Relativistic harmonic oscillator for spin-1/2 particles in the Bethe-Salpeter formalism

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A kernel for the spin-1/2 relativistic Bethe-Salpeter equation is proposed and the equation is solved in the nonrelativistic limit for the mass spectrum in the case of vector coupling. Numerical results are also presented for the fully relativistic case. It is found that in addition to the ordinary, evenly spaced energy levels of the nonrelativistic harmonic oscillator, the spin coupling produces another family of states with level spacing proportional to n, the principal quantum number. It is also found that in this model the phenomenology of the mesons can be reproduced qualitatively only for couplings strong enough to produce significant relativistic effects in the lowest-lying levels.

I. INTRODUCTION

In a previous paper,[1] a kernel was proposed for the spinless relativistic Bethe-Salpeter equation which reduced in the appropriate nonrelativistic limits to the Schrődinger equation for the simple harmonic oscillator. Our model is similar to one proposed by Droz-Vincent[2] using relativistic wave equations for spin-zero particles. Generalizing to the case with spin yields some interesting results in these models and will be the subject of this paper.

In Sec. II a brief review of the construction of the Bethe-Salpeter kernel in the spinless case is presented. In Sec. III we review the separation of the fermion equation into covariant parts and the decomposition into partial waves. These first two sections allow us the opportunity to establish a notation as well as to lay the foundation for the spin-$\frac{1}{2}$ problem. In Sec. IV we construct the actual set of coupled equations we wish to solve, and the nonrelativistic limit of these equations is discussed. Finally, in Sec. V we present the results of numerically integrating the fully relativistic equation and show some of the phenomenology of this model.

II. THE SPIN-ZERO RELATIVISTIC HARMONIC OSCILLATOR

The spin-zero Bethe-Salpeter equation is given by[3]

$$[(\tfrac{1}{2}P+p)^2-m^2+i\eta][(\tfrac{1}{2}P-p)^2-m^2+i\eta]\chi(p) = \int V(p,p')\chi(p')d^4p' . \quad (1)$$

In the center-of-mass coordinate system,

$$p=(\omega,\vec{k}) ,\qquad P=(M,\vec{0}), \quad (2)$$

and the partial-wave decomposition is defined by

$$\chi(p)=\sum_{l\,m}\frac{1}{k}\chi_l(k,\omega)Y_l^m(\hat{k}) . \quad (3)$$

Thus, in the center-of-mass frame,

$$[(-\omega^2+k^2+m^2-\tfrac{1}{4}M^2-i\eta)^2-M^2\omega^2]\chi_l(k,\omega) = \int V_l(k,\omega;k',\omega')\chi_l(k',\omega')d\omega'dk' . \quad (4)$$

If one takes

$$V(p,p')=\lim_{\lambda\to 0} 2ig^2\left(\frac{12\lambda}{\pi^2}\right)\frac{-(p'-p)^2-\lambda^2-i\eta}{[-(p'-p)^2+\lambda^2-i\eta]^4} \quad (5)$$

it can be shown that

$$V_l(k,\omega;k',\omega')=-ig^2\left[\delta_\lambda''(p_1)-\frac{l(l+1)}{k^2}\delta_\lambda(p_1)\right] + i(-1)^l g^2\left[\delta_\lambda''(p_2)-\frac{l(l+1)}{k^2}\delta_\lambda(p_2)\right] , \quad (6)$$

where

$$p_1{}^2=(k'-k)^2-(\omega'-\omega)^2 ,\qquad p_2{}^2=(k'+k)^2-(\omega'-\omega)^2 , \quad (7)$$
$$\delta_\lambda(p)=\frac{\lambda}{\pi}\frac{1}{p^2+\lambda^2} ,$$

and

$$\delta_\lambda''(p)=\frac{\partial^2}{\partial p^2}\delta_\lambda(p)$$

(cf. Ref. 1). Because of the δ-function structure of this kernel, this equation can be separated if we take

$$\chi_l(k,\omega)=\delta(\omega-\omega_0)W_l(k) , \quad (8)$$

where

$$\omega_0 = \tfrac{1}{2}M - (k^2+m^2)^{1/2}\,. \tag{9}$$

The resulting equation for W_l is

$$M[\tfrac{1}{2}M - (k^2+m^2)^{1/2}]W_l(k) = -\frac{g^2}{m^3}\left[(k^2+m^2)\frac{\partial^2}{\partial k^2} + 3k\frac{\partial}{\partial k} + \left(\frac{3}{4} - \frac{m^2}{k^2}l(l+1)\right)\right]W_l(k)\,. \tag{10}$$

This ordinary differential equation is the subject of Ref. 1 and no more needs to be said about it here except that in the appropriate nonrelativistic regime of weak coupling and low-lying states, this equation reduces to the Schrödinger equation for the simple harmonic oscillator with a classical spring constant g^2/m^2. In the strong-coupling limit, the solutions to this equation depart significantly from those of the Schrödinger equation.

III. THE FERMION BETHE-SALPETER EQUATION

The Bethe-Salpeter amplitude for spin-$\frac{1}{2}$ particles is a 16-component bispinor whose various constituents are coupled. As a result, the equation for spin-$\frac{1}{2}$ particles is very much more complicated than that for the spin-0 case. In shorthand notation, we write

$$[\gamma^\mu(p_\mu+\tfrac{1}{2}P_\mu)-m]\chi(p)[\gamma^\nu(p_\nu-\tfrac{1}{2}P_\nu)-m] = \int g_{\mu\nu}\Gamma^\mu(p,p')\chi(p')\Gamma^\nu(p,p')d^4p'\,. \tag{11}$$

Here γ, χ, and Γ are 4×4 matrices which are multiplied together where indicated. Γ characterizes the interaction entirely and the product $g_{\mu\nu}\Gamma^\mu\Gamma^\nu$ forms the Bethe-Salpeter kernel for this problem.

