Comments and Addenda

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Comments on the high-temperature Yang-Mills gas

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The importance of the caloron and other contributions to the thermodynamics of the Yang-Mills gas is discussed. Recent results and outstanding problems are summarized.

I. INTRODUCTION

As emphasized recently by Affleck,[1] one way to make a quantitative test of the instanton method in a scale-invariant field theory is to work at high temperatures. This rigorously and naturally introduces an infrared cutoff which removes the divergence in the integration over instanton scale sizes.[2] For the CP^{N-1} model, Affleck[1] has demonstrated that at high temperatures a dilute gas of calorons[3,4] (finite-temperature instantons) gives the same nonperturbative contribution as a large-N calculation. He argues that similar behavior may be expected in quantum chromodynamics (QCD).

In this note I summarize the consequences which recent work by several people will have on our original estimates[4] of the caloron contribution to the thermodynamics of the four-dimensional Yang-Mills gas [pure SU(2) or SU(3) with no fermions]. In addition, I shall try to answer some questions which were implicit in the earlier work and to emphasize some still unsolved problems.

II. CALORON CONTRIBUTION TO THE PARTITION FUNCTION

In Ref. 4 the finite-temperature infrared cutoff on caloron scale parameters ρ was approximated by a sharp step function at $\rho=\beta=(kT)^{-1}$,

$$\omega(\beta,\rho)\simeq\theta(\beta-\rho)\,. \tag{2.1}$$

Recently, in a beautiful and elegant calculation, Gross, Pisarski, and Yaffe[5] have determined the one-loop fluctuation about the caloron and thus have derived the precise form of the cutoff in an SU(N) theory. For high temperatures (and setting the number of fermions equal to zero), their result becomes

$$\omega(\beta,\rho)=\exp[-(2N-1)\pi^2\rho^2/3\beta^2]\,. \tag{2.2}$$

In addition, a number of authors[6,7,5] have noted that the original value for the one-loop quantum fluctuation around the ($T=0$) instanton was in error (too large) by a factor of 2^{2N}.[8]

Incorporating both of these changes, I have recalculated the caloron contribution to the partition function.[9] Writing $Z=Z_0Z_c$, where Z_c is the dilute-caloron-gas contribution and Z_0 normalizes the result in the no-caloron limit ($Z_c=1$),[10] we have

$$\ln Z_c=2\cos\theta C_N V\beta\int_0^\infty\frac{d\rho}{\rho^5}\left(\frac{4\pi^2}{\bar g^2}\right)^{2N}\times\exp\left(\frac{-8\pi^2}{\bar g^2}\right)\omega(\beta,\rho)\,, \tag{2.3}$$

where $C_2=0.26$ and $C_3=0.097$ for SU(2) and SU(3), respectively, and $\omega(\beta,\rho)$ is given by Eq. (2.2). Inserting the renormalization-group result for $\bar g^2$ in the asymptotically free regime, this may be written

$$\ln Z_c=2\cos\theta b_N V\beta\mu^4\left(\frac{11N}{3}\right)^{2N}\times\int_0^\infty dx\,x^{11N/3-5}\ln^{2N}\frac{1}{x}\exp(-p_N x^2)\,, \tag{2.4}$$

where

$$b_2=0.016\,,\quad b_3=0.0015\,,\quad p_2=(\pi/\mu\beta)^2\,,$$
$$p_3=\tfrac{5}{3}(\pi/\mu\beta)^2\,,\quad x=\mu\rho$$

and μ is the renormalization scale parameter. Equation (2.4) may be evaluated by approximating the integral

$$\int_0^\infty dx\, x^n \ln^m\left(\frac{1}{x}\right) \exp(-px^2)$$
$$\sim \tfrac{1}{2} \ln^m\left(\frac{2p}{n}\right)^{1/2} p^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right). \quad (2.5)$$

From $\ln Z$ all of the thermodynamic functions can be calculated.[4] For example, the ratio of contributions to the pressure from the dilute caloron gas and an ideal SU(N) Bose gas is (setting $\theta = 0$)

$$\frac{p_c}{p_i} \simeq (0.087)(\mu\beta)^{22/3} \ln^4\left[\frac{8.5}{(\mu\beta)^2}\right] \quad \text{for SU(2)}, \quad (2.6a)$$

$$\frac{p_c}{p_i} \simeq (0.0043)(\mu\beta)^{11} \ln^6\left[\frac{5.5}{(\mu\beta)^2}\right] \quad \text{for SU(3)}. \quad (2.6b)$$

