "infrared safe" quantities, i.e., quantities for which the mass singularities cancel. An example is the ratio of two *partial* decay rates which are free of mass singularities before renormalization. After renormalization, each decay rate is proportional to g_0^2, and the mass singularities will cancel when we take the ratio. An example of such a ratio is the fraction of hadronic decays which go into two jets: $f = \Gamma(H \to 2 \text{ jets})/\Gamma(H \text{ hadrons})$. A two-jet event can be defined in the rest frame of the Higgs boson as one in which all but a fraction less than ϵ of the energy M is emitted into a pair of oppositely directed cones of half-angle δ.[6] To order α_s any event which is not a two-jet event is a three-jet event. Ignoring quark masses, the decay rate into three-jet events is

$$\Gamma(3\text{-jet}) = \Gamma_0 \frac{4\alpha_s(M)}{3\pi}\left[(4\ln 2\epsilon + 3)\ln\delta + \frac{\pi^2}{3} + \frac{5}{4}\right], \qquad (18)$$

where we have used $\epsilon, \delta \ll 1$ and $\sin\delta > \epsilon/(1-\epsilon)$.[7] This has the same form as the result for $e^+e^- \to 3$ jets except for the constant term. The fraction $f \simeq 1 - \Gamma(3\text{-jet})/\Gamma_0$ has no mass singularities and hence the order-α_s QCD corrections to this quantity are reliable.

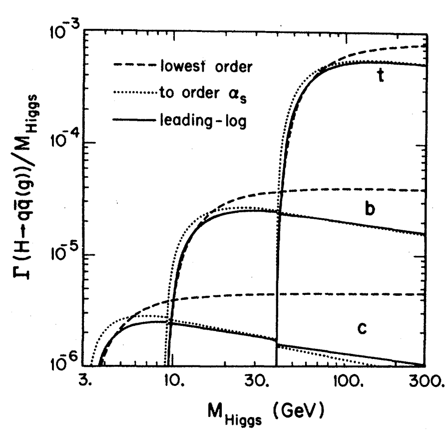

FIG. 2. Decay rate of the Higgs boson into a quark-antiquark pair, in units of the Higgs-boson mass, for c quark, b quark, and a 20-GeV t quark. The discontinuity at each new flavor threshold comes from the simple use of formula Eq. (16) and is not physical. The real rate interpolates smoothly.

V. LEADING-LOGARITHM APPROXIMATION IN QCD

Another approach to the problem of the large order-α_s QCD corrections is to sum, to all orders of perturbation theory, the leading logarithms, i.e., terms of the form $\alpha_s^n \ln^n(M/m)$. We will in fact deduce the leading-logarithm QCD corrections to the Higgs-boson decay rate to all orders without calculating any additional diagrams, simply by using Kinoshita's theorem and the renormalization properties of the Higgs-boson vertex.

From Sec. III, we know that the renormalized Higgs-boson decay rate Γ can be obtained from the unrenormalized decay rate Γ_u by the substitutions $g \to g_0$, $\alpha_s \to \alpha_{s0}$, where α_{s0} is the unrenormalized QCD coupling constant. Since we are only considering the lowest order in g, Γ_u is proportional to g^2:

$$\Gamma_u \simeq \frac{3}{8\pi} g^2 M A\left(\alpha_s \ln\frac{\Lambda}{M}\right), \quad A = 1 + \frac{4}{3}\frac{\alpha_s}{2\pi} 6\ln\frac{\Lambda}{M} + \cdots. \qquad (19)$$

We have obtained A from Eq. (9) by inserting a color factor of $\frac{4}{3}$. For convenience, we have replaced the $1/\epsilon$ from dimensional regularization by an ultraviolet divergence $\ln\Lambda^2$, where Λ is a momentum cutoff parameter. Dimensional considerations prevent any ambiguity in the leading logarithms. The function A can depend only on $\ln\Lambda/M$, as Kinoshita's theorem guarantees the absence of any $\ln m$ singularities in the unrenormalized decay rate.

The renormalized decay rate is then given by

$$\Gamma = \frac{3}{8\pi} g_0^2 M A\left(\alpha_{s0}\ln\frac{\Lambda}{M}\right)$$
$$= \frac{3}{8\pi} g^2 M A\left(\alpha_{s0}\ln\frac{\Lambda}{M}\right) B\left(\alpha_{s0}\ln\frac{\Lambda}{2m}\right), \qquad (20)$$

where $B=(g_0/g)^2=(1-\delta m/m)^2$. B depends only on the mass counterterm from the QCD corrections to the quark self-energy. Since the Higgs boson does not contribute to the quark self-energy to lowest order in g, the function B can depend only on $\ln(\Lambda/2m)$. The factor of 2 in the logarithm is irrelevant in the leading-logarithm approximation, and is inserted only for convenience because it gives the proper threshold behavior to the final result.