For convenience, we shall write (11) as

$$[(p+\tfrac{1}{2}P)^2-m^2][(p-\tfrac{1}{2}P)^2-m^2]\phi(p) = \int g_{\mu\nu}\Gamma^\mu(p,p')\psi(p')\Gamma^\nu(p,p')d^4p'\,, \tag{12}$$

where

$$\phi(p) = \frac{[\gamma^\mu(p_\mu+\frac{1}{2}P_\mu)-m]}{[(p+\frac{1}{2}P)^2-m^2]}\chi(p)\frac{[\gamma^\nu(p_\nu-\frac{1}{2}P_\nu)-m]}{[(p-\frac{1}{2}P)^2-m^2]} \tag{13}$$

and

$$\psi(p) = [\gamma^\mu(p_\mu+\tfrac{1}{2}P_\mu)+m]\phi(p)[\gamma^\nu(p_\nu-\tfrac{1}{2}P_\nu)+m]\,. \tag{14}$$

Since the wave amplitude ϕ has components which characterize both singlet (scalar) and triplet (vector) states, it is impossible to extract the angular dependence of this equation in as trivial a manner as in Eq. (3). There are two parts to solving this problem. The first is to decompose ϕ into parts that transform as vectors or scalars under rotation and, second, to decompose these separately into scalar and vector spherical harmonics.

We begin this program by writing

$$\phi(p) = \phi^S(p) + \phi^V_\mu(p)\gamma^\mu + \tfrac{1}{2}\phi^T_{\mu\nu}(p)\tau^{\mu\nu} + i\phi^A_\mu(p)\gamma^\mu\gamma^5 + i\phi^P(p)\gamma^5\,, \tag{15}$$

where

$$\tau^{\mu\nu} = \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\,. \tag{16}$$

Similarly,

$$\psi(p) = \psi^S(p) + \psi^V_\mu(p)\gamma^\mu + \psi^T_{\mu\nu}(p)\tau^{\mu\nu} + i\psi^A_\mu(p)\gamma^\mu\gamma^5 + i\psi^P(p)\gamma^5\,. \tag{17}$$

Using the properties of the γ matrices it is easy but laborious to show that[4-6]

$$\psi^S = (p^2-\tfrac{1}{4}M^2+m^2)\phi^S + 2mp^\mu\phi^V_\mu + P^\mu p^\nu\phi^T_{\mu\nu}\,, \tag{18}$$

$$\psi^V_\mu = 2mp_\mu\phi^S + (2p^\sigma p_\mu - \tfrac{1}{2}P^\sigma P_\mu)\phi^V_\sigma - (p^2-\tfrac{1}{4}M^2-m^2)\phi^V_\mu + mP^\sigma\phi^T_{\sigma\mu} - i\epsilon^{\alpha\beta\delta}{}_\mu P_\alpha p_\beta\phi^A_\delta\,, \tag{19}$$

$$\psi^T_{\mu\nu} = P_\mu p_\nu\phi^S + mP_\mu\phi^V_\nu - (2p^\sigma p_\mu - \tfrac{1}{2}P^\sigma P_\mu)\phi^T_{\sigma\nu} + \tfrac{1}{2}(p^2-\tfrac{1}{4}M^2+m^2)\phi^T_{\mu\nu} - im\epsilon^{\alpha\beta}{}_{\mu\nu}(p_\alpha\phi^A_\beta - \tfrac{1}{2}P_\alpha p_\beta\phi^P)\,, \tag{20}$$

$$\psi^A = i\epsilon^{\alpha\beta\delta}{}_\mu(P_\alpha p_\beta\phi^V_\delta - mp_\alpha\phi^T_{\beta\delta}) - (2p^\sigma p_\mu - \tfrac{1}{2}P^\sigma P_\mu)\phi^A_\sigma + (p^2-\tfrac{1}{4}M^2+m^2)\phi^A_\mu + mP_\mu\phi^P\,, \tag{21}$$

$$\psi^P_\mu = \tfrac{1}{2}i\epsilon^{\alpha\beta\delta\mu}P_\alpha p_\beta\phi^T_{\delta\mu} + mP^\sigma\phi^A_\sigma - (p^2-\tfrac{1}{4}M^2-m^2)\phi^P\,. \tag{22}$$

In order to do an angular decomposition of the equation, it is necessary to pick a specific coordinate system and identify the parts of the wave function which are components of vectors and those that are scalars under rotations in three-space. Under such rotation, the zero component of a four-vector transforms like a scalar. The tensor part of the amplitude is antisymmetric and, like the electromagnetic field tensor, can be broken into two three-vector parts. These various components appear explicitly in the block form[7]

$$\psi = \begin{pmatrix} \psi^S + \psi^V_0 + (\vec\psi^G - \vec\psi^A)\cdot\vec\sigma & \psi^P + \psi^A_0 + (\vec\psi^F - \vec\psi^V)\cdot\vec\sigma \\ \psi^P - \psi^A_0 + (\vec\psi^F + \vec\psi^V)\cdot\vec\sigma & \psi^S - \psi^V_0 + (\vec\psi^G + \vec\psi^A)\cdot\vec\sigma \end{pmatrix}\,, \tag{23}$$

where the components $\vec\psi^F$ and $\vec\psi^G$ form the tensor part of the amplitude in the following way:

$$\psi^T = \begin{pmatrix} 0 & \psi_x^F & \psi_y^F & \psi_z^F \\ -\psi_x^F & 0 & i\psi_z^G & -i\psi_y^G \\ -\psi_y^F & -i\psi_z^G & 0 & i\psi_x^G \\ -\psi_z^F & i\psi_y^G & -i\psi_x^G & 0 \end{pmatrix} . \tag{24}$$

Obviously, the center-of-mass coordinate system is the most convenient, since analysis in this particular frame gives the intrinsic properties of the composite system. Thus, in what follows we take

$$P=(M,\vec{0}) \tag{25}$$

and

$$p=(\omega,\vec{k}). \tag{26}$$

Also, to keep the proliferation of indices under control, we shall define the center-of-mass amplitudes

$$\begin{aligned} &S=(\phi^S)_{\vec{K}=\vec{0}}, \quad \vec{F}=(\vec{\phi}^F)_{\vec{K}=\vec{0}},\\ &U=(\phi_0^V)_{\vec{K}=\vec{0}}, \quad B=(\phi_0^A)_{\vec{K}=\vec{0}},\\ &\vec{V}=(\vec{\phi}^V)_{\vec{K}=\vec{0}}, \quad \vec{A}=(\vec{\phi}^A)_{\vec{K}=\vec{0}},\\ &\vec{G}=(\vec{\phi}^G)_{\vec{K}=\vec{0}}, \quad P=(\phi^P)_{\vec{K}=\vec{0}}, \end{aligned} \tag{27}$$