The resulting numerical values are considerably reduced from the estimates in Ref. 4. The greatest change is for SU(3) because both the instanton determinant correction factor 2^{2N} and steepness of the cutoff function Eq. (2.2) increase strongly with N. From Eqs. (2.6) we find that the maximum value for this pressure ratio is 8% for the SU(2) case and 0.2% for the SU(3) gas, if we restrict estimates to the region $\mu\beta < 0.5$ where the dilute-gas approximation is expected to be valid. (This region varies somewhat with N.) As before,[4] the ratio decreases to zero as $T \to \infty$.

III. OTHER CONTRIBUTIONS TO THE PARTITION FUNCTION

One of the most troubling questions which plagues any calculation which approximates a functional integral by just taking some class of minima is how to estimate the importance of what has been ignored. For example, are there other periodic Euclidean field configurations besides the caloron which might make a significant contribution to the partition function? The following remarks are intended to make it plausible that for high temperatures the contributions from the dilute caloron gas and "ideal gas"[11] are probably the dominant configurations in the functional integral.

(1) At high temperature, calorons with topological charge $q = \pm 1$ dominate multicaloron ($|q| > 1$) contributions in the dilute-gas approximation (DGA). The cutoff function Eq. (2.2) and the renormalization group ensure that the important scale sizes and the effective coupling $\bar{g}$ are both small at high temperature, and thus the density of calorons is low enough to guarantee the validity of the DGA. Assuming that fluctuations about the multicaloron solutions behave reasonably, the factor $\exp(-S_{\text{cl}})$ in $\ln Z_c$ (where the classical action $S_{\text{cl}} = 8\pi^2 N/g^2$ for $q = N$) strongly suppresses $q > 1$ contributions relative to those with $q = 1$.

(2) Since our single-caloron solution has the maximum number of free parameters for a $q = 1$ configuration,[12] it is the most general such $T \neq 0$ solution. We base this assertion on the fact that the $q = 1$ caloron reduces to the usual instanton solution at $T = 0$ and on the claim[13] that there are no other solutions with unit winding number. (The $q > 1$ calorons of Ref. 3 are obviously not the most general multicaloron solutions since they do not have the correct number of free parameters.)

(3) The original calorons[3] were derived in singular gauge since we constructed them from the singular-gauge multi-instanton solutions of 't Hooft[14] and Jackiw, Nohl, and Rebbi.[15] This of course is necessary if one intends to consider the superposition of calorons in a DGA. In Ref. 3, starting from the standard ansatz $A_\mu = i\bar{\sigma}_{\mu\nu}\partial_\nu \ln\phi$, we only considered solutions to $(1/\phi)\partial^2\phi = 0$ and ignored the other possibility $\partial^2\phi = c\phi^3$. The $T = 0$ instanton of Belavin *et al.*[16] in regular gauge obeys the equation $\partial^2\phi_{\text{reg}} = -8\rho^2\phi_{\text{reg}}^3$. Transforming this solution to singular gauge one finds that it satisfies $(1/\phi_s)\partial^2\phi_s = 0$. Thus the two equations for ϕ are not independent, and [as also follows from (2) above] we need not search for periodic solutions to $\partial^2\phi = c\phi^3$.

In a somewhat different context [see also (4) below] Rossi[17] has determined the form of the gauge transformation matrix Ω [where $A'_\mu = \Omega^{-1}(A_\mu - i\partial_\mu)\Omega$] which removes the singularity in the $\rho \to \infty$ finite-temperature singular-gauge caloron. It is likely that this same gauge transformation converts the caloron to the regular gauge for arbitrary scale size, since near the singularity ($\bar{r}$, $\bar{r} \to 0$ in the notation of Ref. 3) calorons with any ρ are identical. His gauge transformation reduces for $T = 0$ to the one which transforms the usual instanton[16] from the singular to the regular gauge.[18]

