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Analytic properties of the crossing matrix between kinematic-singularity-free helicity amplitudes for pion production

J. Kinsella
Department of Mathermatical Physics, University College, Belfield, Dublin, 4, Ireland
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We determine the crossing matrix between regularized helicity amplitudes for pion production and show that this matrix is analytic apart from poles in the continued crossed-channel c.m. energy, corresponding to poles in the crossed-channel amplitudes.

I. INTRODUCTION

The crossing matrix relating direct- and crossed-channel helicity amplitudes has been calculated for two-body processes by Trueman and Wick,[1] Muzinich[2] and Cohen-Tannoudji, Morel, and Navalret (CMN).[3] Capella[4] and Chen and Wang[5] found the matrix for $(2\to N)$ processes. The singularity structure of helicity amplitudes was analyzed in the two-body case by CMN,[3] Trueman[6] (cf. Ref. 7 for further references), and in the $(2\to N)$ case by Svensson.[7]

In this paper we consider the process $N\pi\to N\pi\pi$ and its crossed process $N\bar{N}\to\bar{\pi}\pi\pi$. Following a brief summary of Svensson's methods in Sec. II, in Sec. III we write down explicitly the regularized helicity amplitudes (RHA's) in both the direct and crossed channels as linear combinations of helicity amplitudes using the general results of Ref. 7. Section IV contains the crossing matrix for helicity amplitudes together with the explicit forms for the crossing angles for the processes under consideration, based on the work of Capella.[4] Next, in Sec. V we determine the crossing matrix between the RHA's and show it to be analytic apart from poles in the continued crossed-channel c.m. energy, corresponding to the constraints on the crossed-channel RHA's. An appendix contains the definitions of the various kinematic determinants used.

Our results provide a check on the consistency of crossing with regularization, as the crossing relation for spinor amplitudes for the $(2\to N)$ case must be assumed, while that for two-body processes follows from quantum field theory.[8] Moreover, they are an essential first step in calculating sum rules for partial waves corresponding to the work of Modjtehedzadeh[9] for spinless multiparticle processes.

II. SVENSSON'S ANALYSIS OF KINEMATIC SINGULARITIES

In this section we summarize Svensson's method of analyzing the kinematic singularities of helicity amplitudes for n-particle final states. We refer the reader to his paper[7] for further details.

A. Kinematics

We consider here only the case of a three-particle final state, i.e., the process $a+b\to 1+2+3$. A one-particle helicity state $|m,s,\eta;\vec{p}\lambda\rangle$ or $|\vec{p}\lambda\rangle$ is defined by

$$|\vec{p}\lambda\rangle = e^{-i\phi J_z}e^{-i\theta J_y}e^{-i\xi K_z}|\vec{0}\lambda\rangle\,, \tag{2.1}$$

where $|\xi,\theta,\phi\rangle$ are defined by

$$E=m\cosh\xi\,, \tag{2.2a}$$

$$|\vec{p}|=m\sinh\xi\,, \tag{2.2b}$$

and

$$(p_x,p_y,p_z)=(|\vec{p}|\sin\theta\cos\phi,\ |\vec{p}|\sin\theta\sin\phi,\ |\vec{p}|\cos\theta)\,. \tag{2.2c}$$

The matrix elements $T_{\lambda_1\lambda_2\lambda_3;\lambda_a\lambda_{\bar{b}}}$ are taken to be

$$\langle\vec{p}_1\lambda_1;\vec{p}_2\lambda_2;\vec{p}_3\lambda_3|T|\vec{p}_a\lambda_a;p_{\bar{b}}\lambda_b\rangle\,,$$

normalized as in Ref. 7. Following Ref. 7 the set $\{p^2, p\cdot p_1, p_a\cdot p_1, p_a\cdot p_2, p_1\cdot p_2\}$ can be taken as the linearly independent set of scalar variables on which $T_{\{\lambda\}}$ depends. There also exists one pseudoscalar $\epsilon_2=\epsilon_{\mu\nu\rho\sigma}p^\mu p_a^\nu p_1^\rho p_2^\sigma$ on which $T_{\{\lambda\}}$ may depend. Parity invariance gives the relation

$$T_{\{\lambda\}}(Z,-\epsilon_2)=\pi_{\{\lambda\}}T_{-\{\lambda\}}(Z,+\epsilon_2)\,, \tag{2.3}$$

where Z represents the set of scalar variables above and

$$\pi_{\{\lambda\}}=\eta_1\eta_2\eta_3\eta_a\eta_1\prod_{i=a}^{3}(-)^{s_i-\lambda_i}\,. \tag{2.4}$$

For the reaction $N\pi\to N\pi\pi$, $\pi_{\{\lambda\}}=(-1)^{\lambda_1-\lambda_a}$ and for the crossed process we consider later, $N\bar{N}\to\bar{\pi}\pi\pi$, $\pi_{\{\lambda\}}=(-1)^{\lambda_1-\lambda_a}$ as the intrinsic parity of a fermion-antifermion pair is negative. Thus the linear combinations

$$T^{(1,2)}_{\{\lambda\}}=\binom{1}{\epsilon_2}(T_{\{\lambda\}}\pm\pi_{\{\lambda\}}T_{-\{\lambda\}}) \tag{2.5}$$

are even in ϵ_2, and so are regular at $\epsilon_2=0$. The angular parameters of Eq. (2.2) are given in terms of the scalars and ϵ_2 in Table I. The special Gram determinants $\Delta_2(p,q)$, $\Delta_3(p,q,r)$ are defined in the Appendix.

B. Kinematic singularities

By hypothesis,[6] kinematic singularities arise when the helicity states become singular on certain hypersurfaces, namely those defined by the singularities of the angular variables of Table I. Thus $T_{\{\lambda\}}$ may have kinematic singularities on the manifolds

$$\Delta_1(p)=s=0\,, \qquad (2.6a)$$

$$\Delta_2(p,p_a)=0\,, \qquad (2.6b)$$

$$\Delta_3(p,p_a,p_1)=0\,, \qquad (2.6c)$$

$$\Delta_2(p,p_k)=0, \quad k=1,2,3\,, \qquad (2.6d)$$

$$\Delta_3(p,p_a,p_k)=0, \quad k=1,2,3\,, \qquad (2.6e)$$

$$\Delta_4(p,p_a,p_1,p_2)=0\,. \qquad (2.6f)$$

Svensson's method of isolating these singularities, based on that of Trueman,[6] consists of commuting the boost and rotation operators which define the helicity states (2.1) to form combinations of the angular parameters for each particle which are analytic at the singularity surface in question. In general there then remains some boost or rotation operator with singular argument which, either singly or in combination with those for other particles, gives the singularity structure on the particular surface being examined.

Using Svensson's expressions for the singularity structure of the helicity amplitude, in the next Section we write down explicitly the regularized helicity amplitudes (RHA's) for $N\pi\to N\pi\pi$ and the crossed process $N\bar{N}\to\bar{\pi}\pi\pi$, also giving the constraints in the latter case.