where $\vec{\phi}^F$ and $\vec{\phi}^G$ are related to ϕ^T as in (24), and $\vec{K}$ is the spatial part of the center-of-mass momentum P. With these definitions we find the following set of equations for the components of ψ in the center-of-mass coordinate system:

$$\begin{aligned} \psi^S &= (\omega^2-k^2-\tfrac{1}{4}M^2+m^2)S+2m\omega U-2m\vec{k}\cdot\vec{V} \\ &\quad +M\vec{k}\cdot\vec{F},\\ \psi_0^V &= 2m\omega S+(\omega^2+k^2-\tfrac{1}{4}M^2+m^2)U-2\omega\vec{k}\cdot\vec{V},\\ \vec{\psi}_0^V &= -2m\vec{k}S-2\omega\vec{k}U+2\vec{k}(\vec{k}\cdot\vec{V}) \\ &\quad +(\omega^2-k^2-\tfrac{1}{4}M^2-m^2)\vec{V}+mM\vec{F}-iM(\vec{k}\times\vec{A}),\\ \vec{\psi}^F &= -M\vec{k}S-mM\vec{V}-(\omega^2+k^2-\tfrac{1}{4}M^2-m^2)\vec{F} \\ &\quad +2\vec{k}(\vec{k}\cdot\vec{F})+2i\omega(\vec{k}\times\vec{G})-2im(\vec{k}\times\vec{A}),\\ \vec{\psi}^G &= -2i\omega(\vec{k}\times\vec{F})+(\omega^2+k^2-\tfrac{1}{4}M^2+m^2)\vec{G}-2\vec{k}(\vec{k}\cdot\vec{G}) \\ &\quad -2m\omega\vec{A}+2m\vec{k}B+M\vec{k}P,\\ \vec{\psi}^A &= iM(\vec{k}\times\vec{V})+2m\omega\vec{G}-2im(\vec{k}\times\vec{F}) \\ &\quad -(\omega^2-k^2-\tfrac{1}{4}M^2+m^2)\vec{A}-2\vec{k}(\vec{k}\cdot\vec{A})+2\omega\vec{k}B,\\ \psi_0^A &= -2m\vec{k}\cdot\vec{G}+(-\omega^2-k^2+\tfrac{1}{4}M^2+m^2)B \\ &\quad +2\omega\vec{k}\cdot\vec{A}+mMP,\\ \psi^P &= -M\vec{k}\cdot\vec{G}+mMB-(\omega^2-k^2-\tfrac{1}{4}M^2-m^2)P. \end{aligned} \tag{28}$$

The scalar functions ψ^S, ψ_0^V, ψ_0^A, ψ^P, S, U, B, P are now to be expanded in ordinary spherical harmonics and the vector-valued functions $\vec{\psi}^V$, $\vec{\psi}^F$, $\vec{\psi}^G$, $\vec{\psi}^A$, $\vec{V}$, $\vec{F}$, $\vec{G}$ are to be expanded in vector spherical harmonics.[8,9] Thus, for example,

$$\begin{aligned} S(p) &= \frac{1}{k}\sum_{jm} S_j(k,\omega)Y_j^m(\hat{k}),\\ \vec{V}(p) &= \frac{1}{k}\sum_{jm}[V_j^{(-)}(k,\omega)\vec{Y}_{jj-1}^m(\hat{k}) \\ &\quad + V_j^{(=)}(k,\omega)\vec{Y}_{jj}^m(\hat{k})+V_j^{(+)}(k,\omega)\vec{Y}_{jj+1}^m(\hat{k})]. \end{aligned} \tag{29}$$

Also, the interaction is decomposed in the following way:

$$g_{\mu\nu}\Gamma^\mu(p,p')\Gamma^\nu(p,p') = \frac{1}{kk'}\sum_{lm} g_{\mu\nu}V_l^{\mu\nu}(k,\omega;k',\omega') \times Y_l^{m*}(\hat{k}')Y_l^m(\hat{k}). \tag{30}$$

This decomposition results in a set of sixteen coupled equations, which breaks up into two independent sets of eight because of the conservation of parity. These we write in the matrix form

$$\begin{aligned} &[(p+\tfrac{1}{2}P)^2-m^2][(p-\tfrac{1}{2}P)^2-m^2]\Phi_{j\alpha}^{(H)}(k,\omega) \\ &\quad = \sum_\beta \int C_\alpha^{(H)} g_{\mu\nu}V_{l\alpha}^{\mu\nu}(k,\omega;k',\omega')\mathfrak{M}_{j\alpha\beta}^{(H)}(k',\omega') \\ &\qquad \times \Phi_{j\beta}^{(H)}(k',\omega')d\omega' dk'. \end{aligned} \tag{31}$$

The quantities in (31) are defined below in Eqs. (32)–(35). For $H=1$,

$$\begin{aligned} &C_1^{(1)}=C_S,\\ &C_2^{(1)}=C_3^{(1)}=C_4^{(1)}=C_V,\\ &C_5^{(1)}=C_6^{(1)}=C_7^{(1)}=C_T,\\ &C_8^{(1)}=C_A, \end{aligned} \tag{32}$$

and for $H=2$,

$$\begin{aligned} &C_1^{(2)}=C_P,\\ &C_2^{(2)}=C_3^{(2)}=C_4^{(2)}=C_A,\\ &C_5^{(2)}=C_6^{(2)}=C_7^{(2)}=C_T,\\ &C_8^{(2)}=C_V. \end{aligned} \tag{33}$$

C_S, C_V, C_T, C_A, and C_P depend on the type of coupling and are given in Table I. For both $H=1$ and $H=2$, the orbital angular momentum index is given by

$$\begin{aligned} &l_1=l_2=l_7=l_8=j,\\ &l_3=l_5=j-1,\\ &l_4=l_6=j+1. \end{aligned} \tag{34}$$

TABLE I. Relative couplings for the scalar, vector, tensor, axial-vector, and pseudoscalar parts of the equation.

Type coupling	C_S	C_V	C_T	C_A	C_P
Scalar	1	1	1	1	1
Pseudoscalar	−1	1	−1	1	−1
Vector	4	−2	0	2	−4
Axial vector	4	2	0	2	−4

TABLE II. The coupling matrix $\mathfrak{M}_j^{(1)}$ for the $(-1)^j$-parity states $^3J\pm 1_J$.