(4) Among the possible periodic Euclidean solutions are those which are trivially periodic, viz., the static (τ independent) monopole configurations. Manton[19] and others[20] have pointed out that there is an equivalence between the static solutions of the Bogomolny[21] equations for the SU(2) Higgs theory in Prasad-Sommerfield[22] limit and Euclidean solutions of the SU(2) Yang-Mills theory, with the Higgs field reinterpreted as A_0 in the Yang-Mills case. Thus the Prasad-Sommerfield monopole[22] and its accompanying Higgs field furnish us with a static (hence periodic) Euclidean monopole solution:

$$A_0^a = \hat{r}_a(Cr\coth Cr - 1)/r\,,$$
$$A_i^a = \epsilon_{aij}\hat{r}_j(1 - Cr/\sinh Cr)/r\,, \qquad (3.1)$$

where

$$\hat{r}_j \equiv x_j/r\,.$$

The contribution of such monopoles to the functional integral for the partition function would then naturally seem to be of interest. Pisarski and Yaffe[5,23] have considered this rather subtle question. (The following remarks are based on their work.)

As might be expected from (2), the caloron solution in fact includes some of these monopoles, viz., those whose action/time slice $= 8\pi^2/g^2 =$ classical action of a caloron. These correspond to monopoles with boundary behavior

$$A_0^a \xrightarrow[r\to\infty]{} (2\pi\beta^{-1})\hat{r}_a\,,$$

i.e., those with $C = 2\pi\beta^{-1}$ in the solution Eq. (3.1). As Rossi's calculations[17] show, the $\rho\to\infty$ caloron[24] may be directly gauge transformed into these particular monopole solutions.

But what of the monopoles with $C \neq 2\pi\beta^{-1}$ which correspond to objects with nonintegral topological charge? Pisarski and Yaffe[5,23] have argued that quantum fluctuations will eliminate the contribution of such dyonic configurations.

In the mathematical limit $T\to\infty$ (ρ fixed) all calorons are again gauge equivalent to monopoles, but physically we are reducing the caloron density to zero. Rossi's work raises the possibility that calorons may be related generally to monopole configurations. This deserves further study.

IV. CONCLUSIONS

If we accept the arguments in Sec. III that the dominant (high-temperature) contributions to the functional integral come from expanding about the classical saddle points at $A_\mu = 0$ and A_μ^{caloron}, where does that leave the problem of describing the thermodynamics of the Yang-Mills gas (without fermions)?

(1) Using the results of Ref. 5, we have seen in Sec. II that the dilute-caloron-gas contribution can be reliably calculated, modulo the unknown (but presumed small) value of θ.

(2) Surprisingly, the contribution from fluctuations about the perturbative ($A_\mu = 0$) vacuum Z_0 is less certain. The natural assumption is that because of asymptotic freedom at high temperatures, we simply have [for SU(N)] an ideal gas of $N^2 - 1$ massless noninteracting gauge bosons. This contribution Z_i is what we compared the caloron contribution to in Sec. II. But several authors[25,26] have presented arguments that a plasmon mass is generated in QCD at high temperatures.[27] As is obvious from dimensional reasoning, such a mass would be proportional to the temperature,[25] or $T/\ln T$,[26] and thus does not vanish as $T\to\infty$. If this is true, we clearly have a massive interacting gas rather than a massless ideal Bose gas,[28] and we cannot naively use Z_i as an approximation for the actual Z_0. In the limit of *extremely* high temperatures it is still reasonable to expect that the perturbative contribution will dominate that of the caloron and that it should behave as an ideal (i.e., noninteracting) gas, since $\bar{g}^2$ vanishes as $-1/\ln(\mu\beta)$ for $T\to\infty$. For somewhat lower temperatures there may be a region where both contributions must be included.

In addition, it should be noted that the arguments which were given for a plasmon mass are not rigorous—and probably incorrect in at least one of the cases.[25,29]

Linde[29] has summarized the situation and emphasized the importance of determining the infrared behavior which is controlled, at finite temperature, by the static screening limit: $k_0 = 0$, $\vec{k}\to 0$. He distinguishes between the electric and magnetic gauge boson masses as manifested in the corresponding longitudinal and transverse Green's functions $G_{00}(k)$ and $G_{ij}(k)$. In a very recent study, Pisarski[23] has examined the question in detail and concluded that a magnetic mass is dynamically generated at high temperature in a manner similar to that proposed by Polyakov.[26] Pisarski finds that to rigorously demonstrate such a magnetic mass acts as a true infrared cutoff on the theory is a delicate matter requiring control of the theory to all orders in the loop expansion.