III. REGULARIZED HELICITY AMPLITUDES

A. Regularized helicity amplitudes for the s_{ab}-channel process $N_a\pi_b \to N_1\pi_2\pi_3$

Following Svensson[7] we find (for $m_a\neq m_b$) no singularities on the surface $\Delta_1(p)=s=0$, and the only kinematic singularities of $T_{\lambda_1,0,0;\lambda_a,0}$ lie on the surfaces $\Delta_2(p,p_a)=0$, $\Delta_2(p,p_1)=0$, and $\Delta_3(p,p_a,p_1)=0$. We look at each surface separately.

(*a*) $\Delta_2(p,p_a)=\Delta_2(p_a,p_b)=0$. By Eq. 4.37 of Ref. 7 we have

$$T^{(1,2)}_{\lambda_1;\lambda_a}\equiv\begin{pmatrix}1\\ \epsilon_2\end{pmatrix}(T_{\lambda_1;\lambda_a}\pm(-)^{-\lambda_1-\lambda_a}T_{-\lambda_1,-\lambda_a})$$
$$=\begin{pmatrix}1\\ \epsilon_2\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\ \lambda_a & 0 & -\lambda_a\end{pmatrix}\frac{[D_2^{(\pm)}(p_a,p_b)]^{-1/4}}{[(\frac{1}{2}+\lambda_a)!\,(\frac{1}{2}-\lambda_a)!\,]^{1/2}}e^{i\pi(1/2-\lambda_a)/2}\sum_{p=0}^{\infty}\{\lambda_a[D_2^{(\pm)}(p_a,p_{\bar{b}})]^{1/2}\}^p(g^{(\pm)(p)}_{1/2;\lambda_1}\pm(-)^{-\lambda_1+p+1}g^{(I)(p)}_{1/2;-\lambda_1})\,. \qquad (3.1)$$

TABLE I. Kinematics of the process.

	Initial-state particle ($i=a,\ b$)	Final-state particle ($k=1,2\ldots m$)
$\cosh\xi$	$\dfrac{p\cdot p_i}{m_i\sqrt{s}}$	$\dfrac{p\cdot p_k}{m_k\sqrt{s}}$
$\sinh\xi$	$\dfrac{[\Delta_2(p,p_i)]^{1/2}}{m_i\sqrt{s}}$	$\dfrac{[\Delta_2(p,p_k)]^{1/2}}{m_k\sqrt{s}}$
$\cos\theta$	$+1$ for $i=a$; -1 for $i=b$	$\dfrac{-\begin{bmatrix}p & p_a\\ p & p_k\end{bmatrix}}{[\Delta_2(p,p_a)]^{1/2}[\Delta_2(p,p_k)]^{1/2}}$
$\sin\theta$	0	$\dfrac{[s\Delta_3(p,p_a,p_1)]^{1/2}}{[\Delta_2(p,p_a)]^{1/2}[\Delta_2(p,p_k)]^{1/2}}$
$\cos\phi$	1	$\dfrac{\begin{bmatrix}p & p_a & p_1\\ p & p_a & p_k\end{bmatrix}}{[\Delta_3(p,p_a,p_1)]^{1/2}[\Delta_3(p,p_a,p_k)]^{1/2}}$
$\sin\phi$	0	$\dfrac{\epsilon(p,p_a,p_1,p_k)[\Delta_2(p,p_a)]^{1/2}}{[\Delta_3(p,p_a,p_1)]^{1/2}[\Delta_3(p,p_a,p_k)]^{1/2}}$

Thus $[D_2^{(\pm)}(p_a,p_b)]^{1/4}T^{(1;2)}_{\lambda_1;\lambda_a}$ is regular at $D_2^{(\pm)}(p_a,p_b)=0$ [where we define $D_2^{(\pm)}(p,q)$ in the Appendix], so we can write $[\Delta_2(p,p_a)]^{1/4}T^{(1;2)}_{\lambda_1;\lambda_a}$ is regular at $\Delta_2(p,p_a)=0$.

(*b*) $\Delta_2(p,p_1)=0$. By Eq. 4.56 of Ref. 7,

$$T^{(1,2)}_{\lambda_1;\lambda_a}=\begin{pmatrix}1\\ \epsilon_2\end{pmatrix}\frac{[D_2^{(\pm)}(p,p_1)]^{-1/4}}{[(\frac{1}{2}+\lambda_1)!\,(\frac{1}{2}-\lambda_1)!)]^{1/2}}\sum_{p=0}^{\infty}\{\lambda_1[D_2^{(\pm)}(p,p_1)]^{1/2}\}^p(e^{\mp i\pi\lambda_1/2}h_{\lambda_a}^{(\pm)(p)}\pm(-)^{-\lambda_1-\lambda_a+p}e^{\pm i\pi\lambda_1/2}h_{-\lambda_a}^{(\pm)(p)})\,. \tag{3.2}$$

Thus again $[D_2^{(\pm)}(p,p_1)]^{1/4}T^{(1,2)}_{\lambda_1;\lambda_a}$ is regular at $D_2^{(\pm)}(p,p_1)=0$, and so $[\Delta_2(p,p_1)]^{1/4}T^{(1,2)}_{\lambda_1;\lambda_a}$ is regular at $\Delta_2(p,p_1)=0$.

(*c*) $\Delta_3(p,p_a,p_1)=0$. By Eq. 4.50 of Ref. 7, at $\Delta_3(p,p_a,p_1)=0$,

$$T_{\lambda_1;\lambda_a}=A^{(\pm)}_{\lambda_1;\lambda_a}e^{i\phi_2(\lambda_a\mp\lambda_1)}, \tag{3.3}$$

where ($\pm$) corresponds to the cases $D_3^{(\pm)}(p,p_a,p_1)=0$, and ϕ_2 is given in Table I.

There is a sign ambiguity in the expression of ϕ_2 in terms of $D_3^{(\pm)}(p,p_a,p_1)$ due to the presence of ϵ_2 in the numerator of $\sin\phi_2$. However, the two linear combinations $T^{(1;2)}_{\lambda_1;\lambda_a}$ of Eq. 2.5 can be written unambiguously as

$$T_{\lambda_1;\lambda_a}=[D_3^{(\pm)}(p,p_a,p_1)]^{-|\lambda_a\mp\lambda_1|/2}A^{(1,2)(\pm)}_{\lambda_1;\lambda_a}, \tag{3.4}$$

where $A^{(1;2)(\pm)}_{\lambda_1;\lambda_a}$ is regular at $\Delta_3(p,p_a,p_1)=0$ and $D_3^{(\pm)}(p,q,r)$ are defined in the Appendix.

Drawing together the results of (a), (b), and (c) above, we can write the RHA for the sub-channel process as

$$\overline{T}^{(1,2)}_{\lambda_1;\lambda_a}=[\Delta_2(p,p_a)]^{1/4}[\Delta_2(p,p_1)]^{1/4}[D_3^{(+)}(p,p_a,p_1)]^{|\lambda_a-\lambda_1|/2}[D_3^{(-)}(p,p_a,p_1)]^{|\lambda_a+\lambda_1|/2}T^{(1,2)}_{\lambda_1;\lambda_a}, \tag{3.5}$$

which is regular for all real or complex values of the scalar variables and of ϵ_2.