$\omega^2-k^2-\frac{1}{4}M^2+m^2$	$2m\omega$	$-2mkI_j$	$2mkJ_j$	MkI_j	$-MkJ_j$	0	0
$2m\omega$	$\omega^2+k^2-\frac{1}{4}M^2+m^2$	$-2\omega kI_j$	$2\omega kJ_j$	0	0	0	0
$-2mkI_j$	$-2\omega kI_j$	$\omega^2-\frac{k^2}{2j+1}-\frac{M^2}{4}-m^2$	$-2k^2I_jJ_j$	mM	0	0	MkJ_j
$2mkJ_j$	$2\omega kJ_j$	$-2k^2I_jJ_j$	$\omega^2+\frac{k^2}{2j+1}-\frac{M^2}{4}-m^2$	0	mM	0	MkI_j
$-MkI_j$	0	$-mM$	0	$-\omega^2-\frac{k^2}{2j+1}+\frac{M^2}{4}+m^2$	$-2k^2I_jJ_j$	$-2\omega kJ_j$	$2mkJ_j$
MkJ_j	0	0	$-mM$	$-2k^2I_jJ_j$	$-\omega^2+\frac{k^2}{2j+1}+\frac{M^2}{4}+m^2$	$-2\omega kI_j$	$2mkI_j$
0	0	0	0	$2\omega kJ_j$	$-2\omega kI_j$	$\omega^2+k^2-\frac{1}{4}M^2+m^2$	$-2m\omega$
0	0	$-MkJ_j$	$-MkI_j$	$2mkJ_j$	$2mkI_j$	$2m\omega$	$-\omega^2+k^2+\frac{1}{4}M^2-m^2$

TABLE III. The coupling matrix $\mathfrak{M}_j^{(2)}$ for the $(-1)^{j+1}$-parity states $^{1,3}J_J$.

$-\omega^2+k^2+\frac{1}{4}M^2+m^2$	mM	0	0	$-MkI_j$	MkJ_j	0	0
mM	$-\omega^2-k^2+\frac{1}{4}M^2+m^2$	$2\omega kI_j$	$-2\omega kJ_j$	$-2mkI_j$	$2mkJ_j$	0	0
0	$2\omega kI_j$	$-\omega^2+\frac{k^2}{2j+1}+\frac{M^2}{4}-m^2$	$2k^2I_jJ_j$	$2m\omega$	0	$2mkJ_j$	$-MkJ_j$
0	$-2\omega kJ_j$	$2k^2I_jJ_j$	$-\omega^2-\frac{k^2}{2j+1}+\frac{M^2}{4}-m^2$	0	$2m\omega$	$2mkI_j$	$-MkJ_j$
MkI_j	$2mkI_j$	$-2m\omega$	0	$\omega^2+\frac{k^2}{2j+1}-\frac{M^2}{4}+m^2$	$2k^2I_jJ_j$	$2\omega kJ_j$	0
$-MkJ_j$	$-2mkJ_j$	0	$-2m\omega$	$2k^2I_jJ_j$	$\omega^2-\frac{k^2}{2j+1}-\frac{M^2}{4}+m^2$	$2\omega kI_j$	0
0	0	$2mkJ_j$	$-2mkI_j$	$-2\omega kJ_j$	$2\omega kI_j$	$-\omega^2-k^2+\frac{1}{4}M^2+m^2$	$-mM$
0	0	MkJ_j	MkI_j	0	0	mM	$\omega^2-k^2-\frac{1}{4}M^2-m^2$

The Bethe-Salpeter amplitudes $\Phi_j^{(H)}$ are defined as

$$\Phi_j^{(1)} = \begin{bmatrix} S_j \\ U_j \\ V_j^{(-)} \\ V_j^{(+)} \\ F_j^{(-)} \\ F_j^{(+)} \\ G_j^{(=)} \\ A_j^{(=)} \end{bmatrix}, \quad \Phi_j^{(2)} = \begin{bmatrix} P_j \\ B_j \\ A_j^{(-)} \\ A_j^{(+)} \\ G_j^{(-)} \\ G_j^{(+)} \\ F_j^{(=)} \\ V_j^{(=)} \end{bmatrix}. \tag{35}$$

Finally, $\mathfrak{M}_j^{(1)}$ is given in Table II and $\mathfrak{M}_j^{(2)}$ in Table III.

In the tables and what follows,

$$I_j = \left(\frac{j}{2j+1}\right)^{1/2}, \quad J_j = \left(\frac{j+1}{2j+1}\right)^{1/2}. \tag{36}$$

As per the comment above, we note that $\Phi_j^{(1)}$ characterizes the ${}^3J\pm 1_J$ states, which have parity $(-1)^J$ while the amplitudes $\Phi_j^{(2)}$ characterize the $(-1)^{J+1}$-parity ${}^{1,3}J_J$ states.

IV. THE RELATIVISTIC HARMONIC OSCILLATOR WITH SPIN

Despite the fact that everything has been greatly complicated by the addition of spin, the idea of constructing an infrared-singular interaction like that used in Sec. II can be carried over almost verbatim. The case with spin is richer by virtue of the various possible spin couplings, but the core of the interaction kernel, i.e., $g_{\mu\nu}\Gamma^\mu\Gamma^\nu$, remains simple. We shall assume vector coupling so that (following Ref. 1)

$$g_{\mu\nu}\Gamma^\mu(p,p')\Gamma^\nu(p,p') = g_{\mu\nu}\gamma^\mu\gamma^\nu(2ig^2)\left(\frac{12\lambda}{\pi^2}\right) \times \frac{-(p'-p^2)-\lambda^2-i\eta}{[-(p'-p)^2+\lambda^2-i\eta]^4} \tag{37}$$

and, hence,

$$V_l^{\mu\nu}(k,\omega;k',\omega') = \frac{2ig^2}{m^2}(k'^2+m^2)^{1/2} \times\left((k'^2+m^2)\frac{\partial^2}{\partial k'^2}+3k'\frac{\partial}{\partial k'}+\frac{3}{4}-\frac{l(l+1)}{k'^2}\right) \times\delta_\lambda(k'-k)\gamma^\mu\gamma^\nu. \tag{38}$$