In addition, there are other possibilities which must be considered, such as condensation of the vector fields[30] or a phase transition into some other state.[31]

We conclude that further work is necessary before one can confidently describe the thermodynamics of the Yang-Mills gas.

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[1]Ian Affleck, Nucl. Phys. B162, 461 (1980) and Harvard Reports Nos. HUTP-80/A004 and A008, 1980 (unpublished).

[2]Other ways to introduce a natural cutoff include working (a) at finite density or (b) in the presence of an external field. For (a) see J. Kapusta, Phys. Rev. D 20, 989 (1979) and for (b) see C. Callan, R. Dashen, and D. Gross, *ibid.* 17, 2717 (1978). Also E. Shuryak, Phys. Lett. 79B, 135 (1978).

[3]B. Harrington and H. Shepard, Phys. Rev. D 17, 2122 (1978).

[4]B. Harrington and H. Shepard, Phys. Rev. D 18, 2990 (1978).

[5]D. Gross, R. D. Pisarski, and L. Yaffe (unpublished).

[6]G. 't Hooft, Phys. Rev. D 18, 2199 (1978), corrected the error in 14, 3432 (1976).

[7]C. Bernard, Phys. Rev. D 19, 3013 (1979) and UCLA Report No. UCLA/79/TEP/12, 1979 (unpublished).

[8]For general SU(N), Bernard in Ref. 7 notes another correction.

[9]The corrected instanton determinant was also incorporated into some caloron calculations by N. Bilic and D. Miller, Phys. Lett. 87B, 239 (1979) and Bielefeld Report No. BI-TP 80/02, 1980 (unpublished).

[10]This normalization was omitted in Refs. 3 and 4.

[11]This is defined as the term obtained by expanding about the perturbative $A_\mu = 0$ vacuum. See the remarks in Sec. IV below.

[12]R. Jackiw and C. Rebbi, Phys. Lett. 67B, 189 (1977); C. Bernard, N. Christ, A. Guth, and E. Weinberg, Phys. Rev. D 16, 2967 (1977).

[13]M. Atiyah and R. Ward, Commun. Math. Phys. 55, 117 (1977).

[14]G. 't Hooft (unpublished).

[15]R. Jackiw, C. Nohl, and C. Rebbi, Phys. Rev. D 15, 1642 (1977).

[16]A. Belavin *et al.*, Phys. Lett. 59B, 85 (1975); A. Polyakov, *ibid.* 59B, 82 (1975).

[17]P. Rossi, Nucl. Phys. B149, 170 (1979).

[18]L. Brown, R. Carlitz, D. Creamer, and C. Lee, Phys. Rev. D 17, 1583 (1978).

[19]N. Manton, Nucl. Phys. B135, 319 (1978).

[20]M. Lohe, Phys. Lett. 70B, 325 (1977); J. Cervero, Harvard report, 1977 (unpublished).

[21]E. Bogomolny, Yad. Fiz. 24, 861 (1976) [Sov. J. Nucl. Phys. 24, 449 (1976)].

[22]M. Prasad and C. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[23]R. D. Pisarski, private communication.

[24]Incidentally, the periodic Euclidean solutions which were first constructed in Ref. 3 [cf. Eq. (2.14)] from the covariant pseudoparticles of Ref. 15 are clearly just the $\rho\to\infty$ limit of the basic caloron and thus are gauge equivalent to the monopole solutions.

[25]M. Kislinger and P. Morley, Phys. Rev. D 13, 2765 (1976).

[26]A. Polyakov, Phys. Lett. 72B, 477 (1978).

[27]In the two-dimensional CP^{N-1} model, the existence of Debye screening at finite temperature has been proved by Affleck (Ref. 1). See also G. Lazarides, Nucl. Phys. B156, 29 (1979).

[28]A good approximation may be that of a massive noninteracting gas.

[29]Regarding the error in interpreting the results of Ref. 25, see A. Linde, Rep. Prog. Phys. 42, 389 (1979).

[30]A. Linde, Pis'ma Zh. Eksp. Teor. Fiz. 27, 470 (1978) [JETP Lett. 27, 441 (1978)].

[31]For example, the vacuum proposed in the work of Nielsen and Olesen. See N. K. Nielsen and P. Olesen, Nucl. Phys. B144, 376 (1978); Phys. Lett. 79B, 304 (1978); H. Nielsen and P. Olesen, Nucl. Phys. B160, 380 (1979); NBI report No. 79-45 (unpublished).