B. RHA's for the t_{1a}-channel process $N_a\overline{N}_1\to\overline{\pi}_b\pi_2\pi_3$

We note first that now the linear combinations $T^{(1,2)}_{\lambda_a;\lambda_1}$ become

$$T^{(1,2)}_{\lambda_a;\lambda_1}=\begin{pmatrix}1\\ \epsilon_2\end{pmatrix}[T_{\lambda_a;\lambda_1}\mp(-)^{-\lambda_a-\lambda_1}T_{-\lambda_a;-\lambda_1}],$$

as noted above. We label the t_{1a}-channel four momenta by q_i, $i=a,\ldots,3$. The continued four momenta q_i^c will then obey the equality

$$(q_a^c,q_b^c,q_1^c,q_2^c,q_3^c)=(p_a,-p_b,-p_1,p_2,p_3),$$

where the continuation is from the t_{1a} channel to the s_{ab} channel.

As $m_a=m_1$ for nucleon-nuclon annihilation, we have singular behavior at $\Delta_1(Q)=0$, $\Delta_2(Q,q_a)=0$, and $\Delta_3(Q,q_a,q_b)=0$, where $Q=q_a+q_1$. However, as is the case for four particle scattering, the singularity at $\Delta_1(Q)=Q^2=0$ can be interpreted as a pseudothreshold singularity, as indeed $D_2^+(q_a,q_1)=\frac{1}{2}Q^2$. Also, by definition of $\Delta_3(Q,q_a,q_b)$, $\Delta_3(Q,q_a,q_b)=\Delta_3(q_a-q_b,q_a,-q_1)$, so $\Delta_3(Q^c,q_a^c,q_b^c)=\Delta_3(p,p_a,p_1)$. Finally, $\Delta_2(Q^c,q_a^c)=\Delta_2(p_a-p_1,p_a)=\Delta_2(p_a,-p_1)$.

We analyze the singularity structure of the t_{1a}-channel amplitudes in the physical region of the s_{ab}-channel process. Thus the singular surfaces of the continued t_{1a}-channel amplitude are $\Delta_3(p,p_a,p_1)=0$ and $\Delta_2(p_a,-p_1)=0$.

(*a*) $\Delta_2(p_a,-p_1)=0$. By Eq. 4.37 of Ref. 7 we have

$$T^{(1,2)}_{\lambda_a;\lambda_1}=\begin{pmatrix}1\\ \epsilon_2\end{pmatrix}\sum_{j=0}^{1}\sum_{p=0}^{\infty}\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&j\\ \lambda_a&\mp\lambda_1&-\lambda\end{pmatrix}\{\lambda[D_2^{(\pm)}(p_a,-p_1)]^{1/2}\}^p$$
$$\times(e^{i(\pi/2)(j-\lambda)}\mp(-)^{-\lambda_1-\lambda_a+j+1+p}e^{i(\pi/2)(j+\lambda)})g_j^{(\pm)(p)}[D_2^{(\pm)}(p_a,-p_1)]^{-j/2}, \tag{3.6}$$

where $g_j^{(\pm)(p)}$ is regular at $D_2^{(\pm)}(p_a,-p_1)=0$.

Thus we find, at $D_2^{(\pm)}(p_a,-p_1)=0$,

$$T^{(1)}_{++}=2\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&1\\ \frac{1}{2}&-\frac{1}{2}&0\end{pmatrix}e^{i\pi/2}g_1^{(+)0}[D_2^{+}(p_a,-p_1)]^{-1/2}, \tag{3.7}$$

$$T^{(2)}_{++}=2\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\ \frac{1}{2}&-\frac{1}{2}&0\end{pmatrix}g_0^{(+)0}, \tag{3.8}$$

$$T^{(1)}_{+-}=2\frac{\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&1\\ \frac{1}{2}&\frac{1}{2}&-1\end{pmatrix}}{\sqrt{2}}g_1^{(+)0}[D_2^{+}(p_a,-p_1)]^{-1/2}$$
$$+O[D_2^{+}(p_a,-p_1)]^{1/2}), \tag{3.9}$$

$$T^{(2)}_{+-}=2\frac{\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&1\\ \frac{1}{2}&\frac{1}{2}&-1\end{pmatrix}}{\sqrt{2}}g_1^{(+)1}+O(D_2^{+}(p_a,-p_1)), \tag{3.10}$$

and at $D_2^-(p_a, -p_1)=0$,

$$T_{++}^{(1)}=2\frac{\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ \frac{1}{2} & \frac{1}{2} & -1\end{pmatrix}}{\sqrt{2}}\, g_1^{(-)1}+O\left(D_2^-(p_a, -p_1)\right), \qquad (3.11)$$

$$T_{++}^{(2)}=2\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ \frac{1}{2} & \frac{1}{2} & -1\end{pmatrix} g_1^{(-)0}\,[D_2^-(p_a, -p_1)]^{-1/2}$$
$$+O([D_2^-(p_a, -p_1)]^{1/2}), \qquad (3.12)$$

$$T_{+-}^{(1)}=2\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix} g_0^{(-)0}, \qquad (3.13)$$

and

$$T_{+-}^{(2)}=2\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix} e^{i\pi/2} g_1^{(-)0}\,[D_2^-(p_a, -p_1)]^{1/2}. \qquad (3.14)$$

Thus clearly $T_{\lambda_a\lambda_1}^{(1)}[D_2^+(p_a, -p_1)]^{1/2}$ and $T_{\lambda_a\lambda_1}^{(2)}[D_2^-(p_a, -p_1)]^{1/2}$ are regular at $\Delta_2(p_a, -p_1)=0$. Moreover, there exist constraints between the amplitudes

$$[D_2^+(p_a, -p_1)]^{1/2}T_{++}^{(1)}+e^{i\pi/2}[D_2^+(p_a, -p_1)]^{1/2}T_{+-}^{(1)}$$
$$=O(D_2^+(p_a, -p_1)) \qquad (3.15)$$

and

$$[D_2^-(p_a, -p_1)]^{1/2}T_{++}^{(2)}e^{i\pi/2}+[D_2^-(p_a, -p_1)]^{1/2}T_{+-}^{(2)}$$
$$=O(D_2^-(p_a, -p_1)). \qquad (3.16)$$

These are important for our purposes as they lead to poles in $D_2^\pm(p_a, -p_1)$ in the crossing matrix between regularized helicity amplitudes.