The unrenormalized QCD coupling constant α_{s0} can be expressed in terms of the running QCD coupling constant $\alpha_s(\mu)$, where μ is an arbitrary renormalization scale parameter with dimensions of mass. The μ dependence of $\alpha_s(\mu)$ is governed by the β function $\beta(\alpha_s)$ via the renormalization-group equation

$$\frac{\partial}{\partial \ln\mu}\alpha_s(\mu)=\beta[\alpha_s(\mu)]\,. \tag{21}$$

The leading-logarithm behavior of $\alpha_s(\mu)$ is governed by the lowest-order coefficient b in the expansion of the β function

$$\beta(\alpha_s)=-b\alpha_s{}^2+\cdots\,,\quad b=\frac{33-2N_f}{6\pi}\,. \tag{22}$$

The solution to the renormalization-group equation (21) in lowest order is

$$\alpha_s(\mu)=\frac{\alpha_s(\mu_0)}{1+b\alpha_s(\mu_0)\ln(\mu/\mu_0)}\,, \tag{23}$$

where μ_0 is any other renormalization scale parameter. From this equation, we can deduce that the leading-logarithm expression for α_{s0} in terms of $\alpha_s(M)$ is

$$\alpha_{s0}=\frac{\alpha_s(M)}{1+b\alpha_s(M)\ln(\Lambda/M)}\,. \tag{24}$$

Inserting Eq. (24) into Eq. (20) we see that the expression for the renormalized decay rate takes the form

$$\Gamma\left(\alpha_s(M)\ln\frac{M}{2m}\right)=\Gamma_0 A\left(\alpha_s(M)\ln\frac{\Lambda}{M}\right)B\left[\frac{\alpha_s(M)\ln(\Lambda/2m)}{1+b\alpha_s(M)\ln(\Lambda/M)}\right]. \tag{25}$$

Note that renormalizing α_s at the Higgs mass leaves the function A free of mass singularities. This relation is sufficient to determine the functional forms of A, B, and Γ. The solution is derived in an appendix:

$$\Gamma=\Gamma_0\left[1-b\alpha_s(M)\ln\frac{M}{2m}\right]^{c/b}, \tag{26}$$

where c is an unspecified integration constant. It can be determined by expanding to order $\alpha_s(M)$ and comparing with our lowest-order calculation, Eq. (17). This yields $c=4/\pi$. Finally, we use Eq. (16) to write

$$\Gamma=\Gamma_0\left(1-\frac{\ln(M/2m)}{\ln(M/\Lambda_{QCD})}\right)^{4/\pi b}=\Gamma_0\left(\frac{\ln(2m/\Lambda_{QCD})}{\ln(M/\Lambda_{QCD})}\right)^{24/(33-2N_f)} \tag{27}$$

Equation (27) is the leading-logarithm approximation to the Higgs-boson decay rate into a quark-antiquark pair, including QCD corrections to all orders in $\alpha_s\ln(M/m)$.

The result of the leading-logarithm summation is compared to the lowest-order decay rate and the decay rate to order α_s in Fig. 2. For energies moderately high above the quark-antiquark threshold, the leading-logarithm result is very close to the order-α_s decay rate. This is a consequence of the fact that $c\simeq b$ for $N_f=4$ or 5 flavors. As $M\to\infty$, Γ/Γ_0 smoothly approaches 0 in the leading-logarithm approximation, while to order α_s, it goes negative and becomes nonsense.

VI. THE RUNNING MASS

The leading-logarithm approximation to the Higgs-boson decay rate has led to the very simple result given in Eq. (27). We would like a simple physical interpretation of this result. The answer is contained in a renormalization-group analysis of mass parameters.

In a massless theory the coupling constant α must depend on some renormalization scale parameter μ. The μ dependence of this "running coupling constant" $\alpha(\mu)$ is governed by the renormalization-group equation (21). In a massive theory with mass parameter $\tilde{m}$, a mass-independent renormalization procedure can be defined, using dimensional regularization, and defining all counterterms by subtracting only the poles in ϵ from primitively divergent Green's functions.[8] For example, the mass counterterm for the quark to order α_s in QCD would be

$$\delta\tilde{m}=\tilde{m}\frac{4}{3}\frac{\alpha_s}{4\pi}\times 3\times\frac{1}{\epsilon}\,. \tag{28}$$