The left side of Eq. (31) is the same as the spinless case and the reasoning that drove us to the conclusion that the wave amplitude had a δ-function structure is still valid if (38) is used (cf. Ref. 1). As a result,

$$\Phi_{j\alpha}^{(H)}(k,\omega) = \varphi_{j\alpha}^{(H)}(k)\delta(\omega-\tfrac{1}{2}M+(k^2+m^2)^{1/2}) \tag{39}$$

and the $\varphi_{j\alpha}^{(H)}$ amplitudes solve the following differential equation:

$$-M\left[\frac{M}{2}-(k^2+m^2)^{1/2}\right]\varphi_{j\alpha}^{(H)}(k) = \frac{g^2}{m^3}\left((k^2+m^2)\frac{\partial^2}{\partial k^2}+3k\frac{\partial}{\partial k}+\frac{3}{4}-\frac{l_\alpha(l_\alpha+1)}{k^2}\right)\left(\sum_\beta \frac{C_\alpha^{(H)}}{m^2}\mathfrak{M}_{j\alpha\beta}^{(H)}(k,\omega_0)\varphi_{j\beta}^{(H)}\right), \tag{40}$$

where $\omega_0 = M/2-(k^2+m^2)^{1/2}$. It is this set of equations that is to be solved.

Our first object must be to find the weak-coupling limit of this equation. We find, for $g^2/m^3 \ll 1$,

$$\frac{g^2}{2m^3}\left(\frac{\partial^2}{\partial\xi^2}-\frac{l_\alpha(l_\alpha+1)}{\xi^2}\right)\left(\sum_\beta\frac{C_\alpha^{(H)}}{m^2}\mathfrak{M}_{j\alpha\beta}^{(H)}\varphi_{j\beta}^{(H)}\right) = \left(\frac{\xi^2}{2}-\frac{E_{\rm NR}}{m}\right)\varphi_{j\alpha}^{(H)}, \tag{41}$$

where $\xi = k/m$ and $E_{\rm NR}$ is the nonrelativistic approximation to the binding energy $M-2m$.

To proceed further, it is necessary to assume

$$\varphi_{j\beta}^{(H)}(\xi) = \left(\frac{\zeta}{a}\right)^{(l_\beta+1)/2} e^{-\zeta/2} f_\beta^{(H)}(\zeta), \tag{42}$$

where

$$\zeta = a\xi^2. \tag{43}$$

This results in the equation

$$\sum_\beta\left(\zeta\frac{\partial^2}{\partial\zeta^2}+(l_\alpha+\tfrac{3}{2}-\zeta)\frac{\partial}{\partial\zeta}-\tfrac{1}{2}(l_\alpha+\tfrac{3}{2}-\zeta/2)\right)\mathfrak{M}_{j\alpha\beta}^{(H)}f_\beta^{(H)} = \frac{2m^3}{g^2}\left(\frac{\zeta}{8a^2}-\frac{E_{\rm NR}}{4am}\right)f_\alpha^{(H)}, \tag{44}$$

which, if it were not for the coupling matrix $\mathfrak{M}_j^{(H)}$ would be the equation for Laguerre polynomials. Thus, we expect that since the exponential is present in the amplitude $\varphi_j^{(H)}$, and cuts off everything for $\xi^2 > (g^2/m^3)^{1/2}$, it is only necessary to keep the constant terms in $\mathfrak{M}_j^{(H)}$. The resultant set of equations exhibit some simplicity. First, since $C_T = 0$, φ^F and φ^G are zero. A solution exists where all the amplitudes are zero except $\varphi^{A(+)}$ and $\varphi^{A(=)}$. Since $\varphi^{V(=)}$ is independent of $\varphi^{A(+)}$ in this limit, we can set $a = \frac{1}{2}(M^3/g^2)^{1/2}$ and find

$$\varphi_j^{V(=)} \propto L_n^{j+3/2}((m^3/g^2)^{1/2}\xi^2)e^{-(1/2)(m^3/g^2)^{1/2}\xi^2} \tag{45}$$

and

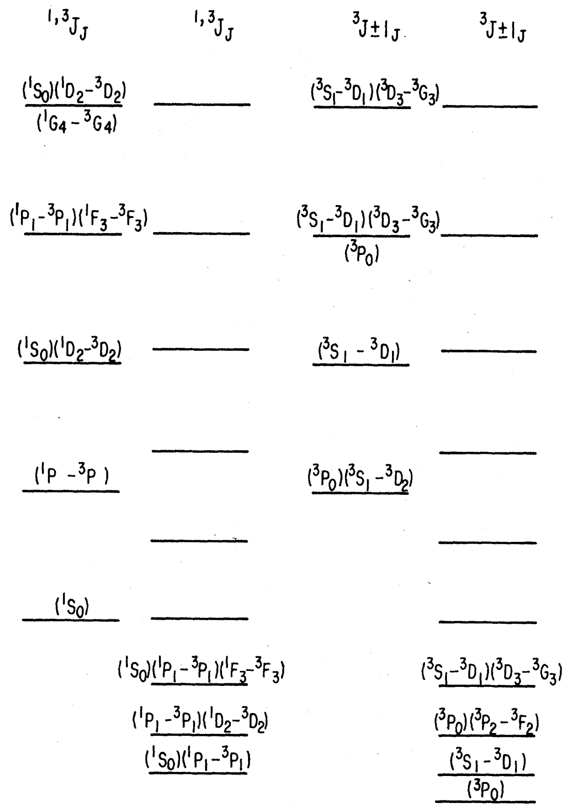

FIG. 1. Schematic energy-level diagram for the nonrelativistic region of g^2/m^3.

$$\frac{(E_{\rm NR})_{n,j}}{2m}=(g^2/m^3)^{1/2}(2n+j+\tfrac{3}{2})\quad n,j=0,1,\ldots . \tag{46}$$

$\varphi_j^{A(+)}$ is complicated because the equation for it contains $\varphi_j^{V(=)}$. Nevertheless, (46) is the expression for the energy levels of a nonrelativistic simple harmonic oscillator, as expected.

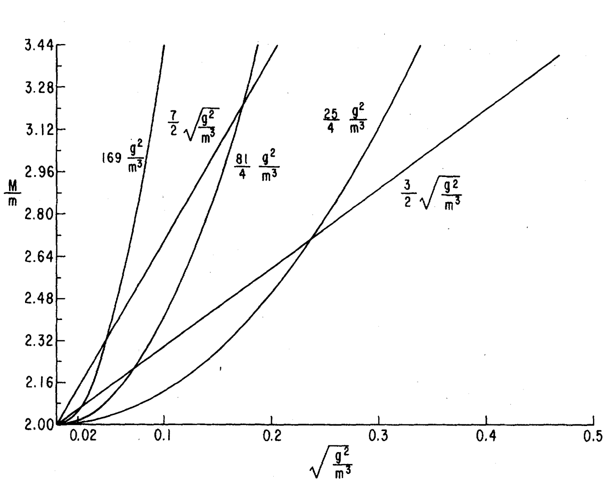

FIG. 2. $^1S_0(0^-)$ energy levels as a function of coupling strengths in the nonrelativistic approximation.