(*b*) $\Delta_3(p, p_a, p_1)=0$. By Eq. 4.50 of Ref. 7 we have

$$T_{\lambda_a\lambda_1}^{(1,2)}=A_{\lambda_a\lambda_1}^{(1,2)}[D_3^+(p, p_a, p_1)D_3^-(p, p_a, p_1)]^{-|\lambda_a-\lambda_1|/2}$$

so $[D_3^+(p, p_a, p_1)D_3^-(p, p_a, p_1)]^{|\lambda_a-\lambda_1|/2}T_{\lambda_a\lambda_1}^{(1,2)}$ is regular at $\Delta_3(p, p_a p_1)=0$. So, finally, we can write the continued t_{1a}-channel RHA's as

$$\overline{T}_{\lambda_a\lambda_1}^{(1)}=T_{\lambda_a\lambda_1}^{(1)}[D_2^-(p_a, -p_1)]^{1/2}$$
$$\times[D_3^+(p, p_a, p_1)D_3^-(p, p_a, p_1)]^{1/2},$$
$$\overline{T}_{\lambda_a\lambda_1}^{(2)}=T_{\lambda_a\lambda_1}^{(2)}[D_2^+(p_a, -p_1)]^{1/2} \qquad (3.17)$$
$$\times[D_3^+(p, p_a, p_1)D_3^-(p, p_a, p_1)]^{1/2}.$$

IV. THE CROSSING MATRIX BETWEEN THE s_{ab} CHANNEL AND THE t_{1a} CHANNEL FOR HELICITY AMPLITUDES

In this section we quote the crossing matrix and crossing angles as calculated by Capella.[4] The method used by Capella and also by Chen and Wang[5] is based on that of Cohen-Tannoudji, Morel, and Navralet.[3] It consists of using the relations between the spinor and helicity amplitudes in the s_{ab} and t_{1a} channels, together with the simple crossing relation for the spinor amplitudes.

Omitting the details of the derivation, the crossing relation (Eq. 2.15 of Ref. 4) is

$$T_{\lambda_1\lambda_a}^{(s_{ab})}=-e^{-i\pi\lambda_a}\sum_{\lambda_1'\lambda_a'} d_{\lambda_1'\lambda_1}^{s_1}(\chi_1)\, d_{\lambda_a'\lambda_a}^{s_a}(\chi_a)\, T_{\lambda_a'\lambda_1'}^{(t_{1a})}, \qquad (4.1)$$

where

$$\cos\chi_a=\frac{\begin{bmatrix}p_a & p\\ p_a & p_1\end{bmatrix}}{[\Delta_2(p, p_a)]^{1/2}[\Delta_2(p_a, p_1)]^{1/2}}, \qquad (4.2\text{a})$$

$$\cos\chi_1=-\frac{\begin{bmatrix}p_1 & p\\ p_1 & p_a\end{bmatrix}}{[\Delta_2(p, p_1)]^{1/2}[\Delta_2(p_a, p_1)]^{1/2}}, \qquad (4.2\text{b})$$

and

$$\sin\chi_i=-\eta_1'^{C}(i)\eta_3(i), \quad i=a \text{ or } 1 \qquad (4.3)$$

defines $\sin\chi_i$. In the above equation,

$$\eta_3(i)=-\frac{(m_i^2 p-p\cdot p_i p_i)}{[\Delta_2(p, p_i)]^{1/2}} \qquad (4.4)$$

is the helicity axis of particle i in the s_{ab} channel and $\eta_1'^{C}(i)$ is the analytic continuation of $\eta_1'(i)$.

Finally,

$$\eta_1'(i)^\mu=\epsilon^\mu{}_{\nu\rho\sigma}\,\eta_2'^{\nu}(i)\eta_3'^{\rho}(i)\,\frac{q_i^\sigma}{M_i}, \qquad (4.5)$$

where

$$\eta_2'(i)^\nu=\frac{\epsilon^\nu_{\alpha\beta\gamma}q_a^\alpha q^\beta q_b^\gamma}{[-\epsilon^\nu_{\nu\rho\sigma}q_a^\nu q_1^\rho q_b^\sigma\epsilon_{\mu\alpha\beta\gamma}q_a^\alpha q_1^\beta q_b^\gamma]^{1/2}} \qquad (4.6)$$

and

$$\eta_3'(i)=-\frac{[m_i^2 Q-Q\cdot q_i q_i]}{m_i[\Delta_2(Q_1, q_i)]^{1/2}}. \qquad (4.7)$$

Thus

$$\eta_1'^{C}(i)^\mu=\frac{\epsilon^\mu_{\nu\rho\sigma}\epsilon^\nu_{\alpha\beta\gamma}p_a^\alpha p_1^\beta p_b^\gamma[(p_a-p_1)\cdot p_i p_i^\rho-m_i^2(p_a^\rho-p_1^\rho)]p_i^\sigma\varepsilon_i}{[-\epsilon^\mu{}_{\nu\rho\sigma}p_a{}^\nu p_1^\rho p_b^\sigma\epsilon_{\mu\alpha\beta\gamma}p_a^\alpha p_1^\beta p_b^\gamma]^{1/2}m_i^2[\Delta_2(p_a, -p_1)]^{1/2}}, \qquad (4.8)$$

where $\epsilon_a=1$, $\epsilon_1=-1$.

This reduces to

$$\eta_1'^{C} = -\frac{\epsilon^{\mu}_{\nu\rho\sigma}\epsilon^{\nu}_{\alpha\beta\gamma}p^{\alpha}_{\omega}p^{\beta}_{1}p^{\gamma}_{b}[p^{\rho}_{a}-p^{\rho}_{1}]p^{\sigma}_{i}\epsilon_{i}}{[-\epsilon^{\mu}_{\nu\rho\sigma}p^{\nu}_{a}p^{\rho}_{1}p^{\sigma}_{b}\epsilon_{\mu\alpha\beta\gamma}p^{\alpha}_{a}p^{\beta}_{1}p^{\gamma}_{b}]^{1/2}[\Delta_2(p_a,-p_1)]^{1/2}}, \tag{4.9}$$

and so

$$\eta_1'^{C}(i)\eta_3(i) = -\frac{\epsilon_i(m_i^2p^{\mu}-p\cdot p_i p^{\mu}_i)(p^{\rho}_a-p^{\rho}_i)p^{\sigma}_i p^{\alpha}_a p^{\beta}_1 p^{\gamma}_b\epsilon_{\mu\nu\rho\sigma}\epsilon^{\nu}_{\alpha\beta\gamma}}{m_i[\Delta_2(p,p_i)]^{1/2}[\Delta_2(p_a,-p_1)]^{1/2}[-\epsilon^{\mu}_{\nu\rho\sigma}p^{\nu}_a p^{\rho}_1 p^{\sigma}_b\epsilon_{\mu\alpha\beta\gamma}p^{\alpha}_a p^{\beta}_1 p^{\gamma}_b]^{1/2}}. \tag{4.10}$$

Therefore,

$$\sin\chi_a = \frac{m_a p^{\mu}_a p^{\rho}_1 p^{\sigma}_b p^{\alpha}_a p^{\beta}_1 p^{\gamma}_b\epsilon_{\mu\nu\rho\sigma}\epsilon^{\nu}_{\alpha\beta\gamma}}{[\Delta_2(p,p_a)]^{1/2}[\Delta_2(p_a,-p_1)]^{1/2}[-\epsilon_{\nu\mu\rho\sigma}\epsilon^{\nu}_{\alpha\beta\gamma}p^{\mu}_a p^{\rho}_1 p^{\sigma}_b p^{\alpha}_a p^{\beta}_1 p^{\gamma}_b]^{1/2}}. \tag{4.11}$$