This should be compared with the mass counterterm δm [Eq. (2)] using on-shell renormalization. The extra factor of $\frac{4}{3}$ in Eq. (28) is a QCD color factor. In a mass-independent renormalization procedure, the parameter $\tilde{m}$ is *not* the physical mass, which, as usual, is defined by the position of the pole in the propagator. The physical mass is now a calculable function of α, $\tilde{m}$, and the renormalization parameter μ. The value of $\tilde{m}$ will, along with the value of α, depend on μ, and the μ dependence of this "running mass" $\tilde{m}(\mu)$ is governed by a renormalization-group equation:

$$\frac{\partial}{\partial \ln\mu}\bar{m}(\mu)=\gamma_m[\alpha(\mu)]\bar{m}(\mu), \tag{29}$$

where $\gamma_m(\alpha)$ is defined in terms of the mass-renormalization constant $Z_m(\alpha)=(\bar{m}-\delta\bar{m})/\bar{m}$:

$$\gamma_m(\alpha)=-\frac{1}{Z_m}\frac{\partial}{\partial\ln\mu}\bigg|_{\alpha_0} Z_m(\alpha_0\mu^{-2\epsilon}). \tag{30}$$

In QCD, the mass-renormalization constant for quarks is given to order α_s by Eq. (28):

$$Z_m(\alpha_s)=1-\frac{1}{\pi}\alpha_s\frac{1}{\epsilon}. \tag{31}$$

Hence

$$\gamma_m(\alpha_s)=-\frac{2}{\pi}\alpha_s+\cdots . \tag{32}$$

The solution to the renormalization-group equation (29) for the running mass $\bar{m}(\mu)$ is

$$\bar{m}(\mu)=\bar{m}(\mu_0)\exp\left[\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha\frac{\gamma_m(\alpha)}{\beta(\alpha)}\right]. \tag{33}$$

Using the lowest-order forms for β and γ_m we find that the running mass has the behavior

$$\bar{m}(\mu)=\bar{m}(\mu_0)\exp\left[\frac{2}{\pi b}\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{d\alpha}{\alpha}\right]$$
$$=\bar{m}(\mu_0)\left[\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]^{2/\pi b}. \tag{34}$$

Expressing the running coupling constant in terms of the QCD scale parameter $\Lambda_{\rm QCD}$ using Eq. (16), we can write

$$\bar{m}(\mu)=\bar{m}(\mu_0)\left(\frac{\ln\mu_0/\Lambda_{\rm QCD}}{\ln\mu/\Lambda_{\rm QCD}}\right)^{2/\pi b}. \tag{35}$$

Returning to Higgs-boson decay, we see that the leading-logarithm expression for the QCD corrections to the decay rate, Eq. (27), can be expressed in the form

$$\Gamma=\frac{3}{8\pi}(G_F\sqrt{2}\,m^2)M\beta^3\left[\frac{\bar{m}(M)}{\bar{m}(2m)}\right]^2, \tag{36}$$

where β is the threshold factor of Eq. (1). We can identify the running mass at the quark threshold $\bar{m}(2m)$ with the physical mass of the quark m, so we can write the decay rate in the very simple form

$$\Gamma=\frac{3}{8\pi}[G_F\sqrt{2}\,\bar{m}(M)^2]\,M\beta^3. \tag{37}$$

Thus, in the leading-logarithm approximation, the effect of QCD corrections is to make the Higgs-boson hadronic decay rate proportional to the square of the running mass of the quark, evaluated at the mass of the Higgs boson.

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APPENDIX

Defining $x\equiv\alpha_s(M)\ln(\Lambda/2m)$, $y\equiv\alpha_s(M)\ln(\Lambda/M)$, and $z=x/(1+by)$, Eq. (25) takes the form

$$F(x-y)=A(y)B(z). \tag{A1}$$

Differentiating both sides with respect to x and y and using $\partial F/\partial x=-\partial F/\partial y$ leads to

$$(1+by)\frac{\partial}{\partial y}\ln A(y)=-\frac{[1-b(x-y)]}{(1+by)}\frac{\partial}{\partial z}\ln B(z)$$
$$=-(1-bz)\frac{\partial}{\partial z}\ln B(z). \tag{A2}$$

But y and z are independent variables, so (A2) can only be satisfied if both sides are equal to some constant, c. Integrating the two resulting equations trivially gives

$$A(y)=c'(1+by)^{c/b},$$
$$B(z)=c''(1-bz)^{c/b}. \tag{A3}$$

From Eq. (A1) we then obtain

$$F(x-y)=\bar{c}[1-b(x-y)]^{c/b}, \tag{A4}$$

where c', c'', and $\bar{c}$ are integration constants.

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