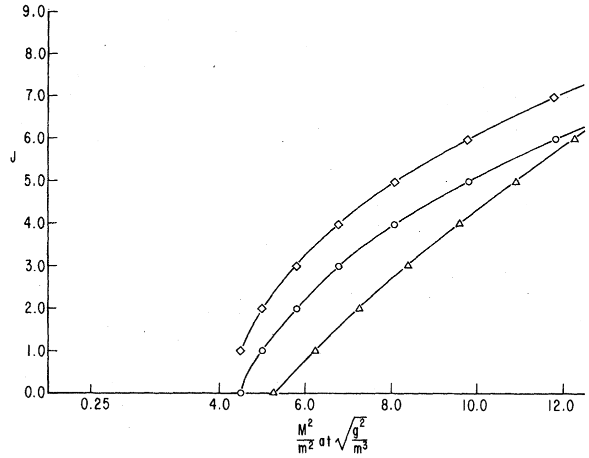

FIG. 3. Angular momentum versus mass squared for the ^1J_J—^2J_J states in the nonrelativistic approximation and for $(g^2/m^3)^{1/2}=0.1$.

Another family of solutions is also easily extracted. The equation for $\varphi^{A(-)}$ is independent of all the other amplitudes, though it appears in the equation for them. Thus we find

$$\varphi^{A(-)}\propto L_n^{j+1/2}\left(\frac{m^3}{2g^2}\,\frac{\xi^2}{2n+j+\frac{1}{2}}\right) \tag{47}$$

and, more importantly, we find the spectrum

$$(E_{\rm NR})_{n,j}=\frac{g^2}{m^3}(2n+j+\tfrac{1}{2})^2\,,$$

$$n=0,1,2,\ldots,\quad j=1,2,3,\ldots . \tag{48}$$

This is a surprising result. The energy spectrum of this family of solutions has spacing proportional to the energy, rather than being evenly spaced. Also $E\propto g^2/m^3$ rather than $(g^2/m^3)^{1/2}$, therefore, these levels lie very much below the ordinary levels represented in Eq. (46). It is easy

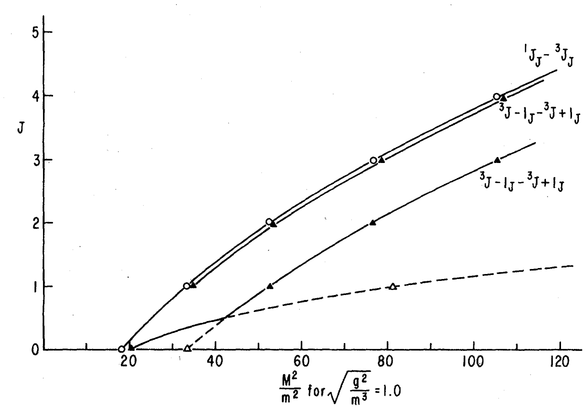

FIG. 4. Regge trajectories in the nonrelativistic approximation for $(g^2/m^3)^{1/2}=1.0$.

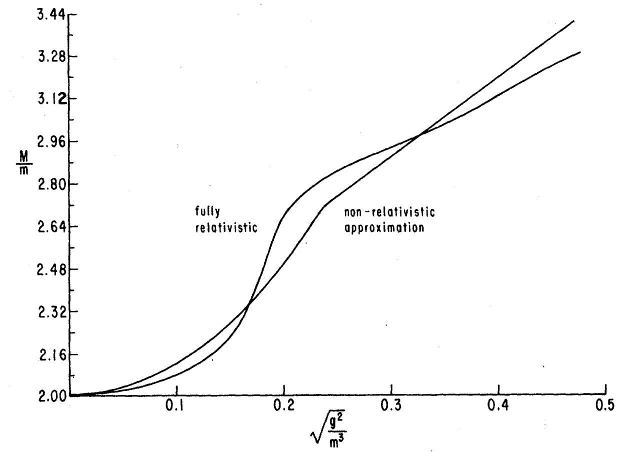

FIG. 5. Comparison of the fully relativistic numerical calculation of the ground-state energy with the nonrelativistic approximation.

to show that (46) and (48) exhaust the possibilities for the energy spectrum in this problem. In general, the wave functions are more difficult to obtain and we shall not concern ourselves with them at this point. The first five figures show the nonrelativistic results explicitly.

Figure 1 is an energy-level diagram for the lowest-lying states in the nonrelativistic regime. Figure 2 shows the $^1S_0(0^-)$ energy levels as a function of coupling constant in the nonrelativistic approximation. Figure 3 shows the relationship between angular momentum and mass for weak coupling. Figure 4 shows the nonrelativistic approximation to this for a coupling that is much stronger but the effect of the anomalous family of energy levels is clearly shown.

V. NUMERICAL RESULTS

Because of the complexity of Eq. (40), it is necessary to resort to numerical techniques to solve it. Luckily, standard approaches work well. If we change variables in Eq. (40) by setting

$$k/m = \tan\zeta \tag{49}$$

then we obtain an equation on the finite interval $[0, \pi/2]$

$$(r\sec\zeta - s)\varphi_\alpha^{(H)} = \left[\cos^2\zeta\frac{\partial^2}{\partial\zeta^2} + \tfrac{1}{2}\sin2\zeta\frac{\partial}{\partial\zeta} + \frac{3}{4} - \frac{l_\alpha(l_\alpha+1)}{\tan^2\zeta}\right] \times\left[\sum_\beta \frac{C_\alpha^{(H)}\mathfrak{M}_{\alpha\beta}^{(H)}}{m^2}\varphi_\beta^{(H)}\right], \tag{50}$$

where

$$r = \frac{m^2M}{g^2}, \quad s = \frac{mM^2}{2g^2}. \tag{51}$$

The boundary condition is that $\varphi_\alpha^{(H)}$ vanish at the end points. We now define a grid $\{\zeta_i\}$ on the interval $[0, \pi/2]$ and arrive at the difference equation