It may be easily checked that

$$-\epsilon_{\nu\mu\rho\sigma}\epsilon^{\nu}_{\alpha\beta\gamma}p^{\mu}_a p^{\rho}_1 p^{\sigma}_b p^{\alpha}_a p^{\beta}_1 p^{\gamma}_b = \Delta_3(p,p_a,p_1), \tag{4.12}$$

so we have

$$\sin\chi_a = -\frac{m_a[\Delta_3(p,p_a,p_1)]^{1/2}}{[\Delta_2(p,p_a)]^{1/2}[\Delta_2(p_a,-p_1)]^{1/2}}. \tag{4.13}$$

Similarly, we find that

$$\sin\chi_1 = \frac{+m_1[\Delta_3(p,p_a,p_1)]^{1/2}}{[\Delta_2(p,p_1)]^{1/2}[\Delta_2(p_a,-p_1)]^{1/2}}. \tag{4.14}$$

It is convenient to take

$$\psi_1 = \pi - \chi_1 \tag{4.15a}$$

$$\psi_a = -\chi_a. \tag{4.15b}$$

So

$$\cos\psi_1 = \frac{\begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix}}{[\Delta_2(p,p_1)]^{1/2}[\Delta_2(p_a,-p_1)]^{1/2}}, \tag{4.16a}$$

$$\sin\psi_1 = \frac{+m_1[\Delta_3(p,p_a,p_1)]^{1/2}}{[\Delta_2(p,p)]^{1/2}[\Delta_2(p_a,-p_1)]^{1/2}}. \tag{4.16b}$$

$$\cos\psi_a = \frac{\begin{bmatrix} p_0 & p \\ p_0 & p_1 \end{bmatrix}}{[\Delta_2(p,p_a)]^{1/2}[\Delta_2(p_a,-p_1)]^{1/2}}, \tag{4.17a}$$

and

$$\sin\psi_a = \frac{m_a[\Delta_3(p,p_a,p_1)]^{1/2}}{[\Delta_2(p,p_a)]^{1/2}[\Delta_2(p_a,-p_1)]^{1/2}}. \tag{4.17b}$$

Thus the crossing relation Eq. (4.1) becomes

$$T^{(sa\bar{b})}_{\lambda,\lambda_a} = \sum_{\lambda_1'\lambda_a'}(-)^{s_1+\lambda_1'-\lambda_a'}d^{s_a}_{\lambda_a',\lambda_a}(\psi_a)d^{s_1}_{\lambda_1',-\lambda_1}(\psi_1)T^{(t1a)}_{\lambda_a'\lambda_1'}. \tag{4.18}$$

$$\begin{pmatrix} T_{++} \\ T_{+-} \\ T_{-+} \\ T_{--} \end{pmatrix} = -i\begin{pmatrix} -\cos\frac{1}{2}\psi_a\sin\frac{1}{2}\psi_1 & -\cos\frac{1}{2}\psi_a\cos\frac{1}{2}\psi_1 & \sin\frac{1}{2}\psi_a\sin\frac{1}{2}\psi_1 & \sin\frac{1}{2}\psi_a\cos\frac{1}{2}\psi_1 \\ \sin\frac{1}{2}\psi_a\sin\frac{1}{2}\psi_1 & \sin\frac{1}{2}\psi_a\cos\frac{1}{2}\psi_1 & \cos\frac{1}{2}\psi_a\sin\frac{1}{2}\psi_1 & \cos\frac{1}{2}\psi_a\cos\frac{1}{2}\psi_1 \\ \cos\frac{1}{2}\psi_a\cos\frac{1}{2}\psi_1 & -\cos\frac{1}{2}\psi_a\sin\frac{1}{2}\psi_1 & -\sin\frac{1}{2}\psi_a\cos\frac{1}{2}\psi_1 & \sin\frac{1}{2}\psi_a\sin\frac{1}{2}\psi_1 \\ -\sin\frac{1}{2}\psi_a\cos\frac{1}{2}\psi_1 & \sin\frac{1}{2}\psi_a\sin\frac{1}{2}\psi_1 & -\cos\frac{1}{2}\psi_a\cos\frac{1}{2}\psi_1 & \cos\frac{1}{2}\psi_a\sin\frac{1}{2}\psi_1 \end{pmatrix}\begin{pmatrix} T_{++} \\ T_{+-} \\ T_{-+} \\ T_{--} \end{pmatrix}. \tag{4.19}$$

Next we form the linear combinations $T^{(1,2)(s_{a\bar{b}})}_{\lambda_1\lambda_a}$, and $T^{(1,2)(t_{1a})}_{\lambda_a\lambda_1}$ with the corresponding crossing matrix

$$\begin{pmatrix} T^{(1)}_{++} \\ T^{(1)}_{+-} \\ T^{(2)}_{++} \\ T^{(2)}_{+-} \end{pmatrix} = -i \begin{pmatrix} \sin\frac{1}{2}(\psi_a-\psi_1) & -\cos\frac{1}{2}(\psi_a-\psi_1) & 0 & 0 \\ \cos\frac{1}{2}(\psi_a-\psi_1) & \sin\frac{1}{2}(\psi_a-\psi_1) & 0 & 0 \\ 0 & 0 & \sin\frac{1}{2}(\psi_a+\psi_1) & \cos\frac{1}{2}(\psi_a+\psi_1) \\ 0 & 0 & \cos\frac{1}{2}(\psi_a+\psi_1) & -\sin\frac{1}{2}(\psi_a+\psi_1) \end{pmatrix} \begin{pmatrix} T^{(1)}_{++} \\ T^{(1)}_{+-} \\ T^{(2)}_{++} \\ T^{(2)}_{+-} \end{pmatrix}, \tag{4.20}$$

where

$$\cos\tfrac{1}{2}\psi_1 \equiv (\tfrac{1}{2}1+\cos\psi_1)^{1/2} = \frac{[D_3^+(p_1,p,p_a)]^{1/2}}{\sqrt{2}\,[\Delta_2(p,p_1)\Delta_2(p_0,p_1)]^{1/4}}, \tag{4.21a}$$

$$\sin\tfrac{1}{2}\psi_1 \equiv (\tfrac{1}{2}1-\cos\psi_1)^{1/2} = \frac{[D_3^-(p_1,p,p_a)]}{\sqrt{2}\,[\Delta_2(p,p_1)\Delta_2(p_a,p_1)]^{1/4}}, \tag{4.21b}$$

and similarly,

$$\begin{bmatrix} \cos \\ \sin \end{bmatrix} \tfrac{1}{2}(\psi_a) = \frac{[D_3^\pm(p_a,p,p_1)]^{1/2}}{\sqrt{2}\,[\Delta_2(p,p_a)\Delta_2(p_a,p_1)]^{1/4}}. \tag{4.21c, d}$$

V. THE CROSSING MATRIX FOR REGULARIZED HELICITY AMPLITUDES

We calculate in this section the crossing matrix for the RHA's of Sec. III, and check that it is regular for all values of the scalar variables apart from poles at $D_2^\pm(p_a, -p_1)$ corresponding to the constraints of Eqs. (3.15) and (3.16). Using Eqs. (3.5), (3.17) and (4.20), we can write the regularized crossing matrix as:

$$\begin{bmatrix} \dfrac{[D_3^-(p,p_a,p_1)]^{1/2}}{[D_2^+(p_a,-p_1)]^{1/2}}\times\sin\tfrac{1}{2}(\psi_a-\psi_1)A & \dfrac{-\cos\frac{1}{2}(\psi_a-\psi_1)A}{[D_2^+(p_a,-p_1)]^{1/2}[D_3^+(p,p_a,p_1)]^{1/2}} & 0 & 0 \\ \dfrac{[D_3^+(p,p_a,p_1)]^{1/2}}{[D_2^+(p_a,-p_1)]^{1/2}}\times\cos\tfrac{1}{2}(\psi_a-\psi_1)A & \dfrac{\sin\frac{1}{2}(\psi_a-\psi_1)A}{[D_2^+(p_a,-p_1)]^{1/2}[D_3^-(p,p_a,p_1)]^{1/2}} & 0 & 0 \\ 0 & 0 & \dfrac{A[D_3^-(p,p_a,p_1)]^{1/2}}{[D_2^-(p_a,-p_1)]^{1/2}}\times\sin\tfrac{1}{2}(\psi_1+\psi_a) & \dfrac{A\cos\frac{1}{2}(\psi_1+\psi_a)}{[D_2^+(p_a,-p_1)]^{1/2}[D_3^+(p,p_a,p_1)]^{1/2}} \\ 0 & 0 & \dfrac{A[D_3^+(p,p_a,p_1)]^{1/2}}{[D_2^-(p_a,-p_1)]^{1/2}}\times\cos\tfrac{1}{2}(\psi_1+\psi_a) & \dfrac{-A\sin\frac{1}{2}(\psi_1+\psi_a)}{[D_2^-(p_a,-p_1)]^{1/2}[D_3^+(p,p_a,p_1)]^{1/2}} \end{bmatrix} \tag{5.1}$$

where

$$A = [\Delta_2(p,p_a)]^{1/4}[\Delta_2(p,p_1)]^{1/4}. \tag{5.2}$$

A. Behavior of regularized crossing matrix at $\Delta_2(p,p_a)=0$ and $\Delta_2(p,p_1)=0$

By Eqs. (4.21a)–(4.21d) it may be seen that $\cos\frac{1}{2}(\psi_a\pm\psi_1)$, and $\sin\frac{1}{2}(\psi_a\pm\psi_1)$ have a common denominator of 2A; thus the regularized crossing matrix (RCM) is regular at $\Delta_2(p,p_a)=0$ and at $\Delta_2(p,p_1)=0$.

B. Behavior of RCM at $\Delta_2(P_a, -P_1)=0$

We first check that each nonzero entry of the RCM has a pole at either $D_2^+(p_a, -p_1)$ or $D_2^-(p_a, -p_1)=0$. First we write down $\cos\frac{1}{2}(\psi_a \pm \psi_1)$ and $\sin\frac{1}{2}(\psi_a \pm \psi_1)$:

$$\cos\tfrac{1}{2}(\psi_a \pm \psi_1) = \frac{1}{2A}\left\{\frac{[D_3^+(p_a,p,p_1)]^{1/2}[D_3^+(p_1,p,p_a)]^{1/2} \mp [D_3^-(p_a,p,p_1)]^{1/2}[D_3^-(p_1,p,p_a)]^{1/2}}{[\Delta_2(p_a,\ -p_1)]^{1/2}}\right\} \tag{5.3}$$

and

$$\sin\tfrac{1}{2}(\psi_a \pm \psi_1) = \frac{1}{2A}\left\{\frac{[D_3^-(p_a,p,p_1)]^{1/2}[D_3^+(p_1,p,p_a)]^{1/2} \pm [D_3^+(p_a,p,p_1)]^{1/2}[D_3^-(p_1,p,p_a)]^{1/2}}{[\Delta_2(p_a,\ -p_1)]^{1/2}}\right\}. \tag{5.4}$$

Thus we must ensure at $D_2^+(p_a, -p_1) \to 0$ that $\sin\frac{1}{2}(\psi_a - \psi_1)$ and $\cos\frac{1}{2}(\psi_a - \psi_1)$ are both $O([D_2^+(p_a, -p_1)]^{-1/2})$, and at $D_2^-(p_a, -p_1)$ that both are finite. Similarly, at $D_2^-(p_a, -p_1) \to 0$ we must have $\sin\frac{1}{2}(\psi_a + \psi_1)$ and $\cos\frac{1}{2}(\psi_a + \psi_1) \sim O([D_2^-(p_a, -p_1)]^{-1/2})$, and at $D_2^+(p_a, -p_1) \to 0$ both must be finite.

1. $D_2^+(P_a, -P_1) \to 0$ $(P_a \cdot P_1 \to m_a m_1)$

We have

$$\cos\tfrac{1}{2}(\psi_a - \psi_1) \cong \frac{1}{2A}\left\{\frac{\begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix}^{1/2}\begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix}^{1/2} + \left(-\begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix}\right)^{1/2}\left(-\begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix}\right)^{1/2}}{[\Delta_2(p_a,\ -p_1)]^{1/2}}\right\}, \tag{5.5}$$

but

$$\begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix} \equiv m_a{}^2 p\cdot p_1 - p\cdot p_a p_a \cdot p_1 = m_a{}^2(p\cdot p_1 - p\cdot p_a) \tag{5.6}$$

at $D_2^+(p_a, -p_1)=0$ (taking $m_a = m_1$). Also

$$\begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix} = m_a{}^2(p\cdot p_a - p\cdot p_1), \tag{5.7}$$

at $D_2^+(p_a, -p_1)=0$. So on the singularity surface we have

$$\begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix} = -\begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix}.$$

Thus at $D_2^+(p_a, -p_1)=0$,

$$\cos\tfrac{1}{2}(\psi_a - \psi_1) \cong \frac{i}{A}\,\frac{\begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix}}{[\Delta_2(p_a,\ -p_1)]^{1/2}}. \tag{5.8}$$

Similarly at $D_2^+(p_a, -p_1)=0$,

$$\sin\tfrac{1}{2}(\psi_a - \psi_1) \cong -\frac{\begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix}}{A[\Delta_2(p_a,\ -p_1)]^{1/2}}. \tag{5.9}$$

We also have

$$\cos\tfrac{1}{2}(\psi_a + \psi_1) \cong \frac{1}{2A}\left\{\frac{\begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix}^{1/2}\begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix}^{1/2} - \left(-\begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix}\right)^{1/2}\left(-\begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix}\right)^{1/2}}{[\Delta_2(p_a,\ -p_1)]^{1/2}}\right\}. \tag{5.10}$$

Now both numerator and denominator are $O([D_2^+(p_a, -p_1)]^{1/2})$, so their ratio is finite as $D_2^+(p_a, -p_1) \to 0$.

Similarly, $\sin\frac{1}{2}(\psi_a+\psi_1)$ is finite at $D_2^+(p_a, -p_1)\to 0$.