$$\begin{aligned}(r\sec\zeta_i - s)\varphi_\alpha^{(H)}(\zeta_i) = {}&\left(\frac{2\cos^2\zeta_i}{\Delta_i(\Delta_{i+1}+\Delta_i)} - \frac{\sin2\zeta_i}{4\Delta_i}\right)\sum_\beta\frac{C_\alpha^{(H)}}{m^2}\mathfrak{M}_{\alpha\beta}^{(H)}(\zeta_{i-1})\varphi_\beta^{(H)}(\zeta_{i-1})\\ &-\left(\frac{2\cos\zeta_i}{\Delta_i\Delta_{i+1}} - \frac{\sin2\zeta_i(\Delta_{i+1}-\Delta_i)}{4\Delta_i\Delta_{i+1}} + \frac{3}{4} - \frac{l_\alpha(l_\alpha+1)}{\tan^2\zeta_i}\right)\sum_\beta\frac{C_\alpha^{(H)}}{m^2}\mathfrak{M}_{\alpha\beta}^{(H)}(\zeta_i)\varphi_\beta^{(H)}(\zeta_i)\\ &+\left(\frac{2\cos^2\zeta_i}{\Delta_{i+1}(\Delta_{i+1}+\Delta_i)} + \frac{\sin2\zeta_i}{4\Delta_i}\right)\sum_\beta\frac{C_\alpha^{(H)}}{m^2}\mathfrak{M}_{\alpha\beta}^{(H)}(\zeta_{i+1})\varphi_\beta^{(H)}(\zeta_{i+1}),\end{aligned} \tag{52}$$

where $\Delta_i = \zeta_i - \zeta_{i-1}$.

To put this into a form suitable for computation, it is necessary to stack the components of $\varphi_\alpha^{(H)}$. Define the vector $q^{(H)}$ such that

$$q_{8(i-1)+\alpha}^{(H)} = \varphi_\alpha^{(H)}(\zeta_i). \tag{53}$$

Also define the matrices $Q^{(H)}$ and K,

$$Q_{8(i-1)+\alpha,8(i-2)+\beta}^{(H)} = \left(\frac{2\cos^2\zeta_i}{(\Delta_{i+1}+\Delta_i)\Delta_i} - \frac{\sin2\zeta_i}{4\Delta_i}\right)\frac{C_\alpha^{(H)}}{m^2}\mathfrak{M}_{\alpha\beta}^{(H)}(\zeta_{i-1}), \tag{54}$$

$$Q_{8(i-1)+\alpha,8(i-1)+\beta}^{(H)} = -\left(\frac{2\cos^2\zeta_i}{\Delta_i\Delta_{i+1}} - \frac{\sin2\zeta_i(\Delta_{i+1}-\Delta_i)}{4\Delta_i\Delta_{i+1}} + \frac{3}{4} - \frac{l_\alpha(l_\alpha+1)}{\tan^2\zeta_i}\right)\frac{C_\alpha^{(H)}}{m^2}\mathfrak{M}_{\alpha\beta}^{(H)}(\zeta_i), \tag{55}$$

$$Q_{8(i-1)+\alpha,8i+\beta}^{(H)} = \left(\frac{2\cos^2\zeta_i}{\Delta_{i+1}(\Delta_{i+1}+\Delta_i)} + \frac{\sin2\zeta_i}{4\Delta_i}\right)\frac{C_\alpha^{(H)}}{m^2}\mathfrak{M}_{\alpha\beta}^{(H)}(\zeta_{i+1}), \tag{56}$$

$$K_{8(i-1)+\alpha,8(i-1)+\beta} = \left(\sec\zeta_i - \frac{M}{2m}\right)\delta^{\alpha\beta}, \tag{57}$$

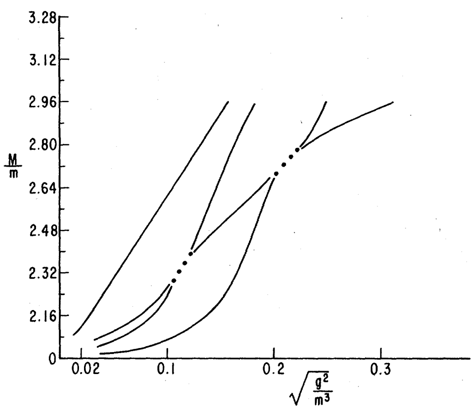

FIG. 6. The four lowest-lying $^1S_0$ states as function of coupling constant in the fully relativistic equation. The dotted section where energy levels cross are places where the numerical technique yielded complex values for g^2, though with small imaginary part. Such crossing phenomena are no surprise and are a symptom of the inherent approximations made in reducing the continuous differential equation to a finite difference equation.

and all other components of $Q^{(H)}$ and K are zero. These definitions make Eq. (52) into the generalized eigenvalue problem

$$\sum_b Q_{ab}^{(H)} q_b^{(H)} = \frac{m^3}{g^2} \frac{M}{m} \sum_c K_{ac} q_c^{(H)} . \tag{58}$$

The result of performing this calculation is shown in the Figs. 5–9. Figure 5 shows the ground-state energy as a function of coupling constant and compares the result of numerically integrating the fully relativistic equation with the nonrelativistic approximation discussed above. The

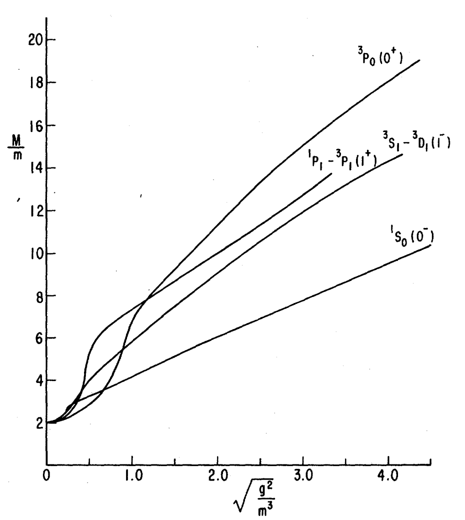

FIG. 7. Lowest-lying $J=0$ and $J=1$ energy levels as a function of coupling constant.