2. $D_2^-(P_a, -P_1)\to 0\ (P_a\cdot P_1\to -m_a^2)$

Similar to Sec. V B 1 above, we find both $\cos\frac{1}{2}(\psi_a-\psi_1)$ and $\sin\frac{1}{2}(\psi_a-\psi_1)$ are finite at $D_2^-(p_a, -p_1)\to 0$. Also, we have

$$\cos\tfrac{1}{2}(\psi_a+\psi_1)\cong\frac{\begin{bmatrix}p_a & p\\ p_a & p_1\end{bmatrix}}{A[\Delta_2(p_a, -p_1)]^{1/2}} \tag{5.11}$$

and

$$\sin\tfrac{1}{2}(\psi_a+\psi_1)\cong\frac{i\begin{bmatrix}p_a & p\\ p_a & p_1\end{bmatrix}}{A[\Delta_2(p_a, -p_1)]^{1/2}}\,. \tag{5.12}$$

Thus the four entries in the upper left-hand block have simple poles in $D_2^+(p_a, -p_1)$, and similarly the four entries in the lower right-hand block have simple poles in $D_2^-(p_a, -p_1)$.

It now remains to be shown that these poles are consistent with the constraints of Eqs. (3.15) and (3.16), i.e., that the s_{ab}-channel RHA's are finite at $\Delta_2(p_a, -p_1)=0$. Thus by Eq. (5.1),

$$\frac{\overline{T}_{++}^{(1)s a\bar{b}}}{[D_3^+(p,p_a,p_1)D_3^-(p,p_a,p_1)]^{1/2}}=-i\left\{\frac{A\sin\frac{1}{2}(\psi_a-\psi_1)}{[D_2^+(p_a, -p_1)]^{1/2}}T_{++}^{(1)t_{1a}}[D_2^+(p_a, -p_1)]^{1/2} - \frac{A\cos\frac{1}{2}(\psi_a-\psi_1)}{[D_2^+(p_a, -p_1)]^{1/2}}T_{+-}^{(1)t_{1a}}[D_2^+(p_a, -p_1)]^{1/2}\right\}. \tag{5.13}$$

Therefore, at $D_2^+(p_a, -p_1)=0$, by Eqs. (5.8) and (5.9),

$$\frac{\overline{T}_{++}^{(1)s a\bar{b}}}{[D_3^+(p,p_a,p_1)D_3^-(p,p_a,p_1)]^{1/2}}\cong -i\left\{\begin{bmatrix}p_a & p\\ p_a & p_1\end{bmatrix}\left[\frac{-T_{++}^{(1)t_{1a}}[D_2^+(p_a, -p_1)]^{1/2}+iT_{+-}^{(1)t_{1a}}[D_2^+(p_a, -p_1)]^{1/2}}{D_2^+(p_a, -p_1)}\right]\right\}. \tag{5.14}$$

However, the constraint Eq. (3.15) tells us that the top line is $O(D_2^+(p_a, -p_1))$. Thus, as expected, $\overline{T}_{++}^{(1)s a\bar{b}}$ is finite at $\Delta_2(p_a, -p_1)=0$. A similar check can be made on $\overline{T}_{+-}^{(1)s a\bar{b}}$, $\overline{T}_{++}^{(2)s a\bar{b}}$, and $\overline{T}_{+-}^{(2)s a\bar{b}}$ with the same results.

C. Behavior of the RCM at $\Delta_3(P,P_a,P_1)=0$

By Eq. (4.13) and (4.14), at $\Delta_3(p,p_a,p_1)=0$, $\sin\psi_a=\sin\psi_1=0$. Therefore, ψ_1 and $\psi_a=0$ or $\pm\pi$ independently. For the RCM Eq. (5.1) to be finite at $\Delta_3(p,p_a,p_1)=0$, we must have (a) at $D_3^+(p,p_a,p_1)\to 0$; $\cos\frac{1}{2}(\psi_a\pm\psi_1)\to 0$, and (b) at $D_3^-(p,p_a,p_1)\to 0$; $\sin\frac{1}{2}(\psi_a\pm\psi_1)\to 0$.

(a) At $\Delta_3(p,p_a,p_1)=0$, $\cos\frac{1}{2}(\psi_a+\psi_1)=0$ if $\cos\frac{1}{2}(\psi_a-\psi_1)=0$.

Thus, referring to Eqs. (5.3) and (5.4) for $\cos\frac{1}{2}(\psi_a\pm\psi_1)$, it suffices to show that both $D_3^+(p_a,p,p_1)D_3^+(p_1,p,p_a)$ and $D_3^-(p_a,p,p_1)D_3^-(p_1,p,p_a)=0$ at $D_3^+(p,p_a,p_1)=0$. Now

$$D_3^\pm(p_a,p,p_1)D_3^\pm(p_1,p,p_a)=\Delta_2(p_a,-p_1)[\Delta_2(p,p_a)]^{1/2}[\Delta_2(p,p_1)]^{1/2}+\begin{bmatrix}p_a & p\\ p_a & p_1\end{bmatrix}\begin{bmatrix}p_1 & p\\ p_1 & p_a\end{bmatrix} \pm\left\{\begin{bmatrix}p_a & p\\ p_a & p_1\end{bmatrix}[\Delta_2(p,p_1)]^{1/2}[\Delta_2(p_a,-p_1)]^{1/2}\pm\begin{bmatrix}p_1 & p\\ p_1 & p_a\end{bmatrix}[\Delta_2(p,p_a)]^{1/2}[\Delta_2(p_a,-p_1)]^{1/2}\right\}, \tag{5.15}$$

so finally the problem reduces to showing at $D_3^+(p,p_a,p_1)=0$ that

$$\Delta_2(p_a,-p_1)[\Delta_2(p,p_a)]^{1/2}[\Delta_2(p,p_1)]^{1/2}+\begin{bmatrix}p_a & p\\ p_a & p_1\end{bmatrix}\begin{bmatrix}p_1 & p\\ p_1 & p_a\end{bmatrix}=0\,, \tag{5.16}$$

and also that

$$[\Delta_2(p_a,-p_1)]^{1/2}\left(\begin{bmatrix}p_a & p\\ p_a & p_1\end{bmatrix}[\Delta_2(p,p_1)]^{1/2}+\begin{bmatrix}p_1 & p\\ p_1 & p_a\end{bmatrix}[\Delta_2(p,p_a)]^{1/2}\right)=0\,. \tag{5.17}$$

But

$$D_3^+(p,p_a,p_1) \equiv [\Delta_2(p,p_a)]^{1/2}[\Delta_2(p,p_1)]^{1/2} + \begin{bmatrix} p & p_a \\ p & p_1 \end{bmatrix} = 0\,, \tag{5.18}$$

so

$$[\Delta_2(p,p_a)]^{1/2}[\Delta_2(p,p_1)]^{1/2} = -\begin{bmatrix} p & p_a \\ p & p_1 \end{bmatrix}. \tag{5.19}$$

Thus,

$$\Delta_2(p_a,-p_1)[\Delta_2(p,p_a)]^{1/2}[\Delta_2(p,p_1)]^{1/2} + \begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix}\begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix}$$
$$= -\Delta_2(p_a,-p_1)\begin{bmatrix} p & p_a \\ p & p_1 \end{bmatrix} + \begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix}\begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix}. \tag{5.20}$$