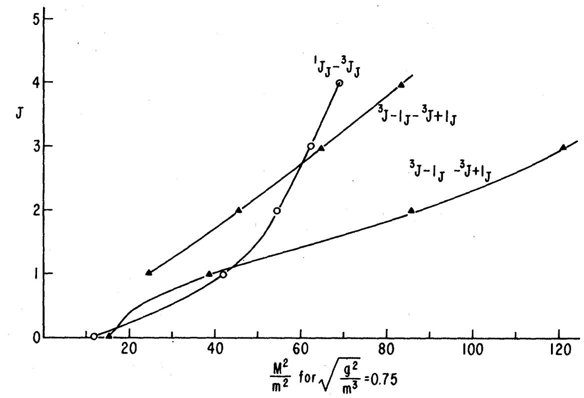

FIG. 8. Regge trajectories for $(g^2/m^3)^{1/2} = 0.75$.

agreement is quite good over a surprisingly large range in g^2/m^3. Figure 6 shows the numerical results for the lowest-lying $^1S_0$ states and Fig. 7 shows the lowest-lying states in both the $^{1,3}J_J$ and $^3J \pm 1_J$ families.

To discuss the phenomenology that arises from these calculations, we return to the nonrelativistic result for a moment. From Eqs. (46) and (48), it is apparent that the lowest-lying $^3P_0$ state lies below the $^1S_0$ for couplings up to $g^2/m^3 = 1$. (This is a value for which the weak-coupling approximation does not apply; nevertheless, the point is that up to very strong coupling, it is impossible to keep the pion lower in mass than the δ.) This means that in order to apply this model to the mesons, very strong coupling is required. Figure 8 shows that a coupling of $(g^2/m^3)^{1/2} = 0.75$ is just sufficient to drop the $^1S_0$ state below the $^3P_0$, but that the $^3S_1$-$^3D_1$ state lies above $^3P_0$, which would place the ϵ lower than the ρ.

In Fig. 9, we have gone still higher in coupling, to $(g^2/m^3)^{1/2} = 1$. At this coupling we find, at last, that the model can be brought into qualitative agreement with the known meson spectrum. The

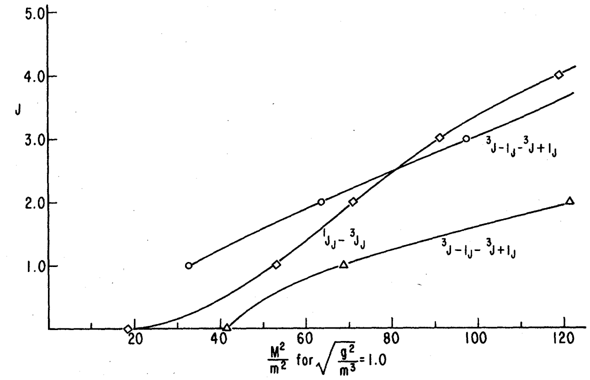

FIG. 9. Regge trajectories for $(g^2/m^3)^{1/2} = 1.0$.

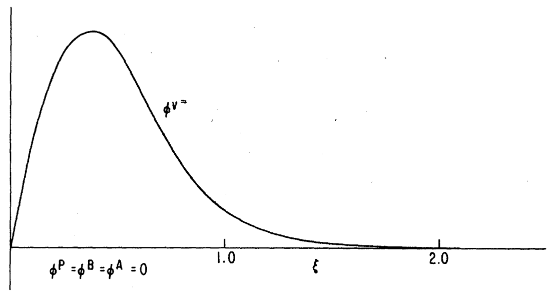

FIG. 10. ${}^1S_0$ amplitude for $(g^2/m^3)^{1/2} = 0.08$ and $M/m = 2.48$.

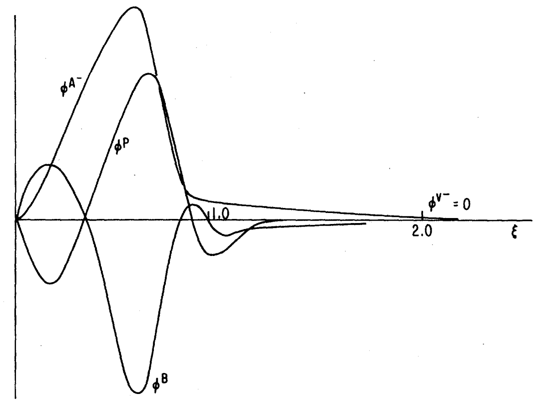

FIG. 11. ${}^1S_0$ amplitude for $(g^2/m^3)^{1/2} = 0.13$ and $M/m = 2.48$.

pion is lower in energy than the ϵ while the ρ and ω lie below the B. Finally, in Figs. 10 and 11 the ${}^1S_0$ amplitudes are plotted. Figure 10 is a plot of the amplitudes when $\phi^{V(-)} \neq 0$. We see for this example that all the other amplitudes are zero. In Fig. 11 the opposite is true. This is one of the anomalous states determined by $\phi^{A(-)}$.

VI. CONCLUSIONS

Despite the inherent complexity of the Bethe-Salpeter formalism for fermion-antifermion systems, we have found it possible to extend a previously derived theory for spin-0 bosons to the case of spin $\frac{1}{2}$ in a fairly straightforward way.

It is surprising, but the addition of spin seems to introduce new structure in the spectrum for even the nonrelativistic domain. The existence of these states is a result only of the way in which we have incorporated spin into the formalism. However, their existence does not prove to be a disaster since they scramble the order of states one expects to see from purely nonrelativistic models and, in fact, at strong enough couplings, bring the model into qualitative agreement with the phenomenology of the known mesons. The disadvantage is, of course, that at the very strong couplings needed to fit the known meson spectrum they require that more than $\frac{2}{3}$ of the mass of even the pion must be in binding energy, and the internal constituents are relativistic for all the mesons.

ACKNOWLEDGMENTS

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[1] J. R. Henley, Phys. Rev. D 20, 2532 (1979).
[2] P. Droz-Vincent, Phys. Rev. D 19, 702 (1979).
[3] H. Bethe and E. Salpeter, Phys. Rev. 84, 1232 (1951).
[4] N. Nakanishi, Prog. Theor. Phys. Suppl. 43, 1 (1969).
[5] A. Swift and B. Lee, Phys. Rev. 131, 1857 (1963).
[6] M. Gourdin, Nuovo Cimento 7, 338 (1958).
[7] M. Gourdin, Ann. Phys. (Paris) 4, 595 (1959).
[8] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1957).
[9] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).