The latter expression is equal to

$$-(p_a\cdot p_1{}^2 - m_a{}^4)(p^2 p_a\cdot p_1 - p\cdot p_1 p\cdot p_a) + (m_a{}^2 p\cdot p_1 - p\cdot p_a p_a\cdot p_1)(m_a{}^2 p\cdot p_a - p\cdot p_1 p_a\cdot p_1)\,. \tag{5.21}$$

This equals

$$p_a\cdot p_1[m_a{}^4 p^2 - p^2 p_a\cdot p_1{}^2 - m_a{}^2(p\cdot p_a{}^2) - m_a{}^2(p\cdot p_1)^2 + 2p\cdot p_a p\cdot p_1 p_a\cdot p_1] = p_a\cdot p_1\Delta_3(p,p_a,p_1) \tag{5.22}$$
$$=0\,. \tag{5.23}$$

Similarly,

$$\begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix}[\Delta_2(p,p_1)]^{1/2} + \begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix}[\Delta_2(p,p_a)]^{1/2} = \frac{1}{[\Delta_2(p,p_a)]^{1/2}}\left\{-\begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix}\begin{bmatrix} p & p_a \\ p & p_1 \end{bmatrix} + \begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix}\Delta_2(p,p_a)\right\} \tag{5.24}$$

$$= -\frac{p\cdot p_a}{[\Delta_2(p,p_a)]^{1/2}}\,[m_a{}^4 p^2 - p^2(p_a\cdot p_1)^2 - m_a{}^2(p\cdot p_a)^2 - m_a{}^2(p_a\cdot p_1)^2$$
$$+ 2p\cdot p_a p\cdot p_1 p_a\cdot p_1]\,, \tag{5.25}$$

$$= \frac{-p\cdot p_a\Delta_3(p,p_a,p_1)}{[\Delta_2(p,p_a)]^{1/2}} = 0\,. \tag{5.26}$$

Thus,

$$\cos\tfrac{1}{2}(\psi_0 \pm \psi_1) = 0 \text{ at } D_3^+(p,p_a,p_1)\,. \tag{5.27}$$

(b) Similarly, at $D_3^-(p,p_a,p_1)=0$, the problem reduces to showing $D_3^\pm(p_a,p,p_1)D_3^\mp(p_1,p,p_a)=0$:

$$D_3^\pm(p_a,p,p_1)D_3^\mp(p_1,p,p_a) = \Delta_2(p_a,-p_1)[\Delta_2(p,p_a)]^{1/2}[\Delta_2(p,p_1)]^{1/2} - \begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix}\begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix}$$
$$\pm [\Delta_2(p_a,-p_1)]^{1/2}\left\{[\Delta_2(p,p_1)]^{1/2}\begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix} - [\Delta_2(p,p_a)]^{1/2}\begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix}\right\}. \tag{5.28}$$

Now $D_3^-(p,p_a,p_1)=0 \to [\Delta_2(p,p_a)]^{1/2}[\Delta_2(p,p_1)]^{1/2} = \begin{bmatrix} p & p_a \\ p & p_1 \end{bmatrix}$. Thus, on substitution into Eq. 5.28, the first two terms become

$$\Delta_2(p_a,-p_1)\begin{bmatrix} p & p_a \\ p & p_1 \end{bmatrix} - \begin{bmatrix} p_a & p \\ p_a & p_1 \end{bmatrix}\begin{bmatrix} p_1 & p \\ p_1 & p_a \end{bmatrix} = 0$$

by Eq. (5.23). Similarly, the second two terms are equal to zero by Eq. (5.27). So we have

$$\sin\tfrac{1}{2}(\psi_a \pm \psi_i) = 0 \text{ at } D_3^-(p,p_a,p_1) = 0\,. \tag{5.29}$$

We have therefore established, as required, that the RCM of Eq. (5.1) is regular for all values of the scalar variables apart from poles at $D_2^\pm(p_a,-p_1)=0$.

VI. CONCLUSIONS

Using the work of Svensson,[7] we have determined the singularity-free helicity amplitudes for $N\pi \to N\pi\pi$ and the crossed process $N\bar{N} \to \bar{\pi}\pi\pi$. Then, using the results of Capella,[4] we calculated explicitly the crossing matrix between these regularized helicity amplitudes. Finally, we checked that this regularized crossing matrix (RCM) is analytic apart from poles at $D_2^{\pm}(p_a, -p_1)=0$, corresponding to the constraints on the crossed-channel regularized helicity amplitudes.

We plan to use this RCM and those for the other crossed-channel processes to obtain sum rules for partial waves analogous to those obtained by Modjtehedzadeh[9] for spinless $2\to 3$ processes.

APPENDIX

We define the special Gram determinants $\Delta_2(p,q)$, $\Delta_3(p,q,r)$, and $\Delta_4(p,d,r,s)$ using the notation of Ref. 1. We define

$$\Delta_2(p,q)=-\begin{bmatrix} p & q \\ p & q \end{bmatrix},$$

where

$$\begin{bmatrix} q_1 & q_2\cdots q_n \\ r_1 & r_2\cdots r_n \end{bmatrix}=\det(q_i\cdot r_k),$$

and $(q_i\cdot r_k)$ is the $n\times n$ matrix whose (i,k) entry is $q_i\cdot r_k$. Similarly,

$$\Delta_3(p,q,r)=\begin{bmatrix} p & q & r \\ p & q & r \end{bmatrix}$$

and

$$\Delta_4(p,q,r,s)=[\epsilon(p,q,r,s)]^2=-\begin{bmatrix} p & q & r & s \\ p & q & r & s \end{bmatrix}.$$

Then $D_n^{\pm}(p_1,p_2,\ldots,p_{n-2};p_{n-1},p_n)$ are defined by

$$\Delta_{n-2}(p_1,\ldots,p_{n-2})\Delta_n(p_1,\ldots,p_n)=D_n^{(+)}(p_1,\ldots,p_{n-2};\,p_{n-1},p_n)D_n^{(-)}(p_1,\ldots,p_{n-2};\,p_{n-1},p_n),$$

where

$$D_n^{(\pm)}(p_1,\ldots,p_{n-2};p_{n-1},p_n)=[\Delta_{n-1}(p_1,\ldots,p_{n-2},p_{n-1})]^{1/2}[\Delta_{n-1}(p_1,\ldots,p_{n-2},p_n)]^{1/2}\pm\begin{bmatrix} p_1 & \cdots & p_{n-2} & p_{n-1} \\ p_1 & \cdots & p_{n-2} & p_n \end{bmatrix}_r.$$

Thus,

$$D_2^{(\pm)}(p,q)=p\cdot q\pm\sqrt{p^2}\sqrt{q^2}$$

and

$$D_3^{(\pm)}(p,q,r)=[\Delta_2(p,q)]^{1/2}[\Delta_2(p,r)]^{1/2}\pm\begin{bmatrix} p & q \\ p & r \end{bmatrix}.$$

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